

On multi-coupled points of continuous mappings in generalized metric spaces

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Abstract

During this paper, a new definition of $\mathcal{M}^\ominus$ -continuous mapping, $\mathcal{M}^\ominus$ -mixed monotone and multi-coupled points are introduced. Also, we proved the multi-coupled point theorem in complete S.metric space.

Subject Classification: Primary 46B20, 46C50, Secondary 39B82.

Keywords: Metric space, Fixed point, Partially ordered set, Continuous mapping.

1. Introduction

Fixed point theory is one of the theories that help to develop mathematical approaches and benefit from them in both pure and applied mathematics. Therefore, many authors have resorted to studying it to create an appropriate framework for modeling a variety of known practical problems. In mathematics and other sciences, its study has also continued in metric and standard spaces, being extremely important in finding the best solutions, as several generalizations appeared to it during the past few decades, see [1-8]. Now, we recall some of the initial concepts found in [5].

Definition 1.1: Let $\mathcal{X}^\ominus \neq \emptyset$, Let $\mathcal{G}: \mathcal{X}^\ominus \times \mathcal{X}^\ominus \times \mathcal{X}^\ominus \rightarrow [0, \infty)$ be a mapping satisfy the following:

- i. $\mathcal{G}(x, y, z) = 0$ if $x = y = z = 0$,
- ii. $\mathcal{G}(x, y, z) > 0, \forall x, y, z \in \mathcal{X}^\ominus, x \neq y$
- iii. $\mathcal{G}(x, x, y) \leq \mathcal{G}(x, y, z), \forall x, y, z \in \mathcal{X}^\ominus, z \neq y$

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- iv. $\mathfrak{S}(x, x, y) = \mathfrak{S}(x, z, y) = \mathfrak{S}(y, z, x) = \dots$
 v. $\mathfrak{S}(x, y, z) \leq \mathfrak{S}(x, v, v) + \mathfrak{S}(v, y, z), \forall x, y, z, v \in \mathcal{X}^{\mathfrak{S}}$

Then the pair $(\mathcal{X}^{\mathfrak{S}}, \mathfrak{S})$ is called generalized metric space $(\mathfrak{S}, \mathcal{M}, \mathcal{S})$.

2. Main Results

This section contains the definition of $\mathcal{M}^{\mathfrak{S}}$ –continuous mapping, $\mathcal{M}^{\mathfrak{S}}$ –mixed monotone and multicoupled fixed point. Also, we prove the existence and uniqueness of the multi-coupled theorem in $\mathfrak{S}$ metric space.

Definition 2.1: Let $(\mathcal{X}^{\mathfrak{S}}, \leq)$ be a partially ordered set $(\mathcal{P}, \mathcal{O}, \mathcal{S})$, we say that the mapping $\Gamma: \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \rightarrow \mathcal{X}^{\mathfrak{S}}$ is

1. $\mathcal{M}^{\mathfrak{S}}$ – continuous mapping if for all $\varepsilon > 0$, there exist $\zeta > 0$ such that for all $e_i, \sigma_i, \varsigma_i \in \mathcal{X}^{\mathfrak{S}}, i = 1, 2, 3, 4$,
 If $\mathfrak{S}((e_1, e_2, e_3, e_4), (\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)) < \zeta$, then
 $\mathfrak{S}(\Gamma(e_1, e_2, e_3, e_4), \Gamma(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \Gamma(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)) < \varepsilon$
 or, for any sequences $\langle \tilde{q}_{ni} \rangle$ convergent to a_i respectively where, $i = 1, 2, 3, 4$ then, we have $\langle \Gamma \tilde{q}_{ni} \rangle$ convergent to Γa_i respectively, for all $s_1, \tau_1, s_2, \tau_2, e, v \in \mathcal{X}^{\mathfrak{S}}$, such. that, $r_1 \leq r_2$, and $\tau_1 \leq \tau_2 \in \mathcal{X}^{\mathfrak{S}}$, then we have $(s_1, \tau, e, v) \leq (s_2, \tau, e, v)$ and $(s, \tau_1, e, v) \leq (s, \tau_2, e, v)$. On the other hand, for all $s, \tau, e_1, v_1, e_2, v_2 \in \mathcal{X}^{\mathfrak{S}}$ if $e_1 \geq e_2$, and $v_1 \geq v_2 \in \mathcal{X}^{\mathfrak{S}}$, then we have $(s, \tau, u_1, v) \geq (s, \tau, u_2, v)$ and $(s, \tau, e, v_1) \geq (s, \tau, e, v_2)$. We denote by $\mathfrak{D}$ the set of all continuous mappings having mixed monotone properties.
2. Appoint (s, τ, e, v) lies in $\mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}}$ is called multi-coupled fixed point $(m.c.f.\mathcal{P})$ of Γ if satisfy $s = F(s, \tau, e, v), F(\tau, s, e, v) = \tau, F(e, v, s, \tau) = u$ and $F(v, e, s, \tau) = v$.

Theorem 2.2: Let $(\mathcal{X}^{\mathfrak{S}}, \mathfrak{S})$ be a complete $(\mathfrak{S}, \mathcal{M}, \mathcal{S})$. and $\Gamma: \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \times \mathcal{X}^{\mathfrak{S}} \rightarrow \mathcal{X}^{\mathfrak{S}}$ be a mapping lies in $\mathfrak{D}$. Define the following. For all $s, e, \tau, v, \varsigma, u, l, \rho, v, k, q \in \mathcal{X}^{\mathfrak{S}}$ such that $s \geq e \geq \varepsilon, \tau \geq v \geq \varsigma, u \leq l \leq \rho$ and $v \leq k \leq q$, then the inequality holds, $\mathfrak{S}(\Gamma(s, \tau, e, v), \Gamma(e, v, l, k), \Gamma(\varepsilon, \varsigma, \rho, q))$

$$\leq \frac{k}{2} [\mathfrak{S}(s, e, \varepsilon) + \mathfrak{S}(\tau, v, \varsigma) + \mathfrak{S}(e, l, \rho) + \mathfrak{S}(v, k, q)] \quad (1)$$

for some $k \in [0, 1)$. Then the set of $(m.c.f.\mathcal{P})$ of $\Gamma \neq \emptyset$.

Proof: By the condition of the theorem their exist $s_0, v_0 \in \mathcal{X}$ such that

$s_0 \leq \Gamma(s_0, \tau_0, e_0, v_0), \tau_0 \leq \Gamma(\tau_0, s_0, e_0, v_0), e_0 \geq \Gamma(e_0, v_0, s_0, \tau_0)$ and

$v_0 \geq \Gamma(v_0, e_0, \tau_0, s_0)$, we define $s_1, \tau_1, e_1, v_1 \in \mathcal{X}^{\mathfrak{S}}$ as:

$s_1 = \Gamma(s_0, \tau_0, e_0, v_0) \geq s_0, \tau_1 = \Gamma(\tau_0, s_0, e_0, v_0) \geq \tau_0, e_1 = \Gamma(e_0, v_0, s_0, \tau_0) \leq e_0$ and

$v_1 = \Gamma(v_0, e_0, \tau_0, s_0) \leq v_0$. Let $s_2 = \Gamma(s_1, \tau_1, e_1, v_1), \tau_2 = \Gamma(\tau_1, s_1, e_1, v_1) \geq \tau_1, e_2 = \Gamma(e_1, v_1, s_1, \tau_1) \leq e_1$ and $v_2 = \Gamma(v_1, e_1, \tau_1, s_1)$. we write,

$$\begin{aligned}
& \Gamma^2(s_0, \tau_0, e_0, v_0) = \\
& \Gamma(F(s_0, \tau_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \Gamma(e_0, v_0, s_0, \tau_0), \Gamma(v_0, e_0, \tau_0, s_0)) = \\
& \Gamma((s_1, \tau_1, e_1, v_1)) = s_2. \text{ And, } \Gamma^2(\tau_0, s_0, e_0, v_0) = \\
& \Gamma(\Gamma(e_0, s_0, \tau_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), \Gamma(e_0, v_0, s_0, \tau_0), \Gamma(v_0, e_0, \tau_0, s_0)) = \\
& \Gamma(\tau_1, s_1, e, v) = \tau. \text{ Now, we have, } s_2 = \Gamma^2(s_0, \tau_0, e_0, v_0) = \Gamma(s_1, \tau_1, e_1, v_1) \geq \\
& \Gamma(s_0, \tau_0, \varrho_0, v_0) = s_1 \geq s_0, t_2 = \Gamma^2(\tau_0, s_0, \varrho_0, v_0) = \Gamma(\tau_1, s_1, \varrho_1, v) \geq \\
& \Gamma(\tau_0, s_0, e_0, v_0) = \tau_1 \geq \tau_0. \text{ And, } \varrho_2 = \Gamma^2(e_0, v_0, s_0, \tau_0) = \Gamma(e_1, v_1, s_1, \tau_1) \leq \\
& \Gamma(e_0, v_0, s_0, \tau_0) = e_1 \leq e_0, v_2 = \Gamma^2(v_0, e_0, \tau_0, s_0) = \Gamma(v_1, e_1, s_1, \tau_1) \leq \\
& \Gamma(v_0, e_0, \tau_0, s_0) = v_1 \leq v_0. \text{ Continuing, } \forall n \geq 0, s_{n+1} = \Gamma^{n+1}(s_0, \tau_0, e_0, v_0) = \\
& \Gamma(\Gamma^n(s_0, t_0, e_0, v_0), \Gamma^n(\tau_0, s_0, e_0, v_0), \Gamma^n(e_0, v_0, s_0, \tau_0), \Gamma^n(v_0, e_0, \tau_0, s_0)) \\
& t_{n+1} = \Gamma^{n+1}(\tau_0, s_0, e_0, v_0) = \\
& \Gamma(\Gamma^n(\tau_0, s_0, e_0, v_0), \Gamma^n(\tau_0, s_0, e_0, v_0), \Gamma^n(e_0, v_0, s_0, \tau_0), \Gamma^n(v_0, e_0, \tau_0, s_0)) \\
& u_{n+1} = \Gamma^{n+1}(\varrho_0, v_0, s_0, \tau_0) = \\
& \Gamma(\Gamma^n(e_0, v_0, s_0, \tau_0), \Gamma^n(v_0, e_0, \tau_0, s_0), \Gamma^n(s_0, \tau_0, e_0, v_0), \Gamma^n(\tau_0, s_0, e_0, v_0)) \\
& \text{ And, } v_{n+1} = \Gamma^{n+1}(v_0, e_0, \tau_0, s_0) = \\
& \Gamma(\Gamma^n(v_0, e_0, \tau_0, s_0), \Gamma^n(s_0, \tau_0, e_0, v_0), \Gamma^n(\tau_0, s_0, e_0, v_0)) \Rightarrow \forall n \geq 0, s_0 \leq \\
& \Gamma(s_0, \tau_0, e_0, v_0) = s_1 \leq \Gamma^2(s_0, \tau_0, e_0, v_0) = s_2 \leq \dots \leq \\
& \Gamma^{n+1}(s_0, \tau_0, e_0, v_0) = s_{n+1} \leq (2) \\
& \tau_0 \geq \Gamma(\tau_0, s_0, e_0, v_0) = t_1 \leq \Gamma^2(\tau_0, s_0, e_0, v_0) = \tau_2 \geq \dots \geq \\
& \Gamma^{n+1}((\tau_0, s_0, e_0, v_0)) = \tau_{n+1} \geq \dots (3) \\
& \varrho_0 \leq \Gamma(e_0, v_0, s_0, \tau_0) = \varrho_1 \leq \Gamma^2(e_0, v_0, s_0, \tau_0) = e_2 \leq \dots \leq \\
& \Gamma^{n+1}(e_0, v_0, s_0, \tau_0) = u_{n+1} \leq \dots (4) \\
& v_0 \geq \Gamma(v_0, e_0, \tau_0, s_0) = v_1 \leq \Gamma^2(v_0, e_0, \tau_0, s_0) = v_2 \geq \dots \geq \Gamma^{n+1}(v_0, e_0, \tau_0, s_0) \\
& = v_{n+1} \geq \dots (5)
\end{aligned}$$

If $(s_{n+1}, \tau_{n+1}, e_{n+1}, v_{n+1}) = (s_n, \tau_n, e_n, v_n)$. Then the set of $(m.c.f.P)$ of $\Gamma \neq \emptyset$. So, we assume $(s_{n+1}, \tau_{n+1}, e_{n+1}, v_{n+1}) \neq (s_n, \tau_n, e_n, v_n)$ for all $n \geq 0$ that is, $s_{n+1} = \Gamma(s_n, \tau_n, e_n, v_n) \neq s_n$ or $\tau_{n+1} = \Gamma(\tau_n, s_n, e_n, v_n) \neq \tau_n$,
 $v_{n+1} = \Gamma(e_n, v_n, s_n, \tau_n) \neq v_n$ or $v_{n+1} = \Gamma(v_n, e_n, s_n, \tau_n) \neq v_n$.

We claim, $\mathfrak{S}(\Gamma^{n+1}(s_0, \tau_0, e_0, v_0), \Gamma^{n+1}(s_0, \tau_0, e_0, v_0), \Gamma^n(s_0, \tau_0, e_0, v_0)) \leq$

$$\begin{aligned}
& \frac{k^n}{2} [\mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0), \Gamma(v_0, e_0, \tau_0, s_0), v_0)] (6) \\
& \mathfrak{S}(\Gamma^{n+1}(\tau_0, s_0, e_0, v_0), \Gamma^{n+1}(\tau_0, s_0, e_0, v_0), \Gamma^n(\tau_0, s_0, e_0, v_0)) \leq
\end{aligned}$$

$$\begin{aligned}
& \frac{k^n}{2} [\mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0), \Gamma(v_0, e_0, \tau_0, s_0), v_0)] (7) \\
& (\Gamma^{n+1}\Gamma(e_0, v_0, s_0, \tau_0), \Gamma^{n+1}\Gamma(e_0, v_0, s_0, \tau_0), \Gamma^n\Gamma(e_0, v_0, s_0, \tau_0)) \leq
\end{aligned}$$

$$\begin{aligned}
& \frac{k^n}{2} [\mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0), \Gamma(v_0, e_0, \tau_0, s_0), v_0) +
\end{aligned}$$

$$\begin{aligned}
& \mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) \quad (8) \\
& (\Gamma^{n+1}(v_0, e_0, \tau_0, s_0), \Gamma^{n+1}(v_0, e_0, \tau_0, s_0), \Gamma^n(v_0, e_0, \tau_0, s_0)) \leq \\
& \frac{k^n}{2} [\mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0), \Gamma(v_0, e_0, \tau_0, s_0), w_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0)] \quad (9). \text{ For } = 1, \text{ we have} \\
& \mathfrak{S}(\Gamma^2(s_0, \tau_0, e_0, v_0), \Gamma^2(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0)) = \\
& \mathfrak{S} \left(\begin{array}{l} \Gamma(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \Gamma(e_0, v_0, s_0, \tau_0), \Gamma(v_0, e_0, \tau_0, s_0)), \\ \Gamma(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \Gamma(e_0, v_0, s_0, \tau_0), \Gamma(v_0, e_0, \tau_0, s_0)) \end{array} \right) \\
& \quad , \Gamma(s_0, \tau_0, e_0, v_0) \\
& \leq \\
& \frac{k}{2} \mathfrak{S}[(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0), \Gamma(v_0, e_0, \tau_0, s_0), v_0)] \\
& \text{Since, } \Gamma(s_0, \tau_0, e_0, v_0) \geq s_0, \Gamma(\tau_0, s_0, e_0, v_0) \geq t_0, \Gamma(e_0, v_0, s_0, \tau_0) \geq \\
& u_0, \Gamma(v_0, e_0, \tau_0, s_0) \leq t_0 \\
& \text{Similarly, we prove (6),(7),(8), and (9) hold for } n = 1. \text{ Now, we assume that} \\
& (6), (7), (8), \text{ and (9) hold for } n = m. \text{ Using the facts that } \Gamma^{m+1}(s_0, \tau_0, e_0, v_0) \geq \\
& \Gamma^m(s_0, \tau_0, e_0, v_0) \text{ and } \Gamma^{m+1}(\tau_0, s_0, e_0, v_0) \leq \Gamma^m(\tau_0, s_0, e_0, v_0), \text{ we get} \\
& \mathfrak{S}(\Gamma^{m+2}(s_0, \tau_0, e_0, v_0), \Gamma^{m+2}(s_0, \tau_0, e_0, v_0), \Gamma^{m+1}(s_0, \tau_0, e_0, v_0)) \\
& = \mathfrak{S}(\Gamma(\Gamma^{m+1}(s_0, \tau_0, e_0, v_0), \Gamma^{m+1}(\tau_0, s_0, e_0, v_0)), \Gamma^{m+1}(e_0, v_0, s_0, \tau_0), \Gamma^{m+1}(v_0, e_0, \tau_0, s_0), \\
& \Gamma(\Gamma^{m+1}(s_0, \tau_0, e_0, v_0), \Gamma^{m+1}(\tau_0, s_0, e_0, v_0)), \Gamma^{m+1}(e_0, v_0, s_0, \tau_0), \Gamma^{m+1}(v_0, e_0, \tau_0, s_0), \\
& \Gamma(\Gamma^m(s_0, \tau_0, e_0, v_0), \Gamma^m(\tau_0, s_0, e_0, v_0), \Gamma^{m+1}(e_0, v_0, s_0, \tau_0), \Gamma^{m+1}(v_0, e_0, \tau_0, s_0))) \leq \\
& \frac{k}{2} [\mathfrak{S}(\Gamma^{m+1}(s_0, \tau_0, e_0, v_0), \Gamma^{m+1}(s_0, \tau_0, e_0, v_0), \Gamma^m(s_0, \tau_0, e_0, v_0)) + \\
& \mathfrak{S}(\Gamma^{m+1}(\tau_0, s_0, e_0, v_0), \Gamma^{m+1}(\tau_0, s_0, e_0, v_0), \Gamma^m(\tau_0, s_0, e_0, v_0)) + \\
& \mathfrak{S}(\Gamma^{m+1}(e_0, v_0, s_0, \tau_0), \Gamma^{m+1}(e_0, v_0, s_0, \tau_0), \Gamma^m(e_0, v_0, s_0, \tau_0)) + \\
& \mathfrak{S}(\Gamma^{m+1}(v_0, e_0, \tau_0, s_0), \Gamma^{m+1}(v_0, e_0, \tau_0, s_0), \Gamma^m(v_0, e_0, \tau_0, s_0))] \leq \\
& \frac{k}{2} \left[\frac{k^m}{2} [\mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \right. \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0) + \\
& \Gamma(v_0, e_0, \tau_0, s_0), v_0)] \leq \frac{k^{m+1}}{2} [\mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0) + \\
& \Gamma(v_0, e_0, \tau_0, s_0), v_0)]. \text{ Similarly, prove the rest} \\
& \mathfrak{S}(\Gamma^m(s_0, \tau_0, e_0, v_0), \Gamma^m(s_0, \tau_0, e_0, v_0), \Gamma^n(s_0, \tau_0, e_0, v_0)) \leq \\
& \mathfrak{S}(\Gamma^m(s_0, \tau_0, e_0, v_0), \Gamma^m(s_0, \tau_0, e_0, v_0), \Gamma^{m-1}(s_0, \tau_0, e_0, v_0)) + \\
& \mathfrak{S}(\Gamma^{m-1}(s_0, \tau_0, e_0, v_0), \Gamma^{m-1}(s_0, \tau_0, e_0, v_0), \Gamma^{m-2}(s_0, \tau_0, e_0, v_0)) + \dots +
\end{aligned}$$

$$\begin{aligned}
& \mathfrak{S}(\Gamma^{n+1}(s_0, \tau_0, e_0, v_0), \Gamma^{n+1}(s_0, \tau_0, e_0, v_0), \Gamma^n(s_0, \tau_0, e_0, v_0)) \leq \\
& \frac{k^{m-1} + \dots + k^n}{2} [\mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(F(v_0, e_0, \tau_0, s_0) + F(v_0, e_0, \tau_0, s_0), v_0)] = \\
& \frac{k^n(1+k+\dots+k^{m-n-1})}{2} [G(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& G(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& G(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(u_0, v_0, s_0, \tau_0), u_0) + G(\Gamma(v_0, e_0, \tau_0, s_0) + \\
& \Gamma(v_0, u_0, \tau_0, s_0), v_0)] < \\
& \frac{k^2}{2(1-k)} [\mathfrak{S}(\Gamma(s_0, \tau_0, e_0, v_0), \Gamma(s_0, \tau_0, e_0, v_0), s_0) + \\
& \mathfrak{S}(\Gamma(\tau_0, s_0, e_0, v_0), \Gamma(\tau_0, s_0, e_0, v_0), \tau_0) + \\
& \mathfrak{S}(\Gamma(e_0, v_0, s_0, \tau_0), \Gamma(e_0, v_0, s_0, \tau_0), e_0) + \\
& \mathfrak{S}(\Gamma(v_0, e_0, \tau_0, s_0) + \Gamma(v_0, e_0, \tau_0, s_0), v_0)]. \text{ That is,} \\
& \mathfrak{S}(s_m, s_m, s_n) < \frac{k^n}{2(1-k)} [\mathfrak{S}(s_1, s_1, s_0) + \mathfrak{S}(\tau_1, \tau_1, \tau_0) + \mathfrak{S}(e_1, e_1, e_0) + \\
& \mathfrak{S}(v_1, v_1, v_0)]. \\
& \text{Thus } \mathfrak{S} \lim_{m,n \rightarrow \infty} (s_m, s_m, s_n) = 0
\end{aligned}$$

Thus by Lemma 4, and by $\langle s_n \rangle$ is $\mathfrak{S}$ -Cauchy so, it is convergent in the complete $(\mathfrak{S}, \mathcal{M}, \mathcal{S})$. Let $s_n \rightarrow s$ as $n \rightarrow \infty$. Similarly $\langle \tau_n \rangle, \langle e_n \rangle, \langle v_n \rangle$ also $\mathfrak{S}$ -Cauchy and hence is convergent in the complete $(\mathfrak{S}, \mathcal{M}, \mathcal{S})$.

Let, $\tau_n \rightarrow \tau, e_n \rightarrow e, v_n \rightarrow v$ as $n \rightarrow \infty$,

$$\begin{aligned}
& \mathfrak{S}(\Gamma(s_n, \tau_n, e_n, v_n), \Gamma(s_n, \tau_n, e_n, v_n), s_n) \leq \frac{k^n}{2} [\mathfrak{S}(s_1, s_1, s_0) + \mathfrak{S}(\tau_1, \tau_1, \tau_0) + \\
& \mathfrak{S}(e_1, e_1, e_0) + \mathfrak{S}(v_1, v_1, v_0)]. \text{ Taking the limit as } n \rightarrow \infty \text{ and since } \Gamma \text{ is} \\
& \mathfrak{S}\text{-continuous we have, } \mathfrak{S}(\Gamma(s, \tau, e, v), \Gamma(s, \tau, e, v), s) \leq 0, \text{ which implies that} \\
& \Gamma(s, \tau, e, v) = s. \text{ Similarly,} \\
& \Gamma(\tau, s, e, v) = \tau, \Gamma(e, v, s, \tau) = e \text{ and } \Gamma(v, e, s, \tau) = v. \quad \text{Then the set} \\
& \text{of } (m.c.f.\mathcal{P}) \text{ of } \Gamma \neq \emptyset
\end{aligned}$$

3. Discussion and Conclusion

We introduced new concepts such as $\mathcal{M}^\mathfrak{S}$ -continuous mapping, $\mathcal{M}^\mathfrak{S}$ -mixed monotone and multi-coupled point and studied the possibility of obtaining the multi-coupled point in $(\mathfrak{S}, \mathcal{M}, \mathcal{S})$.

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Received December, 2021

Revised February, 2022