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New results and application of differential quasi subordinations for higher-order derivatives of meromorphic multivalent functions

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Abstract

The primary purpose of this study is to look at the differential quasi-subordination result for meromorphic multivalent analytic functions. We show how higher-order differential quasi-subordination discoveries can be applied in a variety of ways in an open unit disk. We provide a new class $S^{*}_{\wp,\mathcal{q}}(\xi,\ell,\varphi)$. to represent higher-order derivatives of meromorphic multivalent analytic functions associated with the operator. We have some findings for this class.

Subject Classification: *30C45.*

Keywords: *Multivalent analytic function, Quasi subordination, Convolution.*

1. Introduction

Γ_p class of meromorphic multivalent function $\mathcal{T}$ by

$$\mathcal{T}(z) = \frac{1}{z^{\wp}} + \sum_{n=0}^{\infty} \ell_{n+\wp} z^{n+\wp}, \tag{1.1}$$

which are analytic in $U = \{z \in \mathbb{C}: |z| < 1\}$.

A function $\mathcal{T} \in \delta(\wp, m)$ the multivalent starlike function of order $\mathscr{s}$, if (see[4]) $Re\left\{\frac{z\,\mathcal{T}'(z)}{\mathcal{T}(z)}\right\} > \mathscr{s}, (z \in U, 0 \le \mathscr{s} < \wp)$.

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Denote the class of multivalent starlike functions by $S_p^*(\alpha)$. In particular the class $S_p^*(0) = S_p^*$.

Definition 1.1: [9] If $\mathcal{T}$ and g are analytic functions in U and if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$, we say that $\mathcal{T}$ is subordinate to g ($\mathcal{T} \prec g$ or $\mathcal{T}(z) \prec g(z), z \in U$) or g superordinate to $\mathcal{T}$, such that $\mathcal{T}(z) = g(w(z))$. The term "quasi-subordination" was coined by Robertson [7]. If there exists an analytic function $|J(z)| \leq 1$ such that $\frac{\mathcal{T}(z)}{J(z)}$ analytic in $\mathcal{U}$ and $\frac{\mathcal{T}(z)}{J(z)} \prec g(z),\ z \in \mathcal{U}$, that is, there exists $w(z)$ Schwarz function then $\mathcal{T}(z) = J(z)(w(z))$.

Albehbah and Darus [1] defined a linear operator $\mathcal{M}_p^*(\ell, j)$ on Γ_p by

$$\mathcal{M}_p^*(\ell, j)\, \mathcal{T}(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \left|\frac{(\ell)_{n+2}}{(j)_{n+2}}\right| \ell_{n+p} z^{n+p}, \tag{1.2}$$

where

$(\ell)_n = \frac{\Gamma(\ell+n)}{\Gamma(\ell)} = \ell(\ell+1)\dots(\ell+n-1), \qquad (j \neq 0, -1, -2, \dots,\ \ell \in \mathbb{C}/\{0\}),$

to simplify the notation, we write $\mathcal{M}_p^*(\ell, j) = \mathcal{M}_p^*(\ell)$.

It follows from (1.1) that

$$z\left(\mathcal{M}_p^*(\ell)\, \mathcal{T}(z)\right)' = \ell \mathcal{M}_p^*(\ell+1)\, \mathcal{T}(z) - (\ell + p)\mathcal{M}_p^*(\ell)\, \mathcal{T}(z). \tag{1.3}$$

We differenting (1.3) for $(\varepsilon + n - 1)$ times, we get

$$z\left(\mathcal{M}_p^*(\ell)\, \mathcal{T}(z)\right)^{\varepsilon} = \ell\left(\mathcal{M}_p^*(\ell+1)\, \mathcal{T}(z)\right)^{\varepsilon-1} - (\ell + p - (\varepsilon - 1))\left(\mathcal{M}_p^*(\ell)\, \mathcal{T}(z)\right)^{\varepsilon-1}$$

$$z\left(\mathcal{M}_p^*(\ell)\, \mathcal{T}(z)\right)^{\varepsilon+n} = \ell\left(\mathcal{M}_p^*(\ell+1)\, \mathcal{T}(z)\right)^{\varepsilon+n-1} - (\ell + p - (\varepsilon + n - 1)\left(\mathcal{M}_p^*(\ell)\, \mathcal{T}(z)\right)^{\varepsilon+n-1} \tag{1.4}$$

Lemma 1.1: [7] *Let $q(z)$ univalent function in U, θ and φ be analytic in domain D containing $q(U)$ with $t(w) \neq 0$ where $w \in q(U)$. Let*
$Q(z) = zq'(z)t(q(z))$ and $r(z) = \theta(q(z)) + O(z)$.
Let

- $Q(z)$ starlike univalent in ,
- $Re\left(\frac{zr'(z)}{O(z)}\right) > 0$.

If $p(z)$ is analytic , then

$$\theta(q(z)) + zp'(z)t(p(z)) \prec \theta(q(z)) + zq'(z)\, r(q(z)),$$

implies $p(z) \prec q(z)$ and $q(z)$ is best dominant of above equation.

Suppose that $\mathcal{K}$ class of analytic and starlike univalent functions k in U with $k(0) = 1$.

2. Results and Application

Definition 2.1: $\mathcal{T} \in \delta(\wp, m)$ is function in the class $S^*_{\wp,q}(\xi, \ell, \varphi)$ if it meets the quasi-subordination requirement:

$$(\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} \prec_q \varphi(z), \qquad (2.1)$$

where $\xi \in \mathbb{C}, \wp, \varepsilon \in \mathbb{N}$ and $\varphi \in \mathcal{K}$.

Several authors studied differential subordinations for higher- order derivatives for another classes, such as [2], and another studied differential subordination of a class of multivalent functions for other classes, like [5,6,8].

Theorem 2.1: *Let* $0 \le \xi < \eta$.

Then $$S^*_{\wp,q}(\eta, \ell, m; \varphi) \subset S^*_{\wp,q}(\xi, \ell, m; \varphi).$$

Assume that

$$g(z) = \frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}. \qquad (2.2)$$

Then, given $g(0) = 1$, the function g is analytic in U.

Since $\mathcal{T} \in S^*_{\wp,q}(\eta, \ell, m; \varphi)$, then, we have

$$(\eta - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(q+n-1)}} - \frac{\eta(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} \prec_q \varphi(z), \qquad (2.3)$$

We use (2.3) and take derivatives on both sides of (2.2) in terms of z.

$$(\eta - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\eta(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}$$
$$= g(z) - \frac{\eta}{(\varepsilon+n-1)}zg'(z) \prec_q \varphi(z).$$

By application of Lemma(1.1), we have

$$g(z) \prec_q \varphi(z). \qquad (2.4)$$

Nothing that $0 \le \frac{\xi}{\eta} < 1$ and that φ is starlike univalent in U, as a result of (2.2), (2.3), and (2.4),

$$(\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} = \frac{\xi}{\eta}\left[(\eta - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\eta(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_{\wp}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}\right] + \left(1 - \frac{\xi}{\eta}\right)g(z) \prec_q \varphi(z).$$

Therefore, $\mathcal{F} \in S^*_{\wp,q}(\xi, \ell, m; \varphi)$.

Theorem 2.2: *Let $Re\{\ell\} \geq 0$ and $a \neq 0$. Then*

$$S^*_{p,q}(\xi, \ell+1, m; \varphi) \subset S^*_{p,q}(\xi, \ell, m; \varphi)$$

Proof: Let $f \in S^*_{p,q}(\eta, \ell+1, m; \varphi)$ and suppose that

$$g(z) = (\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}. \tag{2.5}$$

Then, (1.4) and (2.5) is equivalent to

$$g(z) = \left(\frac{\xi\ell}{\varepsilon+n-1} - 1\right)\frac{\ell(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\ell\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}. \tag{2.6}$$

Differentiating (2.6) and using (1.4), we get

$$\begin{aligned} g(z) + zg'(z) &= \frac{\ell\xi}{\varepsilon+n-1} - 1\left(\frac{\ell(\varepsilon+n-1)!}{\varepsilon!}\,\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{(\varepsilon+n-1)z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}\right) - \\ &\quad \frac{\ell\xi(\varepsilon+n)!}{\varepsilon!}\left(\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\,\right)^{(\varepsilon+n-1)}} + \frac{(q+n-1)z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}\right). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we get

$$\ell g(z) + zg'(z) = (\xi - 1)\frac{\ell(\varepsilon+n-1)!}{\varepsilon!}\,\frac{z\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\ell\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell+1)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}},$$

that is

$$g(z) + \frac{1}{\ell}zg'(z) = (\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{\varepsilon+n}}. \tag{2.8}$$

Since $\mathcal{T} \in S^*_{p,q}(\eta, \ell+1, m; \varphi)$, then it follows from (2.8)

$$g(z) + \frac{1}{a}zg'(z) \prec_q \varphi(z), \qquad (Re\{\ell\} \geq 0, \ell \neq 0).$$

As application of Lemma (1.1) and (2.5) we get $g(z) \prec_q \varphi(z)$.

$$(\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} \prec_q \varphi(z).$$

This shows that $\mathcal{T} \in S^*_{p,q}(\xi, \ell, m; \varphi)$ and the proof is completed.

Theorem 3.3: *Let* $\xi > 0, \beta > 0$ *and* $\mathcal{T} \in S^*_{p,q}(\xi, \ell, m; \beta\varphi + 1 - \beta)$. *If* $\beta \leq \beta_0$, *where*

$$\beta_0 = \frac{1}{2}\left(1 - \frac{(p-q)}{\xi}\int_0^1 \frac{u^{\frac{p-q}{\xi}-1}}{1+u}du\right)^{-1}, \tag{2.9}$$

then $\mathcal{T} \in S^*_{p,q}(0, \ell, m; \varphi)$. *The bound* β_0 *is the sharp when* $\varphi(z) = \frac{1}{1-z}$.

Proof: Suppose that

$$g(z) = \frac{(\varepsilon+n-1)!}{q!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}. \tag{2.10}$$

Let $\mathcal{T} \in S^*_{p,q}(\xi, \ell, m; \beta\varphi + 1 - \beta)$ with $\xi > 0$ and $\beta > 0$. Then, we have

$$g(z) + \frac{\xi}{\varepsilon + n - 1}zg'(z)$$

$$= (\xi - 1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} \prec_q \beta\varphi(z) + 1 - \beta.$$

By Lemma (2.1), we obtain

$$g(z) \prec_q \frac{\beta(\varepsilon+n-1)}{\xi}z^{-\frac{(\varepsilon+n-1)}{\xi}}\int_0^z t^{\frac{(q+n-1)}{\xi}-1}\varphi(t)dt + 1 - \beta = (\beta * \theta)(z), \tag{2.11}$$

where $\theta(z) = \frac{\beta(\varepsilon+n-1)}{\xi}z^{-\frac{(\varepsilon+n-1)}{\xi}}\int_0^z \frac{t^{\frac{(\varepsilon+n-1)}{\xi}-1}}{1-t}dt + 1 - \beta.$ (2.12)

If $0 < \beta \leq \beta_0$, where $\beta_0 > 1$ is given by (2.9) , then it follows from (2.12) that

$$Re(\theta(z)) = \frac{\beta(\varepsilon + n - 1)}{\xi}\int_0^z u^{\frac{(\varepsilon+n-1)}{\xi}-1}Re\left(\frac{1}{1 - uz}\right)du + 1 - \beta$$

$$> \frac{\beta(\varepsilon + n - 1)}{\xi}\int_0^z \frac{u^{\frac{(\varepsilon+n-1)}{\xi}-1}}{1 - u}du + 1 - \beta \geq \frac{1}{2}.$$

By using the Herglotz representation of $\theta(z)$, from (2.10) and (2.11), obtained

$$\frac{(\varepsilon + n - 1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} \prec_q (\varphi * \theta)(z) \prec_q \varphi(z).$$

Since φ is starlike univalent in U, then $\mathcal{F} \in S^*_{p,q}(0, a, m; \varphi)$.

For $\varphi(z) = \frac{1}{1-z}$ and $\mathcal{T} \in \delta(p, m)$ defined by

$$\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}^*_p(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} = \frac{\beta(\varepsilon+n-1)}{\xi}z^{-\frac{(\varepsilon+n-1)}{\xi}}\int_0^z \frac{t^{\frac{(\varepsilon+n-1)}{\xi}-1}}{1-t}dt + 1 - \beta,$$

We have

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}=\beta\varphi(z)\,+1-\beta.$$

Thus

$\mathcal{T}\in S_{p,q}^{*}(\xi,\ell,m;\beta\varphi+1-\beta)$. Also, for $\beta>\beta_0$, we have

$$Re\left\{\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}\right\}\to\frac{\beta(\varepsilon+n-1)}{\xi}\int_0^z\frac{u^{\frac{(\varepsilon+n-1)}{\xi}-1}}{1-u}du+1-\beta<\frac{1}{2},$$

which implies that $\mathcal{T}\notin S_{p,q}^{*}(0,\ell,m;\varphi)$. therefore the bound β_0 cannot be increased where $\varphi(z)=\frac{1}{1-z}$.

Theorem 2.4 *Let $\mathcal{T}\in S_{p,q}^{*}(\xi,\mathcal{T},m;\varphi)$ be defined as in (1.2). Then the function k defined by*

$$k(z)=\frac{e+p}{z^e}\int_0^z t^{e-1}\,\mathcal{T}(t)dt,\quad (Re(e)>-p),\tag{2.13}$$

is also in $S_{p,q}^{*}(\xi,\ell,m;\varphi)$.

Proof: Let $\mathcal{T}\in S_{p,q}^{*}(\xi,\ell,m;\varphi)$ be defined as in (1.2). Then , we get

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}\prec_q\varphi(z).\tag{2.14}$$

For $\mathcal{T}\in\delta(p,m)$ and $Re(e)>-p$, from (3.13) that $k\in\delta(p,m)$ and

$$\mathcal{T}(z)=\frac{ek(z)+zk'(z)}{e+p}\tag{2.15}$$

Define function $\gamma(z)$:

$$\gamma(z)=(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)k(z)\,\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)k(z)\,\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n)}}\tag{2.16}$$

Differentiating (2.16) and using (2.14) and (2.15),we get

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}=(\xi-$$

$$1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\left(\frac{ek(z)+zk'(z)}{e+p}\right)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\left(\frac{ek(z)+zk'(z)}{e+p}\right)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\left(\frac{ek(z)+zk'(z)}{e+p}\right)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)\left(\frac{ek(z)+zk'(z)}{e+p}\right)\right)^{(\varepsilon+n)}}=$$

$$\frac{e}{e+p}\left((\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)k(z)\right)^{(\varepsilon+n)}}\right)+$$

$$\frac{1}{e+p}\left((\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\left(zk'(z)\right)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{p}^{*}(\ell)\left(zk'(z)\right)\right)^{(\varepsilon+n-1)}}-\frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{p}^{*}(\ell)\left(zk'(z)\right)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{p}^{*}(\ell)\left(zk'(z)\right)\right)^{(\varepsilon+n)}}\right)$$

$$= \frac{e}{e+\wp}\gamma(z) + \frac{1}{e+\wp}\big(z\gamma'(z) + \wp\gamma(z)\big) = \gamma(z) + \frac{1}{e+\wp} z\gamma'(z) \prec_q \varphi(z).$$

By application of Lemma (1.1) , we obtain $\gamma(z) \prec_q \varphi(z)$.

By using (2.16), we get

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,k(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)k(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,k(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)k(z)\right)^{(\varepsilon+n)}} \prec_q \varphi(z),$$

implies that $k \in S_{\wp,q}^{*}(\xi,\ell,m;\varphi)$.

Theorem 2.5: *Let* $\mathcal{T} \in S_{\wp,q}^{*}(\xi,\ell,m;\varphi)$, $g \in \delta(\wp,m)$*and*

$$Re\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{1}{2}. \tag{2.17}$$

Write $\mathcal{G} = \frac{zg'(z)}{g(z)}$

Then $\mathcal{T} * \mathcal{G} \in S_{\wp,q}^{*}(\xi,\ell,m;\varphi)$.

Proof: Let $\mathcal{T} \in S_{\wp,q}^{*}(\xi,a,m;\varphi)$, and $\mathcal{G} \in \delta(\wp,m)$. Then we have

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n)}} =$$

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\mathcal{G}(z) * \frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\mathcal{G}(z) * \frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}$$

$$= \mathcal{G}(z) * \phi(z), \tag{2.18}$$

Where

$$\phi(z) = (\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi\varepsilon+n!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}(z)\right)^{(\varepsilon+n)}} \prec_q \varphi(z). \tag{2.19}$$

From (2.17) , $\mathcal{G}(z)$ has the Heglotz representation:

$$\mathcal{G}(z) = \int_{|x|=1}\frac{d\omega(x)}{1-xz} \quad (z \in U), \tag{2.20}$$

where $\int_{|x|=1} d\omega(x) = 1$ and $\omega(x)$ is a probability measure on unit circle $|x| = 1$.

Since φ is starlike univalent in U, so from (2.18),(2.19) and (2.20) that

$$(\xi-1)\frac{(\varepsilon+n-1)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n-1)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n-1)}} - \frac{\xi(\varepsilon+n)!}{\varepsilon!}\frac{z\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n)}}{\left(\mathcal{M}_{\wp}^{*}(\ell)\,\mathcal{T}*\mathcal{G}(z)\right)^{(\varepsilon+n)}} =$$

$$\int_{|x|=1}\phi(xz)d\omega(x) \prec_q \varphi(z).$$

This shows that $\mathcal{T} * \mathcal{G} \in S_{\wp,q}^{*}(\xi,\ell,m;\varphi)$.

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