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Hamilton formulation for the electrodynamics of generalized Maxwell using fractional derivatives

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Abstract

A comparative analysis of the Hamiltonian and Lagrangian equations for the Maxwell field was conducted, and it was demonstrated that both methods are equivalent. Dirac's and Euler's techniques were employed to handle the Hamiltonian approach. Additionally, a novel fractional Hamilton formulation was developed for the Maxwell field using fractional derivatives. This formulation yielded a fractional Riemann-Liouville derivative operator and a fractional Hamilton function in terms of the variables A_i, A_j, and A_0. The effectiveness of this

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approach was verified by employing it to examine Maxwell's electrodynamic equation, and the results obtained were in perfect agreement, confirming the validity of the study

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1. Introduction

The most well-known fractional integrals and derivatives of non-integer orders were introduced by Liouville and Riemann, Grnwald and Letnikov, Riesz, Erdelyi and Kober, and Caputo [1,2,3,4]. Over the last 300 years, fraction integrals and derivatives have evolved dramatically [5,6,7,8]. Non-integer order operators come prove handy in a variety of scenarios (See, for example, the eight-volume encyclopedia on fractional calculus and its applications, which was released in 2019 [9]). There are many practical uses involving the phenomena of partial relaxation and oscillation, non-locality in space, long-lasting but weakening memory, openness and dissipation in systems, interactions over great distances, and variations in power based on spatial and frequency factors. Physical [10,11,12,13,14], geometric [13,14,15,16,17], and treatments for the economic notion in terms of fractional derivatives and allow the proposed economic interpretation is based on a generalization of average count and marginal value of economic indicators [18,19], where employ the notions of which evaluates economic performance with the presence fractional derivative operator of order rather than integer order derivative. Interpretations for fraction differentiation and integration in informatics [20] have all been proposed. It is vital to understand how to use probability to evaluate fractional derivatives and fractional integrals [21,22]. On the other hand, Foukrach [23] employed the Adomian decomposition approach to solve a Bürgers system with fractional derivatives in time and space. Numerical solutions presented in the format of quick converging series were identified using symbolic computation and graphically shown in Matlab to demonstrate the method's efficacy and dependability. Alawaideh achieved a significant breakthrough in another study by developing fractional equations of Euler-Lagrange and Hamilton for the Lee-Wick field. The approach relied on utilizing a Riemann-Liouville fractional derivative [24] to formulate a Lagrangian density field. Very recently, Dubey et. al. [25] developed the generalized fractional ABR derivative using the well-known k-Wright function ${}_p\psi_q^k(z)$ for the variables $Z \in C$ C, a_i, $b_j \in C$, β_j

and the innovative ABR operator of generalized k-Wright functions. The Hamilton construction of Proca fields using fractional calculus, as well as recasting Hamilton equations and fraction Euler-Lagrange equations with fractional derivatives, were described in a recent study [26].

Fractional derivatives have been employed to study the effect of fractional derivatives on mathematical models in various fields, including electricity, optics, and radio technology. Applications of this research include power generation, wireless communication, lenses, and radar systems. By investigating the mechanisms by which charges, currents, and field variations lead to the production of electric and magnetic fields, researchers have successfully utilized fractional derivatives to reframe the fractional Maxwell electrodynamics equation and derive equations of motion. These capabilities highlight the potential of fractional derivatives to improve our understanding and analysis of various technological systems.

The purpose of this study is twofold: to introduce a novel Hamiltonian formulation for the Maxwell equation, and to prove the equivalence of Hamiltonian and Lagrangian formulations. The study is organized as follows: Section 2 briefly explains a definitions of fractional derivatives, followed by Section 3 which presents a fractional form regarding the Euler-Lagrangian equation. Section 4 covers to the formulas of movement expressed according to Hamiltonian density expressed as a fraction. An illustrative example is examined in Section 5. Section 6 derives fractional Maxwell's formulas applying the Euler-Lagrange formulas . lastly, we conclude with some remarks in Section 7.

The objective of this research is twofold. Firstly, it aims to introduce a new Hamiltonian formulation for the Maxwell equation, and secondly, it aims to demonstrate the equivalence of Hamiltonian and Lagrangian formulations. The study is structured as follows: Section 2 provides a brief explanation of fractional derivatives, followed by Section 3, which presents the fractional version of the Euler-Lagrangian equation. Section 4 is dedicated to the equations of motion presented in terms of Hamiltonian density in fractional form. Section 5 examines an demonstrative example. In Section 6, fractional Maxwell's equations are derived utilizing the fundamental equations of variational calculus. The research concludes with a few comments in Section 7.

2. Fundamental Explanations

This section of the study provides a brief overview of key properties and definitions utilized in the work. Specifically, we introduce the left Riemann-Liouville fractional derivative, which is defined as follows:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau. \tag{2.1} $$

The right Riemann-Liouville fractional derivative can be described as a mathematical concept that is defined in a specific manner.

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha+1} f(\tau)d\tau. \tag{2.2} $$

The equation involves the Gamma function represented by the symbol Γ, and the parameter α which represents the order of the derivative. It is assumed that α is not an integer, and satisfies the inequality $n-1<\alpha<n$ If α happens to be an integer, then the derivatives are defined in the usual way, which is commonly understood.

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x) \tag{2.3} $$

$$ {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(t) \quad \alpha = 1,2,.. \tag{2.4} $$

3. A Fractional Euler Method with Maxwell's Lagrangian Density

The given system is continuous, and its Lagrangian density is expressed using the dynamical field variables, as well as the generalized coordinate and its respective derivative

$$ \mathcal{L} = \mathcal{L}[\psi_\mu, {}_aD_{x_\mu}^\alpha \psi_\rho, {}_{x_\mu}D_b^\beta \psi_\rho] \tag{3.1} $$

Euler-Lagrange equation for such Lagrangian density in fractional form can be given as

$$ \left[\frac{\partial \mathcal{L}}{\partial \psi_\rho} + \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha \psi_\rho} + \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha \psi_\rho}\right] = 0 \tag{3.2} $$

We can express the variational principle as follows:

$$ \delta S = \int \delta \mathcal{L}\ d^4x = 0 \tag{3} $$

One way to state the variational principle is as follows:

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\psi_\mu}\delta\psi_\rho + \frac{\partial\mathcal{L}}{\partial\, {}_aD^\alpha_{x_\mu}\psi_\rho}\delta\, {}_aD^\alpha_{x_\mu}\psi_\rho + \frac{\partial\mathcal{L}}{\partial\, {}_{x_\mu}D^\beta_b\psi_\rho}\delta\, {}_{x_\mu}D^\beta_b\psi_\rho \right] d^3x \tag{3.4}$$

By using the equation provided in (3.4) in the equation given in (3.3), and utilizing the equations given in (3.5), we can rewrite the expression

$$\begin{bmatrix} \delta\, {}_aD^\alpha_{x_\mu}\psi_\rho = {}_aD^\alpha_{x_\mu}\delta\psi_\rho \\ \delta\, {}_{x_\mu}D^\beta_b\psi_\rho = {}_{x_\mu}D^\beta_b\delta\psi_\rho \end{bmatrix} \tag{3.5}$$

we get,

$$\int \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial\psi_\rho}\delta\psi_\rho + \frac{\partial\mathcal{L}}{\partial\, {}_aD^\alpha_{x_\mu}\psi_\rho}\delta\, {}_aD^\alpha_{x_\mu}\psi_\rho \\ + \frac{\partial\mathcal{L}}{\partial\, {}_{x_\mu}D^\beta_b}\delta\, {}_{x_\mu}D^\beta_b\psi_\rho \end{bmatrix} d^4x = 0 \tag{3.6}$$

It was assumed that $({}_aD^\alpha_{x_\mu}\psi_\rho(x,t))$ is dependent on $(\psi_\rho(x,t))$ in order to find Eq.(3.6), hence $\delta\, {}_aD^\alpha_{x_\mu}\psi_\rho(x,t) = {}_aD^\alpha_{x_\mu}\delta\psi_\rho(x,t)$. We get the Euler-Lagrange equations of motion by applying $(\delta\mathcal{L} = 0)$.

$$\left[\frac{\partial\mathcal{L}}{\partial\psi_\rho} - {}_aD^\alpha_{x_\mu}\frac{\partial\mathcal{L}}{\partial\, {}_aD^\alpha_{x_\mu}\psi_\rho} - {}_{x_\mu}D^\beta_b\frac{\partial\mathcal{L}}{\partial\, {}_{x_\mu}D^\beta_b\psi_\rho} \right] = 0 \tag{3.7}$$

The fraction form of the Equations with lagrangian density is represented by the above equation.

It is worth mentioning that for $\alpha = \beta = 1$, ${}_aD^\alpha_{x_\mu} = \partial_\mu$, ${}_{x_\mu}D^\beta_b = -\partial_\mu$, Eq. (16) reduces to the usual Euler- Lagrange equation for the classical fields [20]

$$\frac{\partial\mathcal{L}}{\partial\psi_\rho} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\rho)} = 0 \tag{3.8}$$

4. Equations of Motion in terms of Hamiltonian Formulation

The Lagrangian of classical field in terms containing fractional partial derivatives is a function of the form

$$L = \mathcal{L}[\psi_\mu, {}_aD^\alpha_{x_\mu}\psi_\rho, {}_{x_\mu}D^\beta_b\psi_\rho] \tag{4.1}$$

Where, $(\mu = 0, i)$, we can express Eq. (18) in the following manner

$$L = \mathcal{L}[\psi_\rho, {}_aD^\alpha_t\psi_\rho, {}_aD^\alpha_{x_i}\psi_\rho, {}_aD^\alpha_t\psi_\rho, {}_tD^\beta_b\psi_\rho, {}_{x_i}D^\beta_b\psi_\rho] \tag{4.2}$$

In numerous instances, we adopt $\pi_\beta = 0$ as we establish (in the Lagrangian density and the Hamiltonian density) the time-rate component on the right- hand side as ${}_aD_t^\alpha\psi$, so that $\pi_\beta = \frac{\partial\mathcal{L}}{\partial\, {}_tD_b^\beta\psi} = 0$ Therefore take $\pi_\beta = 0.$ To review Hamilton's formalism, define a generalized momentum π_α^1 conjugate to $\psi_\rho(x,t)$ as [24].

$$\pi_\alpha^1 = \frac{\partial\mathcal{L}}{\partial\, {}_aD_t^\alpha\psi_\rho}, \tag{4.3}$$

The definition of the canonical Hamiltonian (H) is as follows:

$$H = \pi_\alpha^1\, {}_aD_t^\alpha\psi_\rho - L[\psi_\rho, {}_aD_t^\alpha\psi_\rho, {}_aD_{x_i}^\alpha\psi_\rho] \tag{4.4}$$

If we calculate the total change in the Hamiltonian, we can express it as the following equation.

$$dH = \begin{bmatrix} d\pi_\alpha^1\, {}_aD_t^\alpha\psi_\rho + \pi_\alpha^1 d\, {}_aD_t^\alpha\psi_\rho - \frac{\partial\mathcal{L}}{\partial\psi_\rho} d\psi_\rho \\ -\frac{\partial\mathcal{L}}{\partial\, {}_aD_t^\alpha\psi_\rho} d\, {}_aD_t^\alpha\psi_\rho - \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\psi_\rho} d\, {}_aD_{x_i}^\alpha\psi_\rho \end{bmatrix} \tag{4.5}$$

Upon substituting the values of the conjugate momenta, the resulting expression is as follows.

$$dH = \left[d\pi_\alpha^1\, {}_aD_t^\alpha\psi_\rho - \frac{\partial\mathcal{L}}{\partial\psi_\rho} d\psi_\rho - \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\psi_\rho} d\, {}_aD_{x_i}^\alpha\psi_\rho \right] \tag{4.6}$$

Therefore, we can infer that the Hamiltonian can be expressed as a function in the following manner.

$$H = H[\pi_\alpha^1, \psi_\rho, {}_aD_{x_i}^\alpha\psi_\rho,]$$

Hence, the total change in the Hamiltonian can be expressed in the following manner.

$$dH = \left[\frac{\partial H}{\partial\pi_\alpha^1} d\pi_\alpha^1 + \frac{\partial H}{\partial\psi_\rho} d\psi_\rho + \frac{\partial H}{\partial\, {}_aD_{x_i}^\alpha\psi_\rho} d\, {}_aD_{x_i}^\alpha\psi_\rho \right] \tag{4.7}$$

Using the differential of the Lagrange function (4.3) and the formulas (4.4), (4.6), and (4.7), derive Hamilton's motion equations (4.8). (4.9).

$$\begin{cases} \frac{\partial H}{\partial\psi_\rho} = -\frac{\partial\mathcal{L}}{\partial\psi_\rho} \end{cases} \tag{4.8}$$

$$\begin{cases} \dfrac{\partial H}{\partial \pi_{\alpha}^{1}} = {}_{a}D_{t}^{\alpha}\psi_{\rho} \\ \dfrac{\partial H}{\partial\, {}_{a}D_{x_i}^{\alpha}\psi_{\rho}} = -\dfrac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x_i}^{\alpha}\psi_{\rho}} \end{cases} \tag{4.9}$$

After rewriting equation (4.8) and utilizing upon utilizing the Euler-Lagrange formula, we arrive at the subsequent formula.

$$\frac{\partial H}{\partial \psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x_\mu}^{\alpha}\psi_{\rho}} \tag{4.10}$$

Expand in relation to , mathematical statement takes on the following shape

$$\frac{\partial H}{\partial \psi_{\rho}} = -{}_{a}D_{t}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{t}^{\alpha}\psi_{\rho}} - {}_{a}D_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_{a}D_{x_i}^{\alpha}\psi_{\rho}} \tag{4.11}$$

Using the subscript (i) in this situation implies that the quantity in question is summed over the spatial dimensions x, y, and z.

5. Demonstrative instance

Fractional Electromagnetic Lagrangian Density

The Lagrange density of electromagnetism is composed of two components: one for the electric charge density that is linked to potential and another for the anti-symmetric field strength tensor of the second order. Fractional differential equations are commonly used in various applications to substitute the time derivative in an evolution with a fractional order derivative. In order to gain a better understanding and analysis of formulas and equations, we studied the Lagrangian density of Maxwell's equation [27].

$$\mathcal{L}_{LW} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_{\mu}A^{\mu} \tag{5.1}$$

Maxwell's Lagrangian density involves a four-dimensional anti-symmetric second rank tensor denoted by $F^{\mu\nu}$ and a four-vector potential denoted by A^{μ}, where the typical four-vector current is represented as $J_{\mu} = (\rho c, j)$. To express Maxwell's Lagrangian density in Riemann-Liouville fractional form using these relationships, we need to rework the equations accordingly.

$$\begin{bmatrix} F_{\mu\nu} = {}_{a}D_{x_\mu}^{\alpha}A_{\nu} - {}_{a}D_{x_\nu}^{\alpha}A_{\mu} \\ F^{\mu\nu} = {}_{a}D_{x^\mu}^{\alpha}A^{\nu} - {}_{a}D_{x^\nu}^{\alpha}A^{\mu} \end{bmatrix} \tag{5.2}$$

$$\begin{bmatrix} \partial_{\mu} = {}_aD^{\alpha}_{x_{\mu}} = ({}_aD^{\alpha}_t, {}_aD^{\alpha}_{x_i}) \\ \partial^{\mu} = {}_aD^{\alpha}_{x^{\mu}} = ({}_aD^{\alpha}_t, -{}_aD^{\alpha}_{x^i}) \end{bmatrix} \tag{5.3}$$

$$F_{\mu v}F^{\mu v} = 2[{}_aD^{\alpha}_{x_{\mu}}A_{v}\,{}_aD^{\alpha}_{x^{\mu}}A^{v} - {}_aD^{\alpha}_{x_{\mu}}A_{v}\,{}_aD^{\alpha}_{x^{v}}A^{\mu}] \tag{5.4}$$

$$\begin{bmatrix} A^{\alpha} = (\phi, \vec{A}) \\ A_{\alpha} = (\phi, -\vec{A}) \end{bmatrix} \tag{5.5}$$

The statement is indicating that when μ is equal to zero and i takes on the values of 1, 2, and 3, and v is also zero while j ranges from 1 to 3.

The fractional electromagnetic equation density formula can be derived using the left Riemann-Liouville fractional formula, which involves expanding the variables μ, v in terms of 0, i, and 0, j.

$$\mathcal{L} = -\frac{2}{4}\begin{bmatrix} -({}_aD^{\alpha}_tA_j)^2 + {}_aD^{\alpha}_tA_j\,{}_aD^{\alpha}_{x_j}\phi \\ -({}_aD^{\alpha}_{x_i}\phi)^2 + {}_aD^{\alpha}_{x_i}\phi\,{}_aD^{\alpha}_tA_i \\ +({}_aD^{\alpha}_{x_i}A_j)^2 - {}_aD^{\alpha}_{x_i}A_j\,{}_aD^{\alpha}_{x_i}A_i \end{bmatrix} + J_0\phi - J_iA_i \tag{5.6}$$

6. Fractional form of Euler-Lagrange Equations of Maxwell's Density

To derive the equations of motion from Maxwell's Lagrangian density, we can start by defining elucidating the Lagrangian fractional density and its practical applying the generalization formula of Euler-Lagrange equation (3.7). Specifically, we can focus on the first field variable ϕ in this process.

$$\left[\frac{\partial\mathcal{L}}{\partial\phi} - {}_aD^{\alpha}_t\frac{\partial\mathcal{L}}{\partial\,{}_aD^{\alpha}_t\phi} - {}_aD^{\alpha}_{x_i}\frac{\partial\mathcal{L}}{\partial\,{}_aD^{\alpha}_{x_i}\phi}\right] = 0 \tag{6.1}$$

Calculating these derivatives ϕ yields to

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial\phi} = J_0 \\ \dfrac{\partial\mathcal{L}}{\partial\,{}_aD^{\alpha}_t\phi} = 0 \end{cases} \tag{6.2}$$

$$\left\{ \frac{\partial\mathcal{L}}{\partial\,{}_aD^{\alpha}_{x_i}\phi} = -(-{}_aD^{\alpha}_{x_i}\phi + {}_aD^{\alpha}_tA_i) \right. \tag{6.3}$$

Equations (6.2 and 6.3) are substituted in equation (6.1) to get the following result:

$$[J_0 = -(-{}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i)] \tag{6.4}$$

The given equation is the first instance of a Heterogeneous Mathematical expression in fractional form. To derive additional equations of motion based on the variables A^i and A^j, we can utilize the general formula (4.11)

$$0 = \left[\frac{\partial\mathcal{L}}{\partial A_i} - {}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial{}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial{}_aD_{x_i}^{\alpha}A_i}\right] \tag{6.5}$$

Calculating these derivatives yields to

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial A_i} = -J_i \\ \dfrac{\partial\mathcal{L}}{\partial{}_aD_t^{\alpha}A_i} = -\frac{2}{4}({}_aD_{x_i}^{\alpha}\phi) \\ \dfrac{\partial\mathcal{L}}{\partial{}_aD_{x_i}^{\alpha}A_i} = 0 \end{cases} \tag{6.6}$$

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial{}_aD_t^{\alpha}A_i} = -\frac{2}{4}({}_aD_{x_i}^{\alpha}\phi) \end{cases} \tag{6.7}$$

Substituting equation (6.7and6.6) in equation (6.5) we get

$$[-J_i = {}_aD_t^{\alpha}({}_aD_{x_i}^{\alpha}\phi)] \tag{6.8}$$

And

$$0 = \left[\frac{\partial\mathcal{L}}{\partial A_j} - {}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial{}_aD_t^{\alpha}A_j} - {}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial{}_aD_{x_i}^{\alpha}A_j}\right] \tag{6.9}$$

Calculating these derivatives yields to

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial A_j} = 0 \end{cases} \tag{6.10}$$

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial{}_aD_t^{\alpha}A_j} = -\frac{2}{4}({}_aD_{x_j}^{\alpha}\phi + 2\,{}_aD_t^{\alpha}A_j) \\ \dfrac{\partial\mathcal{L}}{\partial{}_aD_{x_i}^{\alpha}A_j} = -\frac{2}{4}(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i) \end{cases} \tag{6.11}$$

After inserting equations (6.11 and 6.10) into equation (6.9) we obtain:

$$\begin{bmatrix} 0 = \frac{2}{4}\, {}_aD_t^{\alpha}({}_aD_{x_j}^{\alpha}\phi + 2\,{}_aD_t^{\alpha}A_j) \\ +\frac{2}{4}\, {}_aD_{x_i}^{\alpha}(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i) \end{bmatrix} \tag{6.12}$$

The following results are obtained when equations (6.8) and (6.12) are combined:

$$\begin{bmatrix} -J_i = {}_aD_t^{\alpha}({}_aD_{x_i}^{\alpha}\phi) + \frac{2}{4}\, {}_aD_{x_i}^{\alpha}({}_aD_{x_j}^{\alpha}A_j) \\ +\frac{2}{4}\, {}_aD_{x_i}^{\alpha}(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i) \end{bmatrix} \tag{6.13}$$

The equation presented is the second non-homogeneous equation expressed in fractional form. For $\alpha = 1$, equation (6.13) converges to the standard equations. It is possible to define a conjugate momentum using the following method:

$$\begin{cases} \pi_1^1 = \dfrac{\partial L}{\partial({}_aD_t^{\alpha}\phi)} \\ \pi_1^2 = \dfrac{\partial L}{\partial({}_aD_t^{\alpha}A_i)} \\ \pi_1^3 = \dfrac{\partial L}{\partial({}_aD_t^{\alpha}A_j)} \end{cases} \tag{6.14}$$

We can represent the Hamiltonian density by applying equation (4.4) as follows:

$$H = \frac{2}{4}\,[-({}_aD_t^{\alpha}A_j)^2 + {}_aD_t^{\alpha}A_j\, {}_aD_{x_j}^{\alpha}\phi - ({}_aD_{x_i}^{\alpha}\phi)^2 - {}_aD_t^{\alpha}A_j\, {}_aD_{x_j}^{\alpha}A_i] \tag{6.15}$$

by use the fields' variables (ϕ, A_i, A_j) *so that, we can re-write Eq (4.11),* we get

$$\frac{\partial \mathcal{H}}{\partial \phi} = \left[-{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}\phi} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\phi} \right] \tag{6.16}$$

$$\frac{\partial \mathcal{H}}{\partial A_i} = \left[-{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_i} \right] \tag{6.17}$$

$$\frac{\partial \mathcal{H}}{\partial A_j} = \left[-{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_j} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_j} \right] \tag{6.18}$$

Applying the Hamiltonian equation (4.11) and differentiating with respect to ϕ yields

$$[J_0 = -(-{}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i)] \tag{6.19}$$

The equation (6.19) obtained through the use of Hamiltonian equation (6.17) is identical to the equation derived in fractional form by equation (6.4). Deriving the rate of change with regard to the variable A^i utilizing the Hamiltonian framework (6.17) gives us the following result:

$$[-J_i = {}_aD_t^{\alpha}({}_aD_{x_i}^{\alpha}\phi)] \tag{6.20}$$

And using Eq.(6.18), with respect to A^j, we get:

$$\begin{bmatrix} 0 = \frac{2}{4}\,{}_aD_t^{\alpha}({}_aD_{x_j}^{\alpha}\phi + 2\,{}_aD_t^{\alpha}A_j) \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i) \end{bmatrix} \tag{6.21}$$

The outcome of the above calculation matches the result obtained through the use of Euler-Lagrange equation as demonstrated in Eq. (6.13). To proceed further, we can add equations (6.20) and (6.21) to obtain:

$$\begin{bmatrix} -J_i = {}_aD_t^{\alpha}({}_aD_{x_i}^{\alpha}\phi) + \frac{2}{4}\,{}_aD_{x_i}^{\alpha}({}_aD_{x_j}^{\alpha}A_j) \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i) \end{bmatrix} \tag{6.22}$$

The given equation is the second non-homogeneous equation represented in fractional form.

7. Conclusion

This study builds upon prior research by introducing fractional derivatives into Hamilton's equation for Maxwell's field. Specifically, we establish the Hamilton operator perspective of Maxwell's field systems and provide the relevant Hamilton's equations. Additionally, we construct the Euler-Lagrange equation and derive the Hamilton equations of motion for Maxwell's density. Our findings indicate that the results obtained from the Hamiltonian formulation are comparable to those from the Euler-Lagrange formula. In conclusion, this research contributes to a deeper understanding of the dynamics of Maxwell's field systems and the potential applications of fractional derivatives in Hamiltonian mechanics. This study comes to a close with the next remarks

- We use fractional derivative to generalize our equations so they may be applied to the Maxwell field. The procedure's purpose is to construct a set of Hamilton equations that are generalized. Maxwell extended electrodynamics is the foundation of the method.
- The findings suggest that this approach has the potential to address various mathematical physics problems, such as the Hamilton-Lagrange equations
- This method can be used to create a new formalism for the Maxwell equations at various Riemann- Liouville orders, such as (1/2,2/3,3/2,..), which is the basis for a new research project we're working on.

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