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Generalization of semi-regular modules

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Abstract

We will add new types of modules over a ring $\mathfrak{R}$ called generalized semi-regular modules. Assuming $\mathfrak{R}$ is a ring with identity and $\mathcal{M}$ is a unitary left $\mathfrak{R}$-module which is a duality. We show that our new ideas have characterization and a few properties.

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1. Introduction

"Let R be an associative ring with identity and let M be an unital left R - module a submodule N of M is said to be small in M (briefly N << M), if where M=N+L for L≤M implies L=M [1]. A submodule N of M is called essential in M (briefly N≤ e M), if N∩K≠0 for every submodule K≠0 [1]. A submodule L of M is called generalized small (briefly L≪g M), if for essential submodule T of M with property M= L+T implies that T= M[2]. It is clear that every small submodule is generalized as small but the converse is essential. The intersection of all maximal essential submodules of M is called generalized radical M denoted by Radg(M), if M has no maximal then Rad(M)=M. Radg (M) is the sum of all g - small submodule of M."

"Recall that an element m∈M is called a regular element, if there exists α ϵ M*= Hom (M, R) such that (α(m))m = [3] equivalent every submodule is a projective direct summand. An R -module M is called a regular module if each

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of its elements is regular. "A submodule N of An R -module M is said to lie over projective summand of M if there exists a decomposition where A is a projective submodule of N" and B∩N≪ M. A submodule N of an R -module M is called lie over a projective summand of M, if there exists a decomposition M=A⊕B where A is a projective submodule of N and B ∩ N << M."[4]

"An R-module M is called a semi-regular, if every cyclic submodule of M is lying over projective summand in M. In this paper we introduce g - semi-regular modules some projective of these types of modules will be proved".

2. Generalized Semi-Regular Module

In this section, it is devoted to introducing a generalized semi-regular module as a generalization of the semi-regular module, some properties of this type of module will be proved.

First, we start with the following known definitions:

Definition 2.1: [5]" A submodule $\mathcal{L}$ of $\mathcal{M}$ is called generalized small denoted by $\mathcal{L} <<_g \mathcal{M}$, if for essential submodule $\mathcal{T}$ of $\mathcal{M}$. with the property $\mathcal{M}= \mathcal{L} + \mathcal{T}$ implies that $\mathcal{T} = \mathcal{M}$".

Definition 2.2: [6]" A submodule $\mathcal{N}$ of $\mathcal{M}$ An $\mathfrak{R}$-module $\mathcal{M}$ is said to be g-lie over projective summand of $\mathcal{M}$ = A ⊕ B where A is a projective submodule of $\mathcal{N}$ and $B \cap \mathcal{N} \ll_g \mathcal{M}$".

Remarks and Examples (2.3):

1- "It is clear that every direct summand of a- projective $\mathfrak{R}$ -module $\mathcal{M}$ is g -lie over projective summand of $\mathcal{M}$."

2- "Every non-zero submodule of Z as Z-module is not g -lie over summand of Z to show that? "Let 0≠ nZ be a submodule of Z, since Z is indecomposable, then {0} is the only summand of Z. that contained in nZ. But Z∩nZ = nZ not generalized small in Z, therefore, nZ is not g -lying over a summand of Z".

3- "Every a finitely generated "submodule of Q as Z-module is g -lying over projective summand of Q, Let $\mathcal{N}$ be a finitely generating submodule of Q since Q is indecomposable then {0} is the only projective summand of Q that is contained in $\mathcal{N}$ .It is easy to see that Q∩ $\mathcal{N} = \mathcal{N} <<_g$ Q thus $\mathcal{N}$ is g -lying over projective summand of Q".

Definition 2.4: "An $\mathfrak{R}$ -module $\mathcal{M}$ is called g- semi-regular, if every cyclic submodule of $\mathcal{M}$ *is* g -lying over projective summand in $\mathcal{M}$."

Remark and Example 2.5:

1- "It is clear that every regular is g -semi-regular.
Let $\mathcal{N} \leq \mathcal{M}$, $\mathcal{N}$ is cyclic then $\mathcal{N}$ is projective summand of $\mathcal{M}$ and hence $\mathcal{M} = \mathcal{N} \oplus \mathcal{K}$ for $\mathcal{K} \leq \mathcal{M}$ and $\mathcal{K} \cap \mathcal{N} = 0 <<_g \mathcal{M}$ therefore $\mathcal{M}$ is g -semi-regular. In particular, the module Z_6 as –Z module then is regular.

2- The module "Z as Z-module, Z is not g -semi-regular".
3- Consider the "module Q as Z-module, then Q is g -semi-regular".

Let $x\epsilon$Q by (2.3.3) $< x ><<_g$ Q, and Q = {0}⊕Q where {0} is projective submodule of Q and Q ∩ $< x > = < x ><<_g$ Q.

Proportion 2.6: Let be $\mathcal{M}$ an $\mathfrak{R}$ -module, then $\mathcal{M}$ is regular if and only if $\mathcal{M}$ is g -semi-regular and $Rad_g(\mathcal{M}) = 0$.

Proof: Let $\mathcal{M}$ be a regular module, then by (2.5) $\mathcal{M}$ is g - semi-regular

Now let $0 \neq m \in Rad_g(\mathcal{M})$ then $Rm <<_g \mathcal{M}$ by [6] . But is $\mathcal{M}$ regular so $\mathfrak{R}m$ is a summand of $\mathcal{M}$ which is a contradiction, thus $Rad_g(\mathcal{M}) =0$

Conversely, Let $\mathcal{M}$ be a semi-regular module and x ∈ $\mathcal{M}$ then $\mathfrak{R}$x is g - lie over the projective summand of $\mathcal{M}$ thus there exists a projective summand A of $\mathfrak{R}_x$ such that $\mathcal{M} = A \oplus B$, where $B \leq \mathcal{M}$ and $B \cap \mathfrak{R}_x \leq Radg(\mathcal{M})= 0$, We have $\mathfrak{R}_x = A\oplus (B \cap \mathfrak{R}_x)$.But $B \cap \mathfrak{R}_x =0$ therefore $\mathfrak{R}_x = A$ is a projective summand of $\mathcal{M}$ thus $\mathcal{M}$ is regular.

Corollary 2.7: *Let $\mathfrak{R}$ be a ring with $Rad_g(\mathfrak{R}) = 0$ and $\mathcal{M}$ be a projective $\mathfrak{R}$ -module Then $\mathcal{M}$i s g -semi-regular if and only if $\mathcal{M}$ is regular.*

Proof: Assume that $\mathcal{M}$ is g -semi-regular, since $\mathcal{M}$ is a projective, then by [6] $Rad_g(\mathcal{M}) = Rad_g(\mathfrak{R})\ \mathcal{M} = 0$, $\mathcal{M} = 0$,thus $\mathcal{M}$ is regular by (2.6). The converse is clear by (2.5)(1).

Proportion 2.8: "Let $\mathcal{M}$ be an $\mathfrak{R}$ -module and let x∈, $\mathcal{M}$ then the following statements are equivalent":

1- $\mathfrak{R}x$ is $g-$lying over a projective summand.
2- There exists $\alpha \in \mathcal{M}^*$ such that $(\alpha(x))^2 = \alpha(x)$ and $x- \alpha(x)x \in Rad_g(\mathcal{M})$
3- There exists homomorphism $\beta: \mathcal{M} \to \mathfrak{R}x$ such that $\beta^2 = \beta, \beta\ (\mathcal{M})$ is projective and $x -\beta(x) \in Rad_g(\mathcal{M})$.

Proof: (1) → (2) Let $\mathcal{M}$ = P ⊕ Q where $P \leq \mathfrak{R}_x$ is projective and $\mathfrak{R}x \cap Q <<_g \mathcal{M}$, then P is finitely generated so $Q_1, Q_2, Q_3,\ldots Q_n$, $x_1, x_2 \ldots x_{\cdot n}$ is the dual basis of P.

Write x_t= r_t x and define $\alpha \in P^*$ by $\alpha = \sum Q_\alpha r_t$ extend α to $\mathcal{M}$ by setting $Q_\alpha = 0$.

If $x = P + q , p \in P, Q \in q$ then $(\alpha x)x = (p\ \alpha x) = p$, consequently $(x\alpha)^2 x = p\ \alpha = x\alpha$ and $x - (x\ \alpha)\ x = q \in \mathfrak{R}\ x \cap Q \leq Rad_g (\mathcal{M})$.

(2) ⟹(3)

By [7, Lemma 1.1.11] $x \in \mathcal{M}, \alpha \in$ P ,write y= (αx) x, then (y α) y = y, y $\in \mathcal{M}$ so $\mathcal{M}$ = $\mathfrak{R}$y+ V , where V={ v; (v α)y=0} since $\mathfrak{R}$x ∩ V = $\mathfrak{R}$ (x-y) and since $\mathfrak{R}x$ ∩ V≤ $Rad_g(\mathcal{M})$, Rad(x -y) ≤ $Rad_g(\mathcal{M})$, hence x- (αx) x≤ $Rad_g(\mathcal{M})$.
(3) ⟹ (1) Clear

Remark 2.9: An element ∈ $\mathcal{M}$, $\mathcal{M}$ is said to be g -semi-regular in $\mathcal{M}$, if any of the conditions in proportion (2.8) are satisfied.

Corollary 2.10: *Let $\mathcal{M}$ be an $\mathfrak{R}$ -module and $x, y \in \mathcal{M}$ If $x - y \in Rad_g(\mathcal{M})$ and y is a g -semi-regular element, then x is g -semi-regular.*

We prove the theorem that gives a characterization of semi-simple modules in terms of submodules g -lying over direct summand.

Proportion 2.11: Let $\mathcal{M}$ be an $\mathfrak{R}$-module then $\mathcal{M}$ is semi simple $\mathfrak{R}$ –module if every submodule N of $\mathcal{M}$ is g -lying over summand of $\mathcal{M}$ and $Rad_g(\mathcal{M}) = 0$.

Proof: Let $\mathcal{N}$ be a submodule of $\mathcal{M}$, then by our assumption is g -lying over a summand of $\mathcal{M}$.Thus $\mathcal{M}$ = A ⊕ B where A ≤ $\mathcal{N}$ and ($\mathcal{N}$ ∩ B) ≤ $Rad_g(\mathcal{M})$ = 0 so by modular law.

$$\mathcal{N} = A \oplus (\mathcal{N} \cap B) = A.$$

Thus $\mathcal{N}$ is a summand of $\mathcal{M}$.

Remark 2.12: A finitely generated submodule of the g semi-regular module is g-lying over projective summand. However in general a submodule of g -semi-regular may not g -lie over a projective summand as the following example :

Example 2.13: Let X be an infinite set . Consider the ring $\mathfrak{R}$= ($P(x), \nabla, \cap$), It is clear that $A = A^2$ for every $A \leq X$ and hence every cyclic ideal is a projective summand of P(x). Thus $\mathfrak{R}$ is a regular ring and hence $\mathfrak{R}$ is a g -semi-regular ring.

So every finitely generated ideal of $\mathfrak{R}$, g -lying over a projective summand.

Now Let $I = \{A, A \subseteq X, A$ is finite$\}$, it is an ideal of $\mathfrak{R}$. One can easily not finitely generated.

It is not a direct summand.

Claim that I is not g - lying over projective summand of $\mathfrak{R}$.

A assume not ,then there exists an ideal J of $\mathfrak{R}$ such that $J \leq I$, $\mathfrak{R}$= $J \oplus J_1$ and I ∩ J_1 ≤ $Rad_g(\mathfrak{R})$, but R is regular therefore $Rad_g(\mathfrak{R})$ =0 by (2.6) , $I \cap J_1 = 0$ hence $\mathfrak{R} = I \cap J_1$ a Contradiction.

Proportion 2.14: "Let $\mathcal{M}$ be an $\mathfrak{R}$-module such that $\mathcal{M} = \oplus_{i \in I} \mathcal{M}_i$, then $\mathcal{M}$ is g - semi-regular if and only if $\mathcal{M}$ is g -semi-regular $\forall\ I \epsilon i$.

Proof: Let $\mathcal{N}$ is be a direct summand of $\mathcal{M}$ and let xϵ $\mathcal{N}$, since $\mathrm{Rad}_g(\mathcal{N}) = \mathcal{N} \cap \mathrm{Radg}(\mathcal{M})$ it easily that X is g -semi-regular in $\mathcal{M}$ if and only if X is g -semi-regular in $\mathcal{N}$

Consequently, it is sufficient to prove the proportion for two summands".

Let $\mathcal{M} = \oplus\mathcal{K}$, where $\mathcal{N}$ and $\mathcal{K}$ are g - semi-regular modules.

Consider x + y ϵ $\mathcal{M}$, where x $\epsilon\mathcal{N}$, y $\epsilon\mathcal{K}$, choose α ϵ $\mathcal{N}^*$ such that $(x\alpha)^2 = x\alpha$ and $x - (x\alpha)x \epsilon \mathrm{Rad}_g(\mathcal{N})$. Extend α to $\mathcal{M}$ by defining $\mathcal{K}_\alpha = 0$ then $(x + y)\alpha = x\alpha$ is an idempotent so by (2.13) it is sufficient to show $that\ (x + y) - [(x + y)\alpha](x + y) = (x - (x\alpha)\alpha + (y - (\alpha x)y$ is semi-regular in $\mathcal{M}$ but $x - (x\alpha)\, x\, \epsilon\, \mathrm{Rad}_g(\mathcal{N}) \leq \mathrm{Rad}_g(\mathcal{M})$. and $Y - (x\alpha)\, y$ is g - semi-regular in $\mathcal{K}$ and hence in $\mathcal{M}$ by proportion (2.8).

Proportion (2.15): "Let $\mathfrak{R}$ be a ring and be an $\mathfrak{R}$-module then the following statements are equivalent:

1. $\mathfrak{R}$ is g - semi-regular.
2. Every projective $\mathfrak{R}$ - module $\mathcal{M}$ is g - semi-regular.
3. Every free $\mathfrak{R}$–module $\mathcal{M}$ is g - semi-regular.
4. Every finitely generated projective $\mathfrak{R}$ -module $\mathcal{M}$ is g - semi-regular.
5. Every finitely generated free $\mathfrak{R}$- module $\mathcal{M}$ is g - semi-regular.
6. $\mathfrak{R}\oplus \mathfrak{R}$ is g - semi regular".

Proof: (1) ⇒ (2) Let $\mathfrak{R}$ be a g - semi-regular ring and let $\mathcal{M}$ is a projective $\mathfrak{R}$-module, then by [7, Lemma[1.1.11]] there exists a free $\mathfrak{R}$ –module F such that $F = A \oplus B\ and\ A \cong \mathcal{M}.$

Since F is a free module then by [7, Lemma [1.1.11]] $F \cong \oplus_{i\epsilon I} \mathfrak{R}$, for some index set $I.$

Thus F is a g -semi-regular module by prop. (2.14) but A is a summand of F therefore A is g -semi-regular module, but $A \cong \mathcal{M}$, so M is g - semi-regular.

(2) ⇒ (3) Clear since every free $\mathfrak{R}$ - module is projective.

(3) ⇒ (4) "Let $\mathcal{M}$ be a finitely generated projective module then by[7, Lemma [1.1.11]] there exists a finitely generated free module F such that $F = A \oplus B$ and A $\cong \mathcal{M}$. Since F is g -semi-regular and A is summand, then by prop. (2.14) A is g -semi-regular but $A \cong \mathcal{M}$, so $\mathcal{M}$ is g- semi-regular".

(4) ⇒ (5) Clear since every free $\mathfrak{R}$ -module is projective by [7, Lemma [1.1.11]]

(5) ⇒ (1) Since $\mathfrak{R}$ is isomorphic to the free module generated by one element then $\mathfrak{R}$ is g -semi-regular.

(4) ⇒ (6) and (6) ⇒ (1) Clear

"Recall that an $\mathfrak{R}$ -module $\mathcal{M}$ is called a lifting module or (D1) module, if every submodule $\mathcal{N}$ of $\mathcal{M}$, there exists a decomposition $\mathcal{M} = A \oplus B$, $A \leq \mathcal{N}$, $B \cap \mathcal{N} << B$".

Not every lifting is generalized lifting.

Remark 2.16: "Every $\mathfrak{R}_x$ be a submodule of $\mathcal{M}$. since $\mathcal{M}$ is g -lifting $\mathcal{M} = A \oplus B$, where $A \leq \mathfrak{R}_x$ and $\mathfrak{R}_x \cap B <<_g B$, since $\mathcal{M}$ is projective and A is summand of $\mathcal{M}$ then by proportion (2.6) , A is projective but $B \cap \mathfrak{R}_x <<_g B$, therefore, $\mathfrak{R}_x \cap B <<_g \mathcal{M}$ and hence $\mathfrak{R}_x$ is g -lie over projective summand of $\mathcal{M}$ thus $\mathcal{M}$ is g - semi-regular".

Remark 2.17: A lifting module may not be a g -semi-regular module as the following example shows.

Consider the module Z_4 as Z-module. Z_4 is not projective and indecomposable. Z_4 is not g-lying over summand Z_4 is not g-lying over projective summand.

Proportion 2.18: Let $\mathcal{M}$ be a Noetherian g semi-regular $\mathfrak{R}$–module then $\mathcal{M}$ is g - lifting module.

Proof: "Let $\mathcal{N}$ be a submodule of $\mathcal{M}$ since $\mathcal{M}$ is Noetherian then $\mathcal{N}$ is finitely generated" since $\mathcal{M}$ is $g$semi-regular then there exists a decomposition $\mathcal{M} = A \oplus B$, where A is projective submodule N and $(A \cap B) \leq Rad_g(\mathcal{M})$, since $\mathcal{M}$ is finitely generated then $Rad_g(\mathcal{M}) <<_g \mathcal{M}$, and hence $(\mathcal{N} \cap B) <<_g \mathcal{M}$, but $\mathcal{N} \cap B \leq B$ and B is summand of $\mathcal{M}$, therefore $(\mathcal{N} \cap B) <<_g B$ by [7, [1.1.11]]" thus $\mathcal{M}$ is g - lifting.

"Recall that a module $\mathcal{M}$ is called e-hollow if every proper $\mathcal{N}$ of $\mathcal{M}$ is e-small"

Proportion 2.19: "Let $\mathcal{M}$ be an indecomposable $\mathfrak{R}$ -module if $\mathcal{M}$ is g -semi-regular, then $\mathcal{M}$ is e- hollow. The converse is true if $\mathcal{M}$ is projective".

Proof: "Let $\mathfrak{R}x$ be a proper submodule of $\mathcal{M}$. Since $\mathcal{M}$ is g -semi-regular, then $\mathcal{M} = A \oplus B$, where A is a projective sub-module of $\mathfrak{R}x$ and $B \cap \mathfrak{R}x <<_g \mathcal{M}$. By modular law, we have $\mathfrak{R}x = A \oplus (\mathfrak{R}x \cap B)$. But $\mathcal{M}$ is indecomposable and A is a proper summand of $\mathcal{M}$ " therefore $= 0$, and hence $\mathfrak{R}x = \mathfrak{R}x \cap B <<_g \mathcal{M}$. Thus every cyclic submodule of $\mathcal{M}$ is g - small. Thus $\mathcal{M}$ is e-hollow".

"Conversely, assume that $\mathcal{M}$ is a projective $\mathfrak{R}$-module and let $\mathfrak{R}y$ is a cyclic submodule of $\mathcal{M}$. Assume that $\mathfrak{R}y$ is a proper submodule of $\mathcal{M}$, and then $\mathfrak{R}y$ is g - small. So $\mathcal{M} = \{0\} \oplus \mathcal{M}$, where $\{0\}$ is a projective submodule of

$\Re y$ and $\mathcal{M} \cap \Re y = \Re y <<_g \mathcal{M}$. Now, if $\Re y = \mathcal{M}$, $\mathcal{M} = \mathcal{M} \oplus \{0\}$ $\mathcal{M}$ is projective and $\{0\} << \mathcal{M}$. Thus $\mathcal{M}$ is g - semi-regular".

Note: The converse of (2.19) is not true for, example module Z_4 as Z –module

Corollary 2.20: *Let $\Re$ be an integral domain, then $\Re$ is e- hollow if and only if $\Re$ is g - semi-regular.*

Conclusion:

Finally, we conclude that:

1- new types of modules over a ring R.

2- Those new modules are called "Generalization of Semi-Regular Modules".

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