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General upper bounds for the numerical radius on complex Hilbert space

Mohammed Al-Dolat *
Khaldoun Al-Zoubi †
Department of Mathematics & Statistics
Jordan University of Science and Technology
P. O. Box 3030
Irbid 22110
Jordan

Abstract

In this work, we show that if $\{A_i\}_{i=1}^{m}$ and $\{X_i\}_{i=1}^{m}$ are two sets of bounded linear operators on the complex Hilbert space H, then for every $n \in \mathbb{N}$ and $m > 2$, we have

$$w\Big(A_1^{n-1}\Big(\sum_{i=0}^{m-1} A_{m-i}X_{m-i}A_{i+1}^{*}\Big)(A_1^{*})^{n-1}\Big) \le \|A_1\|^{2n-2}\Big(2\|A_1\|\,\|A_m\| + \sum_{j=2}^{m-1}\|A_j\|^{2}\Big)w(E),$$

and

$$w\big(A_1^{n-1}A_2X_2(A_1^{*})^{n} \pm A_1^{n}X_1A_2^{*}(A_1^{*})^{n-1}\big) = 2\|A_1\|^{2n-1}\|A_2\|\,w\left(\begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix}\right),$$

where $w(.)$ is the numerical radius and $E = \begin{bmatrix} 0 & & X_1 \\ & \dots & \\ X_m & & 0 \end{bmatrix}$. This provides an improvement of Theorem 3 by Fong and Holbrook [3] and a generalization of Theorem 3 by Hirzallah et al. [6]. Moreover, we provide some new upper bounds for the numerical radius of off-diagonal operator matrices and provide a generalization of the main result by Abu-Omar and Kittaneh [17].

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* *E-mail:* **mmaldolat@just.edu.jo (Corresponding Author)**
† *E-mail:* **kfzoubi@just.edu.jo**

1. Introduction

Let $B(H)$ be the C^* algebra of all bounded linear operators on the complex Hilbert space H. The numerical radius $w(.)$, the spectral radius $r(.)$ and the usual operator norm $\|.\|$ for $T \in B(H)$ are respectively defined by $w(T) = \sup\{|\langle Tx,x\rangle| : x \in H, \|x\| = 1\}$, $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$, and $\|T\| = \sup\{\langle Tx,Tx\rangle^{\frac{1}{2}} : x \in H, \|x\| = 1\}$. The numerical radius $w(\cdot)$ defines an equivalent norm for the usual operator norm on $B(H)$ and fulfills the sharp two-sided inequality

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|, \tag{1.1}$$

and the power inequality

$$w(T^n) \le w^n(T), \tag{1.2}$$

for all $T \in B(H)$ and $n \in \mathbb{N}$. In [1], Kittaneh provided an improvement for the upper bound of (1.1) by showing that

$$w(T) \le \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}), \tag{1.3}$$

which is further improved in [2] by Yamazaki as

$$w(T) \le \frac{1}{2}(\|T\| + w(\tilde{T})) \le \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}}), \tag{1.4}$$

where $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ with unitary U and $|T| = (T^*T)^{\frac{1}{2}}$. On the other hand, the authors in [3] proved that if $A, B, X \in B(H)$, then

$$w(A^*X + XA) \le 2\|A\| w(X), \tag{1.5}$$

and

$$w(AB + BA) \le 2\sqrt{2}\|B\| w(A). \tag{1.6}$$

One of the most important properties of the numerical radius is the weak unitary invariance property. That is

$$w(U^*TU) = w(T), \tag{1.7}$$

for every unitary $U \in B(H)$ and $T \in B(H)$. For more interesting results about the numerical radius of operator matrices and its application in finding bounds for the zeros of complex polynomials, one can refer to [4-12].

Let $H_1, H_2, \ldots, H_m$ be complex Hilbert spaces and $H = \bigoplus_{i=1}^{m} H_i$. According to this decomposition, any operator $T \in B(H)$ has an $m \times m$

operator matrix representation of the form $T = [T_{ij}]$ with $T_{ij} \in B(H_j, H_i)$, the space of all bounded linear operators from H_j to H_i. In [18], Hou and Du established the following estimate for the numerical radius

$$w(T) \leq w([t_{ij}]), \tag{1.8}$$

where $t_{ij} = \| T_{ij} \|$, which is considerably improved in [17] by AboOmar and Kittaneh as

$$w(T) \leq w([t_{ij}]), \tag{1.9}$$

where

$$t_{ij} = \begin{cases} w(T_{ii}), & i = j \\ \| T_{ij} \|, & i \neq j. \end{cases}$$

In [6], the authors proved that if $T_1, T_2, T_3, T_4 \in B(H)$, then

$$w(T_1 T_3 T_2^* + T_2 T_4 T_1^*) \leq 2 \| T_1 \| \, \| T_2 \| \, w\left(\begin{bmatrix} 0 & T_3 \\ T_4 & 0 \end{bmatrix}\right). \tag{1.10}$$

Motivated and inspired by the above results, our purpose in this paper is to provide an improvement for the inequality (1.5) and a generalization for the inequalities (1.9) and (1.10). Further, we establish new upper bounds for the numerical radius of the off-diagonal parts of $m \times m$ operator matrices, which gives new upper bounds for the numerical radius and the usual operator norm for the sums and products of operators.

2. The Main Results

In order to establish our main result, Theorem 2.6, we shall only need a few of well-known lemmas and shall recall them here for the sake of convenience.

The first lemma is a direct consequence of celebrated Jensen and Young inequalities (see, for instance, [13]).

Lemma 2.1 : *Let $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $s \geq 1$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left(\frac{a^{ps}}{p} + \frac{b^{qs}}{q}\right)^{\frac{1}{s}}.$$

The next lemma is a result of the spectral Theorem for positive operators together with Jensen's inequality (see [14]).

Lemma 2.2 : *Let* $T \in B(H)$ *be a positive operator and* $x \in H$ *such that* $\|x\| \leq 1$. *Then, for any* $s \geq 1$, *we have* $\langle Tx, x\rangle^s \leq \langle T^s x, x\rangle$.

The next lemma is also a consequence of the spectral Theorem [14].

Lemma 2.3 : *Let* $A, B \in B(H)$ *such that* $|A|B = B^* |A|$. *If* f, g *are nonnegative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t) = t$, *where* $t \geq 0$, *then*

$$|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\| \, \|g(|A^*|)y\|,$$

for every vectors $x, y \in H$.

The next lemma is well known (although it is not usually stated in this form); a proof of it can be found in [15] for (a-b) and [16] for (d). For (c), it follows by applying identity (1.7) to the operator matrices $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ and $U = \begin{bmatrix} I & 0 \\ 0 & iI \end{bmatrix}$.

Lemma 2.4 : *Let* $X, Y \in B(H)$. *Then*

(a) $w\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{w(X), w(Y)\}$,

(b) $w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y)$,

(c) $w\left(\begin{bmatrix} 0 & X \\ -Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)$,

(d) $w(X) = \max_{\theta \in R} \|\operatorname{Re}(e^{i\theta}X)\|$.

The fifth lemma is a simple consequence of the convexity for the function $f(t) = t^s$, where $s \geq 1$ and $t \geq 0$.

Lemma 2.5 : *Let* $a_i \geq 0$, $(i = 1, 2, \ldots, m)$. *Then, for every* $s \geq 1$, *we have*

$$\left(\sum_{i=1}^{m} a_i\right)^s \leq m^{s-1} \sum_{i=1}^{m} a_i^s.$$

We immediately deduce our first main result which can be considered as an improvement of inequality (1.5) and a generalization of inequality (1.10).

Theorem 2.6 : *Let* $\{A_i\}_{i=1}^m, \{X_i\}_{i=1}^m \subseteq B(H)$. *Then, for every* $n \in \mathbb{N}$ *and* $m > 2$, *we have*

$$w\left(A_1^{n-1}\left(\sum_{i=0}^{m-1} A_{m-i} X_{m-i} A_{i+1}^*\right)(A_1^*)^{n-1}\right) \leq \| A_1 \|^{2n-2} \left(2 \| A_1 \| \, \| A_m \| + \sum_{j=2}^{m-1} \| A_j \|^2\right) w(E), \tag{2.1}$$

where $E = \begin{bmatrix} 0 & & & X_1 \\ & & X_2 & \\ & \cdots & & \\ X_m & & & 0 \end{bmatrix}$.

Proof: Let $C = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$. Then by Lemma 2.4 part (a) and the definition of numerical radius we have

$$\begin{aligned} & w\left(A_1^{n-1}\left(\sum_{i=0}^{m-1} A_{m-i} X_{m-i} A_{i+1}^*\right)(A_1^*)^{n-1}\right) \\ & \quad = w\left(\begin{bmatrix} A_1^{n-1}\left(\sum_{i=0}^{m-1} A_{m-i} X_{m-i} A_{i+1}^*\right)(A_1^*)^{n-1} & 0 \\ 0 & 0 \end{bmatrix}\right) \\ & \quad = w(C^n E (C^*)^n) \\ & \quad \leq \| C^n \|^2 \, w(E) \qquad (2.2) \\ & \quad \leq \| A_1 \|^{2n-2} \left(\sum_{j=1}^{m} \| A_j \|^2\right) w(E) \\ & \quad = \left(\| A_1 \|^{2n} + \| A_1 \|^{2n-2} \| A_m \|^2 + \| A_1 \|^{2n-2} \sum_{j=2}^{m-1} \| A_j \|^2\right) w(E). \end{aligned}$$

Now, by replacing A_1 by tA_1, A_m by $\frac{A_m}{t^{2n-1}}$, and A_j, $(j = 2, 3, \ldots, m-1)$ by $\frac{A_j}{t^{n-1}}$ in the last inequality for any $t > 0$, we obtain

$$w\left(A_1^{n-1}\left(\sum_{i=0}^{m-1} A_{m-i} X_{m-i} A_{i+1}^*\right)(A_1^*)^{n-1}\right) \leq \left(t^{2n} \| A_1 \|^{2n} + \frac{\|A_1\| \, \|A_m\|^2}{t^{2n}} + \| A_1 \| \sum_{j=2}^{m-1} \| A_j \|^2\right) w(E).$$

Therefore,

$$\begin{aligned} w\left(A_1^{n-1}\left(\sum_{i=0}^{m-1} A_{m-i} X_{m-i} A_{i+1}^*\right)(A_1^*)^{n-1}\right) & \leq \min_{t>0}\left(\left(t^{2n} \| A_1 \|^{2n} + \frac{\|A_1\| \, \|A_m\|^2}{t^{2n}} + \| A_1 \| \sum_{j=2}^{m-1} \| A_j \|^2\right) w(E)\right) \\ & = \| A_1 \|^{2n-2} \left(2 \| A_1 \| \, \| A_m \| + \sum_{j=2}^{m-1} \| A_j \|^2\right) w(E), \end{aligned}$$

as desired.

Remark 2.7 : If we set $m=2$ in the last Theorem and adopt the above technique with utilizing Lemma 2.4 part (c), we have

$$w(A_1^{n-1}A_2X_2(A_1^*)^n \pm A_1^nX_1A_2^*(A_1^*)^{n-1}) = 2\|A_1\|^{2n-1}\|A_2\|\, w(\begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix}). \qquad (2.3)$$

Also, it is easy to see that if $n=1$, $A_1=A^*$, $A_2=I$ (the identity operator on *H*), and $X_1=X_2=X$ in inequality (2.3), then by Lemma 2.4 part (b) we have inequality (1.5). Moreover, inequality (1.10) can be directly obtained by letting $n=1$ in inequality (2.3).

Next, we give two upper bounds for the numerical radius of $m\times m$ off-diagonal operator matrices.

Theorem 2.8 : *Let* $\{A_i\}_{i=1}^m, \{B_i\}_{i=1}^m \subseteq B(H)$ *such that* $|A_i|B_i = B_i^*|A_i|$, $(i=1,2,\dots,m)$ *and let* $\{f_i\}_{i=1}^m$, $\{g_i\}_{i=1}^m$ *be two sets of nonnegative continues functions on* $[0,\infty)$ *such that* $f_i(t)g_i(t)=t$ $(t\ge 0)$. *Then for every* $s\ge 1$, *we have*

$$w^s(T) \le \begin{cases} \frac{m^{s-1}}{2}\max\limits_{1\le i\le m}(r^s(B_i)).\min(\sum\limits_{i=1}^{\frac{m}{2}}\sqrt{\|K_i\|\,\|M_i\|}, \sum\limits_{i=1}^{\frac{m}{2}}\sqrt{\|L_i\|\,\|N_i\|}), & n \text{ even} \\ \frac{m^{s-1}}{2}\max\limits_{1\le i\le m}(r^s(B_i)).\min(\sum\limits_{i=1}^{\frac{m+1}{2}}\sqrt{\|K_i\|\,\|M_i\|}, \sum\limits_{i=1}^{\frac{m+1}{2}}\sqrt{\|L_i\|\,\|N_i\|}), & n \text{ odd}, \end{cases} \qquad (2.4)$$

where $K_i = f_i^{2s}(|A_i|)+g_{m-i+1}^{2s}(|A_{m-i+1}^*|)$, $M_i = g_i^{2s}(|A_i^*|)+f_{m-i+1}^{2s}(|A_{m-i+1}|)$, $L_i = f_i^{2s}(|A_i|)+f_{m-i+1}^{2s}(|A_{m-i+1}^*|)$, $N_i = g_i^{2s}(|A_i^*|)+g_{m-i+1}^{2s}(|A_{m-i+1}|)$ and

$$T = \begin{bmatrix} 0 & & A_1B_1 \\ & A_2B_2 & \\ \cdots & & \\ A_mB_m & & 0 \end{bmatrix}.$$

Proof: Let m be an even number and $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ be any unit vector in $\bigoplus\limits_{i=1}^m H_i$, where $H_i = H$ $(i=1,2,\dots,m)$. Then we have

$$\begin{aligned} |<Tx,x>|^s &\le m^{s-1}\sum_{i=1}^m |\langle A_iB_ia_{m-i+1}, a\rangle|^s \quad \text{(by Lemma 2.5)} \\ &\le m^{s-1}\sum_{i=1}^m r^s(B_i)\,\|f_i(|A_i|)a_{m-i+1}\|^s\|g_i(|A_i^*|)a_i\|^s \text{ (by Lemma 2.3)} \\ &\le m^{s-1}\max_{1\le i\le m}(r^s(B_i))\sum_{i=1}^m \langle f_i^{2s}(|A_i|)a_{m-i+1}, a_{m-i+1}\rangle^{\frac{1}{2}}\langle g_i^{2s}(|A_i^*|)a_i, a_i\rangle^{\frac{1}{2}} \\ &\qquad \text{(by Lemma 2.2)} \end{aligned}$$

$$\leq m^{s-1} \max_{1\leq i\leq m}(r^s(B_i))\sum_{i=1}^{\frac{m}{2}}\langle K_i a_{m-i+1}, a_{m-i+1}\rangle^{\frac{1}{2}}\langle M_i a_i, a_i\rangle^{\frac{1}{2}}$$

$$\text{(by the Cauchy} - \text{schwarz inequality)}$$

$$\leq m^{s-1} \max_{1\leq i\leq m}(r^s(B_i))\sum_{i=1}^{\frac{m}{2}}\sqrt{\| K_i \| \, \| M_i \|}\, \| a_{m-i+1} \| \, \| a_i \|$$

$$\leq \frac{m^{s-1}}{2} \max_{1\leq i\leq m}(r^s(B_i))\sum_{i=1}^{\frac{m}{2}}\sqrt{\| K_i \| \, \| M_i \|}(\| a_{m-i+1} \|^2 + \| a_i \|^2).$$

Thus,

$$w^s(T) = \sup\{ |\langle Tx, x\rangle|^s : x \in \bigoplus_{i=1}^{m} H_i, \| x \| = \sum_{i=1}^{m} \| a_i \|^2 = 1\}$$
$$\leq \frac{m^{s-1}}{2} \max_{1\leq i\leq m}(r^s(B_i))\sum_{i=1}^{\frac{m}{2}}\sqrt{\| K_i \| \, \| M_i \|}.$$

If n is an odd number, then by a similar argument combined with the arithmetic geometric mean, we have

$$\frac{|\langle Tx, x\rangle|^s}{m^{s-1}\max\limits_{1\leq i\leq m}(r^s(B_i))} \leq \langle f^{2s}_{\frac{m+1}{2}}(|A_{\frac{m+1}{2}}|)a_{\frac{m+1}{2}}, a_{\frac{m+1}{2}}\rangle^{\frac{1}{2}}\langle g^{2s}_{\frac{m+1}{2}}(|A^*_{\frac{m+1}{2}}|)a_{\frac{m+1}{2}}, a_{\frac{m+1}{2}}\rangle^{\frac{1}{2}}$$
$$+\frac{1}{2}\sum_{i=1}^{\frac{m-1}{2}}\sqrt{\| K_i \| \, \| M_i \|}$$
$$\leq \frac{1}{2}\langle (f^{2s}_{\frac{m+1}{2}}(|A_{\frac{m+1}{2}}|) + g^{2s}_{\frac{m+1}{2}}(|A^*_{\frac{m+1}{2}}|))a_{\frac{m+1}{2}}, a_{\frac{m+1}{2}}\rangle$$
$$+\frac{1}{2}\sum_{i=1}^{\frac{m-1}{2}}\sqrt{\| K_i \| \, \| M_i \|}$$
$$\leq \frac{1}{2}\| f^{2s}_{\frac{m+1}{2}}(|A_{\frac{m+1}{2}}|) + g^{2s}_{\frac{m+1}{2}}(|A^*_{\frac{m+1}{2}}|)\| \, \| a_{\frac{m+1}{2}} \|^2 + \frac{1}{2}\sum_{i=1}^{\frac{m-1}{2}}\sqrt{\| K_i \| \, \| M_i \|}$$
$$\leq \frac{1}{2}\sum_{i=1}^{\frac{m+1}{2}}\sqrt{\| K_i \| \, \| M_i \|}.$$

Therefore,

$$w^s(T) = \sup\{ |\langle Tx, x\rangle|^s : x \in \bigoplus_{i=1}^{m} H_i, \| x \| = \sum_{i=1}^{m} \| a_i \|^2 = 1\}$$
$$\leq \frac{m^{s-1}}{2} \max_{1\leq i\leq m}(r^s(B_i))\sum_{i=1}^{\frac{m+1}{2}}\sqrt{\| K_i \| \, \| M_i \|}.$$

Finally, the second bound in each case can be obtained in the same manner.

As a direct consequence of Theorem 2.8, we obtain the following corollary.

Corollary 2.9 : *Let* $X, Y \in B(H)$, $s \geq 1$, *and* $0 \leq \alpha, \beta \leq 1$. *Then*

$$w^s\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq 2^{s-2} \min(\sqrt{\lambda}, \sqrt{\mu}),$$

where $\lambda = \left\| |X|^{2s\alpha} + |Y^*|^{2s(1-\beta)} \right\| \left\| |X^*|^{2s\beta} + |Y|^{2s(1-\alpha)} \right\|$ and

$$\mu = \left\| |X|^{2s\alpha} + |Y^*|^{2s(1-\alpha)} \right\| \left\| |X^*|^{2s\beta} + |Y|^{2s(1-\beta)} \right\|.$$

Proof: The result follows by letting $m = 2$, $B_1 = B_2 = I$, $A_1 = X$, $A_2 = Y$, $f_1(t) = t^{\alpha}$, $f_2(t) = t^{1-\alpha}$, $g_1(t) = t^{\beta}$, and $g_2(t) = t^{1-\beta}$ in Theorem 2.8.

It is worth mentioning here that Theorem 4 in [17] can be obtained from Corollary 2.9 by letting $p = \frac{1}{2}$.

The next corollary represents an extension of inequality 1.3.

Corollary 2.10 : *Let* $X \in B(H)$, $s \geq 1$, *and* f, g *are nonnegative continues functions on* $[0, \infty)$ *satisfying* $f(t)g(t) = t$, $(t \geq 0)$. *Then*

$$w^s(X) \leq \frac{1}{2} \min(\lambda, \sqrt{\mu}),$$

where $\lambda = \left\| f^{2s}(|X|) + g^{2s}(|X^*|) \right\|$ and
$\mu = \left\| f^{2s}(|X|) + f^{2s}(|X^*|) \right\| \left\| g^{2s}(|X|) + g^{2s}(|X^*|) \right\|$. In particular,

$$w^s(X) \leq \frac{1}{2}\left(\left\| |X|^s + |X^*|^s \right\| \right).$$

Proof: Set $m = 1$, $B_1 = I$, $A_1 = X$, $f_1(t) = f(t)$, and $g_1(t) = g(t)$ in Theorem 2.8. For the particular case, set $f_1(t) = g_1(t) = \sqrt{t}$.

The following is a corollary of inequality (2.3).

Corollary 2.11 : *Let* $X, Y \in B(H)$, $s \geq 1$, $p \in [0,1]$, *and k is an even number. Then*

$$w^{\frac{s}{k}}((XY)^{\frac{k}{2}}) \leq 2^{s-2} \sqrt{\left\| |X|^{2sp} + |Y^*|^{2s(1-p)} \right\| \left\| |X^*|^{2sp} + |Y|^{2s(1-p)} \right\|}.$$

In particular,

$$w(XY) \leq \frac{1}{4}\left\| |X| + |Y^*| \right\| \left\| |X^*| + |Y| \right\|.$$

Proof: By letting $m = 2$, $B_1 = B_2 = I$, $A_1 = X$, $A_2 = Y$, $f_1(t) = g_1(t) = t^p$, and $f_2(t) = g_2(t) = t^{1-p}$, we have

$$\begin{aligned}
\max(w^s((XY)^{\frac{k}{2}}), w^s((YX)^{\frac{k}{2}})) &= w^s\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}^k\right) \\
&\leq w^{sk}\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \\
&\leq \left(2^{s-2}\sqrt{\left\| |X|^{2sp} + |Y^*|^{2s(1-p)} \right\| \left\| |X^*|^{2sp} + |Y|^{2s(1-p)} \right\|}\right)^k.
\end{aligned}$$

Thus, the proof is complete by setting $k = 2$, $s = 1$, and $p = \frac{1}{2}$.

Corollary 2.12 : *Let* $X, Y \in B(H)$, $s \geq 1$, *and k is an odd number. then*

$$\left\|(XY)^{\frac{k-1}{2}} X + ((YX)^{\frac{k-1}{2}} Y)^*\right\|^{\frac{s}{k}} \leq 2^{\frac{s}{k}+(r-2)} \left\| |X|^s + |Y^*|^s \right\|^{\frac{1}{2}} \left\| |Y|^s + |X^*|^s \right\|^{\frac{1}{2}}.$$

Proof: Let $T = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in B(H \oplus H)$ and $m = 2$, $B_1 = B_2 = I$, $A_1 = X$, $A_2 = Y$, $f_1(t) = g_1(t) = f_2(t) = g_2(t) = t^{\frac{1}{2}}$. Then, we have

$$\begin{aligned}
\left\|(XY)^{\frac{k-1}{2}} X + ((YX)^{\frac{k-1}{2}} Y)^*\right\|^s &= \left\|T^k + (T^k)^*\right\|^s \\
&\leq 2^s \max_{\theta \in R} \left\|\text{Re }(e^{i\theta} T^k)\right\|^s \\
&= 2^s w^s(T^k) \text{ (by Lemma 2.4(c))} \\
&\leq 2^s (w^s(T))^k \text{ (by the power inequality)} \\
&\leq 2^{s+(s-2)k} \left\| |X|^s + \| Y^* | \right\|^{\frac{k}{2}} \left\| |X|^s + \| Y^* | \right\|^{\frac{k}{2}}.
\end{aligned}$$

The next theorem provides another upper bound of $w(T)$, where

$$T = \begin{bmatrix} 0 & & & A_1B_1 \\ & & A_2B_2 & \\ & \cdots & & \\ A_mB_m & & & 0 \end{bmatrix} \text{ with } |A_i| B_i = B_i^* |A_i|.$$

Theorem 2.13 : *Let* $\{A_i\}_{i=1}^m, \{B_i\}_{i=1}^m \subseteq B(H)$ *such that* $|A_i| B_i = B_i^* |A_i|$ *and* $\{f_i\}_{i=1}^m$, $\{g_i\}_{i=1}^m$ *be two sets of nonnegative continues functions on* $[0,\infty)$ *such that* $f_i(t) g_i(t) = t \quad (t \geq 0)$. *Then for every* $s \geq 1$ *and* $\alpha_i, \beta_i > 1$ *with* $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$, *we have*

$$w^s(T) \leq \begin{cases} m^{s-1} \max\limits_{1\leq i\leq m}(r(B_i)) \min(\sum_{i=1}^{\frac{m}{2}} \lambda_i, \sum_{i=1}^{\frac{m}{2}} \mu_i), & n \text{ even} \\ m^{s-1} \max\limits_{1\leq i\leq m}(r(B_i)).(\sqrt{\| f_{\frac{m+1}{2}}^{2s}(|A_{\frac{m+1}{2}}|) \| \, \| g_{\frac{m+1}{2}}^{2s}(|A_{\frac{m+1}{2}}^*|) \|} & \\ \qquad + \min(\sum_{i=1}^{\frac{m-1}{2}} \lambda_i, \sum_{i=1}^{\frac{m-1}{2}} \mu_i)), & n \text{ odd}, \end{cases} \tag{2.5}$$

where $T = \begin{bmatrix} 0 & & A_1B_1 \\ & A_2B_2 & \\ \ldots & & \\ A_mB_m & & 0 \end{bmatrix}$, $\lambda_i = \sqrt{\frac{\|K_i^{\alpha_i}\|}{\alpha_i} + \frac{\|M_i^{\beta_i}\|}{\beta_i}}$, and $\mu_i = \sqrt{\frac{\|L_i^{\alpha_i}\|}{\alpha_i} + \frac{\|N_i^{\beta_i}\|}{\beta_i}}$

(K_i, M_i, L_i and N_i as in Theorem 2.8).

Proof: Let m be an even number and $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ be any unit vector in $\bigoplus_{i=1}^m H_i$ where $H_i = H$, $(i = 1, 2, \ldots, m)$. Then, we have

$$\begin{aligned} \frac{|\langle Tx, x\rangle|^s}{m^{s-1}} &\leq \max_{1\leq i\leq m}(r^s(B_i)) \sum_{i=1}^{\frac{m}{2}} \langle K_i a_{m-i+1}, a_{m-i+1}\rangle^{\frac{1}{2}} \langle M_i a_i, a_i\rangle^{\frac{1}{2}} \\ &\leq \max_{1\leq i\leq m}(r^s(B_i)) \sum_{i=1}^{\frac{m}{2}} \left(\frac{\langle K_i^{\alpha_i} a_{m-i+1}, a_{m-i+1}\rangle^{\frac{1}{2}}}{\alpha_i} + \frac{\langle M_i^{\beta_i} a_i, a_i\rangle^{\frac{1}{2}}}{\beta_i} \right) \\ &\leq \max_{1\leq i\leq m}(r^s(B_i)) \sum_{i=1}^{\frac{m}{2}} \left(\frac{\sqrt{\|K_i^{\alpha_i}\|}}{\alpha_i} \|a_{m-i+1}\| + \frac{\sqrt{\|M_i^{\beta_i}\|}}{\beta_i} \|a_i\| \right). \end{aligned}$$

Now, by Lagrange Multipliers Theorem, we obtain that

$$\max_{\|a_{m-i+1}\|^2 + \|a_i\| \leq 1} \left(\frac{\sqrt{\|K_i^{\alpha_i}\|}}{\alpha_i} \|a_{m-i+1}\| + \frac{\sqrt{\|M_i^{\beta_i}\|}}{\beta_i} \|a_i\| \right) = \sqrt{\frac{\|K_i^{\alpha_i}\|}{\alpha_i^2} + \frac{\|M_i^{\beta_i}\|}{\beta_i^2}} = \lambda_i.$$

Hence,

$$w^{s}(T)=\sup\left\{|\langle Tx,x\rangle|^{s}: x\in\bigoplus_{i=1}^{m}H_{i},\|x\|=1\right\}$$
$$\le m^{s-1}\max_{1\le i\le m}(r^{s}(B_{i}))\sum_{i=1}^{\frac{m}{2}}\lambda_{i}.$$

The proof of the case where *m* is an even number is finish by follows the same technique above with the same procedure that used in Theorem 2.8. Finally, by an analogous argument, one can easily show that inequality 2.5 also holds when *m* is an odd number.

As a direct consequence of Theorem 2.13, we obtain the following results.

Corollary 2.14 : *Let* $X,Y\in B(H)$, $s\ge 1$, *and* k *is an even number. Then*

$$w^{\frac{2s}{k}}((XY)^{\frac{k}{2}})\le 4^{s-2}(\|(|X|^{s}+|Y^{*}|^{s})^{2}\|+\|(|Y|^{s}+|X^{*}|^{s})^{2}\|).$$

In particular,

$$w(XY)\le\frac{1}{4}(\|(|X|+|Y^{*}|)^{2}\|+\|(|Y|+|X^{*}|)^{2}\|).$$

Proof: The result follows from the power inequality and Theorem 2.13 with $m=2$, $B_{1}=B_{2}=I$, $A_{1}=X$, $A_{2}=Y$, $\alpha_{1}=\beta_{1}=2$, and $f_{1}(t)=f_{2}(t)=g_{1}(t)=g_{2}(t)=\sqrt{t}$.

Corollary 2.15 : *Let* $X,Y\in B(H)$, $r\ge 1$, *and k is an odd number. Then for all* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\|(XY)^{\frac{k-1}{2}}X+((YX)^{\frac{k-1}{2}}Y)^{*}\|^{\frac{2r}{k}}\le 4^{\frac{r}{k}+r-1}\left(\frac{\|(|X|^{r}+|Y^{*}|^{r})^{p}\|}{p^{2}}+\frac{\|(|Y|^{r}+|X^{*}|^{r})^{q}\|}{q^{2}}\right).$$

In particular,

$$\|XYX+(YXY)^{*}\|\le\frac{1}{4}\left(\|(|X|+|Y^{*}|)^{2}\|+\|(|Y|+|X^{*}|)^{2}\|\right)^{\frac{3}{2}}.$$

Finally, the next theorem provides a generalization for AboOmar and Kittaneh estimate (1.9).

Theorem 2.16 : *Let* $H_{1},H_{2},...,H_{m}$ *be complex Hilbert spaces and let* $T=[A_{ij}B_{ij}]$ *be* $m\times m$ *operator matrix with* $B_{ij}\in B(H_{j})$ *and* $A_{ij}\in B(H_{j},H_{i})$ *such that*

$|A_{ij}|B_{ij}=B_{ij}^*|A_{ij}|$ for $i\neq j$. If f and g are nonnegative continues functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geq 0)$, then

$$w(T)\leq w([t_{ij}]),$$

where

$$t_{ij}=\begin{cases} w(A_{ii}B_{ii}), & i=j \\ r(B_{ij})\| f(|A_{ij}|)\|\, \| g(|A_{ij}^*|)\|), & i\neq j. \end{cases}$$

Proof: Let $a=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\in\bigoplus_{i=1}^m H_i$ be any unit vector and let $b=\begin{bmatrix} \|x_1\| \\ \|x_2\| \\ \vdots \\ \|x_m\| \end{bmatrix}$. Then

$$\begin{aligned}
|\langle Ta,a\rangle| &\leq \sum_{i=1}^m |\langle A_{ii}B_{ii}x_i,x_i\rangle| + \sum_{\substack{i,j=1\\ i\neq j}}^m |\langle A_{ij}B_{ij}x_j,x_i\rangle| \\
&\leq \sum_{i=1}^m w(A_{ii}B_{ii})\|x_i\|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^m r(B_{ij})\| f(|A_{ij}|)x_j\|\, \| g(|A_{ij}^*|)x_i\| \\
&\leq \sum_{i=1}^m w(A_{ii}B_{ii})\|x_i\|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^m r(B_{ij})\| f(|A_{ij}|)\|\, \| g(|A_{ij}^*|)\|\, \|x_i\|\, \|x_j\| \\
&= \langle [t_{ij}]b,b\rangle \\
&\leq w([t_{ij}])\|b\|^2 \\
&= w([t_{ij}]).
\end{aligned}$$

Thus,

$$w(T)=\sup_{\|a\|=1}|\langle Ta,a\rangle|\leq w([t_{ij}]),$$

which proves the result.

Clearly, the AboOmar and Kittaneh estimate (1.9) can be derived from Theorem 2.16 by setting $B_{ij}=I_{H_j}$ (the identity operator on H_j) and $f(t)=g(t)=\sqrt{t}$.

Using Theorem 2.16 and the fact that if $A=[a_{ij}]$ is $m\times m$ matrix where $a_{ij}\geq 0$, then $w(A)\leq\frac{1}{2}r([a_{ij}]+[a_{ji}])$ we have the following result.

Corollary 2.17 : *Let H_1,H_2 be complex Hilbert spaces and let $T=\begin{bmatrix} A_{11}B_{11} & A_{12}B_{12} \\ A_{21}B_{21} & A_{22}B_{22} \end{bmatrix}$ be 2×2 operator matrix with $B_{ij}\in B(H_j)$ and $A_{ij}\in B(H_j,H_i)$ such that*

$|A_{12}|B_{12} = B_{12}^*|A_{12}|$ *and* $|A_{21}|B_{21} = B_{21}^*|A_{21}|$. *If* f *and* g *are nonnegative continues functions on* $[0,\infty)$ *satisfying* $f(t)g(t) = t \ (t \geq 0)$, *then*

$$w(T) \leq \frac{w(A_{11}B_{11}) + w(A_{22}B_{22}) + \sqrt{(w(A_{11}B_{11}) - w(A_{22}B_{22}))^2 + \alpha + \beta}}{2}$$

$$\alpha = r(B_{12})\|f(|A_{12}|)\|\,\|g(|A_{12}^*|)\| \text{ and } \beta = r(B_{21})\|f(|A_{21}|)\|\,\|g(|A_{21}^*|)\|.$$

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