Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 4, pp. 585–593
DOI : 10.47974/JIM-1463

Fuzzy differential for subclass of analytic functions defined by linear operator

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Abstract

In this work, we establish some results for fuzzy differential for analytic functions associated with the Hadamard product for generalization Srivastava-Attiya operator is defined in the open unit disc $\mathcal{L} = \{\omega \in \mathbb{C}: |\omega| < 1\}$. In the present paper, the operator is applied to a new class of analytic functions with the Hadamard product. The goal of this article is to study some properties of fuzzy differential subordination and fuzzy differential superordination of analytic functions related to the Hadamard product. Using double theories, a sandwich-type theorem was obtained.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Univalent functions, Analytic functions, Starlike functions, Hadamard product, Fuzzy differential subordination, Fuzzy differential superordination.*

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1. Introduction

Let $\mathcal{H}$ denote the family for analytic functions in $\mathcal{L} = \{\omega \in \mathbb{C}: |\omega| < 1\}$and $\mathcal{H}[a, m]$ represent a subclass of $\mathcal{H}$ that includes the following function:

$$f(\omega) = a + a_m\omega^m + a_{m+1}\omega^{m+1} + \cdots$$

Suppose A is a subclass of $\mathcal{H}$ and $\mathcal{A}$ that includes all functions of the following type:

$$f_j(\omega) = a_{1,j} + a_{2,j}\omega^2 + \cdots = \omega + \sum_{k=2}^{\infty} a_{k,j}\omega^k, (j = 1,2; k \in \mathrm{N}) \qquad (1)$$

In addition, their Hadamard product (or combination) of $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ is given by:

$$(f_1 * f_2)(\omega) = \omega + \sum_{k=2}^{\infty} a_{k,1}a_{k,2}\, \omega^k = (f_2 * f_1)(\omega) \qquad (2)$$

A function $f \in \mathcal{A}$ is named univalent starlike of order τ $(0 \leq \tau < 1)$, if

$$\mathrm{Re}\left\{\frac{\omega f'(\omega)}{f(\omega)}\right\} > \tau \qquad (\omega \in \mathcal{L})$$

Denote this class by $S(\tau)$.

In [6], Srivastava and Gaboury introduced the linear operator $\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f): \Sigma \to \Sigma$ as follows:

$$\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}} f(\omega) = \omega + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j+1)_{k-1}}{\prod_{j=1}^{q}(\mu_j+1)_{k-1}} \cdot \left(\frac{a+1}{a+k}\right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\Lambda(a+1,b,s,\lambda)} a_k \frac{\omega^k}{k!} \qquad (3)$$

Where

$(\lambda_j \in \mathbb{C}(j = 1, \dots, n); s \in \mathbb{C}; a, \mu_j \in \mathbb{C} \setminus \omega_0^-(j = 1, \dots, q); p \leq q + 1; \omega \in \mathcal{L}.$

It's simple to look that $\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}} f(\omega)\right)'$

$$= (\lambda_1 + 1)\mathcal{J}^{s,a,\lambda}_{(\lambda_1+1,\lambda_2,\cdots,\lambda_p),(\mu_q),\mathrm{b}} f(\omega) - \lambda_1\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}} f(\omega).$$

Definition 1 [1]: Assume X is a non-empty set. A fuzzy subset is defined as the application $F: X \to [0, 1]$. Another, more specific definition would be: A try $(\mathcal{S}, F_{\mathcal{S}})$, where "$F_{\mathcal{S}}: X \to [0, 1]$ and $\mathcal{S} = \{x \in X : 0 < F_{\mathcal{S}} \leq 1\} = supp\ (\mathcal{S}, F_{\mathcal{S}})$" be known as a fuzzy subset. The membership function of the fuzzy subset $(\mathcal{S}, F_{\mathcal{S}})$ is named after the function $F_{\mathcal{S}}$.

Definition 2 [2]: Assume that two fuzzy subsets of X, $(\mathcal{M}, F_{\mathcal{M}})$ and $(\mathcal{N}, F_{\mathcal{N}})$. the fuzzy subsets $\mathcal{M}$ and $\mathcal{N}$ are equivalent if and only if $F_{\mathcal{M}}(x) = F_{\mathcal{N}}(x)$, $x \in X$ which we indicate by$(\mathcal{M}, F_{\mathcal{M}}) = (\mathcal{N}, F_{\mathcal{N}})$. The fuzzy subset $(\mathcal{M}, F_{\mathcal{M}})$

contains the fuzzy subset $(\mathcal{N},\ F_{\mathcal{N}})$ if and only if $F_{\mathcal{M}}(x) \leq\ F_{\mathcal{N}}\ (x),\ x \in X$ and the inclusion relation is denoted by $(\mathcal{M}, F_{\mathcal{M}})\ \subseteq (\mathcal{N},\ F_{\mathcal{N}})$. Assume $\mathcal{D}$ is a set in $\mathbb{C}$ and f, g are analytic functions. We usually signified by:

$$f(\mathcal{D})\ =\ supp(\ f(\mathcal{D}), F_{f(\mathcal{D})})\ = \{f\ (\ \omega) : 0\ <\ F_{f(\mathcal{D})}\ (f(\omega))\ \leq\ 1, \omega \in \mathcal{D}\}$$

and $\quad g(\mathcal{D}) = \mathrm{supp}(\ g(\mathcal{D}), F_{g(\mathcal{D})}) = \{g(\omega) : 0 <\ F_{g(\mathcal{D})}\ (g(\omega) \leq 1, \omega \in \mathcal{D}$

Definition 3 [2]: Assume that $\mathcal{D} \subset \mathbb{C}, \omega_0 \in \mathcal{D}$, is a constant value, and the functions $f, g \in \mathcal{H}(\mathcal{D})$. The function f is claimed to be a fuzzy subordinate of g written
as $f \prec_F g$ or $f(\omega) \prec_F g(\omega)$ If the following requirements are satisfied:

$$1. f(\omega_0) = g(\omega_0), \qquad\qquad 2. F_{f(\mathcal{D})}(f(\omega)) \leq F_{g(\mathcal{D})}(g(\omega)),\ \omega \in \mathcal{D}$$

Definition 4 [3]: Let $\psi : \mathrm{C}^3 \times \mathcal{L}\ \rightarrow\ C$ and $\hbar$ are univalent in $\mathcal{L}$. If p is analytic in $\mathcal{L}$ and the (second-order) fuzzy differential subordination is satisfied

$$F_{\psi(\mathbb{C}^3\times\mathcal{L})}(\psi(p(\omega), \omega p'(\omega), \omega^2 p''(\omega); \omega)) \leq F_{\hbar(\mathcal{L})}(\hbar(\omega)), \tag{4}$$
$$\psi(p(\omega), \omega p'(\omega), \omega^2 p''(\omega); \omega) \prec_F\ \hbar(\omega), \qquad \omega \in\ \mathcal{L},$$

then p is named "a fuzzy answer of the fuzzy differential subordination. The univalent function q is named a fuzzy dominant of the fuzzy answer of the fuzzy differential subordination, or simply a fuzzy dominant, if $p(\omega) \prec_F q(\omega), \omega \in \mathcal{L}$ for all p satisfying (4). the fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(\omega) \prec_F q(\omega)$, $\omega \in \mathcal{L}$ for all fuzzy dominant q of (4) is claimed the fuzzy best dominant of (4)".

Definition 5 [5]: Let $\psi: \mathrm{C}^3 \times \mathcal{L} \rightarrow C$ and $\hbar$ be analytic in $\mathcal{L}$. If p and $\psi(p(\omega), \omega p'(\omega), \omega^2 p''(\omega); \omega)$ are univalent in $\mathcal{L}$ and satisfy the (second-order) fuzzy differential superordination.

$$F_{\hbar(\mathcal{L})}(\hbar(\omega))\ \leq F_{\psi(\mathbb{C}^3\times\mathcal{L})}(\psi(p(\omega), \omega p'(\omega), \omega^2 p''(\omega); \omega)) \tag{5}$$
$$\text{i.e. } \hbar(\omega) \prec_F\ (\psi(p(\omega), \omega p'(\omega), \omega^2 p''(\omega); \omega)), \qquad \omega \in \mathcal{L},$$

then p is named "a fuzzy answer of the fuzzy differential superordination. The analytic function q is named fuzzy subordinate of the fuzzy answer of the fuzzy differential superordination, or simply a fuzzy subordination if $q(\omega) \prec_F\ p(\omega)$, $\omega \in \mathcal{L}$ for all p satisfying (5). A univalent fuzzy subordination $\tilde{q}$ that satisfy $q(\omega) \prec_F\ \tilde{q}(\omega)$, for all fuzzy subordinate q of (5) is claimed the fuzzy best subordinate of (5)".

Lemma 1 [4]: *Assume that q is a univalent function for the unit disk $\mathcal{L}$ and Let θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{L})$ with $\phi\ (w) \neq 0$ when $w \in\ q(\mathcal{L})$. Set $Q(\omega)\ =\ \omega q'(\omega)\phi(q(\omega))$ and $\hbar(\omega)\ =\ \theta(q(\omega))\ +\ Q(\omega)$. Assume that:*

1. *Q is a starlike univalent in $\mathcal{L}$ and*
2. $Re\left(\frac{\omega h'(\omega)}{Q(\omega)}\right) > 0$ *for* $\omega \in \mathcal{L}$. *If p it is analytic in $\mathcal{L}$, with $p(0) = q(0)$, $p(\mathcal{L}) \subseteq \mathcal{D}$ and $\psi: \mathbb{C}^2 \times \mathcal{L} \to \mathbb{C}$, $\psi(p(\omega), \omega p'(\omega)) = \theta(p(\omega)) + \omega p'(\omega) \cdot \phi(p(\omega))$ is analytic in $\mathcal{L}$.*
   Then $F_{\psi(\mathbb{C}^2\times\mathcal{L})}[\theta(p(\omega)) + \omega p'(\omega) \cdot \phi(p(\omega))] \le F_{\hbar(\mathcal{L})}\hbar(\omega)$,
   Implying $F_{p(\mathcal{L})}p(\omega) \le F_{q(\mathcal{L})}q(\omega)$
   i.e., $p(\omega) \prec_F q(\omega)$ *where q is the fuzzy best dominant.*

Lemma 2 [5]: *Assume that q is a convex univalent function for the unit disk $\mathcal{L}$ and Let v and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{L})$. Assume that:*

1. $Re\left(\frac{v'(q(\omega))}{\phi(q(\omega))}\right) > 0$ *for $\omega \in \mathcal{L}$ and*
2. $Q(\omega) = \omega q'(\omega)\phi(q(\omega))$ *is starlike univalent in $\mathcal{L}$*
   If $p(\omega) \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(\mathcal{L}) \subseteq \mathcal{D}$ *and* $\psi: \mathbb{C}^2 \times \mathcal{L} \to \mathbb{C}$, $\psi(p(\omega), p'(\omega)) = v(p(\omega)) + \omega p'(\omega) \cdot \phi(p(\omega))$ *are both univalent in* $\mathcal{L}$
   then $F_{\hbar(\mathcal{L})}\hbar(z) \le F_{\psi(\mathbb{C}^2\times\mathcal{L})}[v(p(\omega)) + \omega p'(\omega) \cdot \phi(p(\omega))]$
   implying $F_{q(\mathcal{L})}q(\omega) \le F_{p(\mathcal{L})}p(\omega)$
   i.e., $q(\omega) \prec_F p(\omega)$ *where q is the fuzzy best subordinate.*

2. Main Results

Theorem 1: *Let q be analytic and univalent in the open unit disk $\mathcal{L}$ such that:*

$$q(\omega) \neq 0 \text{ and } \left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)}\right)^{\beta} \in \mathcal{H}(\mathcal{L})$$

for all $\omega \in \mathcal{L}$, where s, a represent complex numbers with $a \neq 0, -1, -2, \ldots$ and $\beta, \lambda > 0$.

Assume that $\frac{\omega q'(\omega)}{q(\omega)}$ is starlike univalent in $\mathcal{L}$. Let:

$$\mathrm{Re}\left(\frac{\omega q''(\omega)}{q'(\omega)} - \frac{\omega q'(\omega)}{q(\omega)} + \frac{2\mu}{\gamma}(q(\omega))^2 + \frac{\eta}{\gamma}q(\omega) + 1\right) > 0 \tag{6}$$

$$\Psi(\alpha, \eta, \mu, \gamma, \beta; \omega) = \alpha + \gamma\beta\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)\right)''}{\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)\right)'} - \frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)(\omega)} + 1\right)$$

$$+\ \mu\left(\frac{\omega\left(J^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{J^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}\right)^{2\beta} + \eta\left(\frac{\omega\left(J^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{J^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}\right)^{\beta} \quad (7)$$

then $F_{\psi(C^2\times\mathcal{L})}[\Psi(\alpha,\eta,\mu,\gamma,\beta;\omega)] \le F_{\psi(C^2\times\mathcal{L})}[\alpha+\gamma\,\frac{\omega q'(\omega)}{q(\omega)}+\mu(q(\omega))^2+\eta q(\omega)]$

$$= F_{\hbar(\mathcal{L})}h(\omega) \quad (8)$$

Implying $F_{\left(\frac{\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}}\right)^{\beta}(\mathcal{L})}\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}\right)^{\beta}$

$$\le F_{q(\mathcal{L})}q(\omega), \quad (9)$$

i.e., $\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}\right)^{\beta} \prec_F q(\omega)$ where q is the fuzzy best dominant

Proof: Define $p(\omega)=\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}\right)^{\beta}, \ \omega\in\mathcal{L},$ (10)

and $\frac{\omega\, p'(\omega)}{p(\omega)}=\beta\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)''}{\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}-\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),b}(f_1*f_2)\ (\omega)}+1\right)$

By obtaining $\theta(w)=\mu w^2+\eta w+\alpha$ and $\phi(w)=\frac{\gamma}{w}$, it is clear that θ and ϕ are analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w)\neq 0,\ w\in\mathbb{C}\backslash\{0\}$.Consider $Q(\omega)=\omega q'(\omega)\phi\left(q(\omega)\right)=\gamma\,\frac{\omega q'(\omega)}{q(\omega)}$ and $h(\omega)=Q(\omega)+\theta\left(q(\omega)\right)=\alpha+\gamma\,\frac{\omega q'(\omega)}{q(\omega)}+\mu\left(q(\omega)\right)^2+\eta q(\omega)$, we derive that Q is starlike univalent in U.

$$\mathrm{Re}\left(\frac{\omega h'(\omega)}{Q(\omega)}\right)=\mathrm{Re}\left(1+\frac{\eta}{\gamma}q(\omega)+(\tfrac{2\mu}{\gamma})(q(\omega))^2-\frac{\omega q'(\omega)}{q(\omega)}+\omega\frac{q''(\omega)}{q(\omega)}\right)>0$$

using (8),we deduce $F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha+\gamma\,\frac{\omega\, p'(\omega)}{p(\omega)}+\mu(p(\omega))^2+\eta\, p(\omega)\right)$

$$\le F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha+\gamma\,\frac{\omega q'(\omega)}{q(\omega)}+\mu(q(\omega))^2+\eta\, q(\omega)\right)$$

Using Lemma 1 as a guide, we get $F_{p(\mathcal{L})}p(\omega) \leq F_{q(\mathcal{L})}q(\omega), \omega \in \mathcal{L}$, By using (10), we obtain

$$F_{\left(\frac{\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q,b}\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q,b}}\right)^{\beta}(\mathcal{L})}\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q,b}\ (f_1 * f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q,b}\ (f_1 * f_2)(\omega)}\right)^{\beta} \leq F_{q(\mathcal{L})}q(\omega)$$

i.e., $\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta} \prec_F q(\omega)$ and q is the fuzzy best dominant.

Corollary 1: Let s , a represent complex numbers with $a \neq 0, -1, -2, \ldots$ and $\beta, \lambda > 0$. Assume that (6) is correct. Then

$$F_{\psi(\mathbb{C}^2\times\mathcal{L})}[\Psi(\alpha,\eta,\mu,\gamma,\beta;\omega)]$$
$$\leq F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha+\gamma\frac{(\mathcal{M}-\mathcal{N})\omega}{(1+\mathcal{M}\omega)(1+\mathcal{N}\omega)}+\mu\left(\frac{1+\mathcal{M}\omega}{1+\mathcal{N}\omega}\right)^2+\eta\frac{1+\mathcal{M}\omega}{1+\mathcal{N}\omega}\right)$$

implies $\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta} \prec_F \frac{1+\mathcal{M}\omega}{1+\mathcal{N}\omega}$ and $\frac{1+\mathcal{M}\omega}{1+\mathcal{N}\omega}$ is the fuzzy best dominant.

Proof: For $q(\omega) = \frac{1+\mathcal{M}\omega}{1+\mathcal{N}\omega}, -1 \leq \mathcal{N} < \mathcal{M} \leq 1$, The given conclusion follows from Theorem 1.

Theorem 2: *Let q is analytic and univalent in $\mathcal{L}$ such that $q(\omega) \neq 0$ and $\frac{\omega q'(\omega)}{q(\omega)}$ be univalent starlike in $\mathcal{L}$. Assume that:*

$$\operatorname{Re}\left(\frac{\eta}{\gamma}q(\omega)q'(\omega)+\frac{2\mu}{\gamma}q^2(\omega)q'(\omega)\right) > 0 \tag{11}$$

Let s, a represent complex number with $a \neq 0, -1, -2, \ldots$ and $\beta, \lambda > 0$.

If $\Psi(\alpha,\eta,\mu,\gamma,\beta;\omega)$ is univalent in $\mathcal{L}$ and $\left(\frac{\omega\left(\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta} \in$ $\mathcal{H}[q(0),1] \cap \mathrm{Q}$ where $\Psi(\alpha,\eta,\mu,\gamma,\beta;\omega)$is defined in (7), then

$$\left(\alpha+\gamma\frac{\omega q'(\omega)}{q(\omega)}+\mu\big(q(\omega)\big)^2+\eta q(\omega)\right) \leq F_{\psi(\mathbb{C}^2\times\mathcal{L})[\Psi(\alpha,\eta,\mu,\gamma,\beta;\omega)]}$$

implying $F_{q(\mathcal{L})}q(\omega) \leq$

$$F_{\left(\frac{\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}}\right)^{\beta}(\mathcal{L})}\left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta},$$

i.e., $q(\omega) \prec_F \left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta}$ and q the fuzzy best subordinate.

Proof: Define $p(\omega) = \left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta}$, $\omega \in \mathcal{L}$,

considering $\nu(w) = \mu w^2 + \eta w + \alpha$ and $\phi(w) = \frac{\gamma}{w}$, it is clear that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$ and $\phi(w) \neq 0$, $w \in \mathbb{C}\setminus\{0\}$.

In this situation,

$$\frac{v'(q(\omega))}{\phi(q(\omega))} = \frac{q'(\omega)[\eta+2\mu q(\omega)]q(\omega)}{\gamma},$$

which implies $\mathrm{Re}\left(\frac{v'(q(\omega))}{\phi(q(\omega))}\right) = \mathrm{Re}\left(\frac{\eta}{\gamma}q(\omega)q'(\omega) + \frac{2\mu}{\gamma}q^2(\omega)q'(\omega)\right) > 0,$

$$for\ \alpha, \eta, \mu, \gamma \in \mathbb{C}, \gamma \neq 0.$$

We obtain $F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha + \gamma\frac{\omega q'(\omega)}{q(\omega)} + \mu\big(q(\omega)\big)^2 + \eta q(\omega)\right)$

$$\leq F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha + \gamma\frac{\omega p'(\omega)}{p(\omega)} + \mu(p(\omega))^2 + \eta p(\omega) + \alpha\right)$$

Using Lemma 1, we drive that

$$F_{q(\mathcal{L})}q(\omega) \leq F_{\left(\frac{\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}}\right)^{\beta}(\mathcal{L})}\left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta}$$

i.e. $q(\omega) \prec_F \left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1*f_2)(\omega)}\right)^{\beta}$ and q the fuzzy best subordinate.

Theorem 1and 2 are combined to form the sandwich theorem shown below.

Theorem 3: *Let* q_1 *and* q_2 *both analytic and univalent in* $\mathcal{L}$ *such that* $q_1(\omega) \neq 0$ *and* $q_2(\omega) \neq$ *for all* $\omega \in \mathcal{L}$, *with* $\frac{\omega q'_1(\omega)}{q_1(\omega)}$ *and* $\frac{\omega q'_2(\omega)}{q_2(\omega)}$ *as univalent starlike . Consider that* q_1 *satisfy* (6), q_2 *satisfy* (11). *Let s,* a *represent complex numbers with* $a \neq 0, 1, 2, \ldots$ *and* $\beta, \lambda > 0$. *If* $\Psi\ (\alpha, \eta, \mu, \gamma, \beta; \omega)$ *is given in (7) as univalent in* $\mathcal{L}$ *and*

$$\left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)}\right)^{\beta} \in \mathcal{H}[q(0), 1] \cap \mathrm{Q}$$

Then $F_{\psi(\mathbb{C}^2\times\mathcal{L})}\left(\alpha + \gamma\frac{\omega q_1'(\omega)}{q_1(\omega)} + \mu\big(q_1(\omega)\big)^2 + \eta q_1(\omega)\right) \leq$

$F_{\psi(\mathbb{C}^2\times\mathcal{L})}[\Psi(\alpha, \eta, \mu, \gamma, \beta; \omega)] \leq \left(\alpha + \gamma\frac{\omega q_1'(\omega)}{q_1(\omega)} + \mu\big(q_1(\omega)\big)^2 + \eta q_1(\omega)\right)$

Implies $\mathrm{F}_{q_1(\mathcal{L})} q_1(\omega) \leq F_{\left(\frac{\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}}\right)^{\beta}(\mathcal{L})} \left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)}\right)^{\beta} \leq$

$\mathrm{F}_{q_2(\mathcal{L})} q_2(\omega)$.

i.e., $$q_1(\omega) \prec_F \left(\frac{\omega\left(\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)\right)'}{\mathcal{J}^{s,\alpha,\lambda}_{(\lambda_p),(\mu_q),\mathrm{b}}(f_1 * f_2)\ (\omega)}\right)^{\beta} \prec_F q_2(\omega)$$

and q_1 and q_2 are also the fuzzy best subordinate and the fuzzy best dominant, respectfully.

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Received January, 2022

Revised March, 2022