Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 4, pp. 747–759
DOI : 10.47974/JIM-1511

First integral blow-up solutions to complex-valued KdV equations

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Abstract

We apply Lie theory to determine the one-parameter point transformations which leave complex-valued Korteweg-de Vries equations invariant. The conserved vectors of the systems are constructed. We provide travelling wave reductions that lead to third-order ordinary differential equations. These equations are highly nonlinear to solve directly and, therefore, we establish their first integrals. The latter is of second-order and facilitates the analysis of the complex-valued system, as well as some interesting blow-up solutions.

Subject Classification: *[2020] 35Q7F, 35A15, 83C20, 35L65.*

Keywords: *Lie symmetries, KdV equations, Conservation laws, Complex equations.*

1. Introduction

The Korteweg-de Vries (KdV) equation needs no introduction. It has been obtained as a fundamental model for wave spread in number of different physical conditions. Research on pole dynamics of KdV equations and work related to it, started with remarks of Kruskal in the 1970s [11], and subsequently, extensive work was then carried out at a later stage [12, 3, 4, 16, 5]. We will concentrate on the research of generalised KdV equations under a complex-valued sense. Such equations, in real-valued terms, have been actively studied in terms of its solitons and solutions. Recently, many studies have paid great attention to construct the complex solutions by different methods [15, 8]. It is predicted that the complex KdV equation, shares the same properties of the real-valued KdV class of

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equations [2], i.e. the integrable properties, the infinitely many symmetries and infinitely many conservation laws [13]. More properties include, the inverse scattering transformation [1], Hirota bilinear method [7] and Darboux transformation [17]. In addition, the resulting coupled system, that has been decomposed into real and imaginary parts, is a special case of the general coupled KdV systems, connected to the two layer fluid dynamical systems and two-component Bose-Eubstein condensates [14]. Such equations are important in dynamics and physics.

Our interest is to find out about the symmetry, conservation law and solution properties of the complex-valued KdV class of equations. Such properties have far reaching consequences [9, 10, 19]. Some recent papers have considered important conservation laws, for example, see [20, 18, 6].

First, we consider the complex-valued KdV-Burgers' equation

$$\phi_t = 2\phi\phi_x + \phi_{xxx} + \alpha\phi_{xx}, \tag{1}$$

where $\phi = \phi(x,t)$. For $\alpha = 0$, we get the complex-valued KdV equation. The complex equation can be decomposed into real and imaginary parts if we suppose that

$$\phi(x,t) = u(x,t) + iv(x,t),$$

is a complex valued dependent variable. Then Eq. (1) is equivalent to the coupled system

$$\begin{aligned} -u_t + 2uu_x - 2vv_x + u_{xxx} - \alpha u_{xx} &= 0, \\ -v_t + 2uv_x + 2vu_x + v_{xxx} - \alpha v_{xx} &= 0. \end{aligned} \tag{2}$$

Similarly, the complex-valued modified KdV equation (mKdV) equation is

$$\phi_t = \phi_{xxx} - 6|\phi|^2\phi_x - \alpha\phi_{xx}, \tag{3}$$

and decomposing into the system of equations

$$\begin{aligned} -u_t + u_{xxx} - 6(u^2+v^2)u_x - \alpha u_{xx} &= 0, \\ -v_t + v_{xxx} - 6(u^2+v^2)v_x - \alpha v_{xx} &= 0. \end{aligned} \tag{4}$$

The plan of the paper is as follows. In Section 2, we briefly describe the mathematical theory necessary for our investigation. Section 3 and 4 deal with the solutions of the complex-valued KdV Burgers' and mKdV systems. Section 5 is the conclusion.

2. Theory

The procedure for determining point symmetries for an arbitrary system of equations is well-known [21]. Consider q unknown functions u^α which depend on p independent variables x^i, i.e. $u=(u^1,\ldots,u^q)$, $x=(x^1,\ldots,x^p)$, with indices $\alpha=1,\ldots,q$ and $i=1,\ldots,p$. Let

$$G_\alpha(x,u^{(k)})=0, \tag{5}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. We consider the following symmetry

$$X=\xi^i\partial_{x^i}+\eta^\alpha\partial_{u^\alpha}, \tag{6}$$

given by

$$X[G_\alpha(x,u^{(k)})]=0,\ \text{when}\ G_\alpha(x,u^{(k)})=0, \tag{7}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation. A current $T=(T^1,\ldots,T^n)$ is conserved if it satisfies

$$D_iT^i=0, \tag{8}$$

along the solutions of the given equation. Eq. (8) is called a local conservation law. The multipliers Λ^α of (5) satisfies the relation [22]

$$D_iT^i=\Lambda^\alpha G^\alpha, \tag{9}$$

for all functions u^α and the overdetermined equations for Λ^α are

$$\frac{\delta}{\delta u^\alpha}[\Lambda^\alpha G^\alpha]=0, \tag{10}$$

that enables us to calculate conserved vectors.

3. The complex-valued KdV Burgers' equation

Suppose that the Lie point symmetry of (2) is of the general form

$$X=\xi^1(x,t,u,v)\partial_x+\xi^2(x,t,u,v)\partial_t+\eta^1(x,t,u,v)\partial_u+\eta^2(x,t,u,v)\partial_v,$$

then solving (7) gives the Lie point symmetries of the system of equations. We note that there are two cases of interest: $\alpha\neq0$ and $\alpha=0$.

We obtain that the symmetries of (2), when $\alpha\neq0$ are

$$X_1^1 = \partial_t, X_2^1 = \partial_x, X_3^1 = t\partial_x - \frac{1}{2}\partial_u. \tag{11}$$

Similarly for $\alpha = 0$, that is, the complex-valued KdV equation, we solve the determining equations to find that the symmetries of (2) are

$$X_1^2 = t\partial_x - \frac{1}{2}\partial_u, X_2^2 = \partial_t, X_3^2 = \partial_x, X_4^2 = x\partial_x + 3t\partial_t - 2u\partial_u - 2v\partial_v, \tag{12}$$

with Lie brackets presented in Table 1.

Table 1

Lie brackets of (2), Left: $\alpha \neq 0$, Right: $\alpha = 0$

[,]	X_1^1	X_2^1	X_3^1
X_1^1	0	0	$-X_2^1$
X_2^1	0	0	0
X_3^1	X_2^1	0	0

[,]	X_1^2	X_2^2	X_3^2	X_4^2
X_1^2	0	X_3^2	0	$2X_1^2$
X_2^2	$-X_3^2$	0	0	$-3X_2^2$
X_3^2	0	0	0	$-X_3^2$
X_4^2	$-2X_1^2$	$3X_2^2$	X_3^2	0

Next we conduct a detailed study of the conservation laws of this system of equations. Suppose that the variable $\overline{\phi}$ denotes the conjugate of ϕ. Each conservation law is presented in the variables u, v and the original variable ϕ.

Therefore, if (T^t, T^x) is a conserved vector of a conservation law, then

$$D_t T^t + D_x T^x = 0,$$

along the solutions of the differential equation, where T^t is the conserved density and T^x is the conserved flux. This case, for $\alpha \neq 0$ admits two conservation laws, viz.

$$T_1^t = u = \frac{1}{2}(\phi + \overline{\phi}),$$

$$T_1^x = \alpha u_x - u^2 + v^2 - u_{xx} = \frac{\alpha}{2}(\phi_x + \overline{\phi}_x) - \frac{1}{2}(\phi_{xx} + \overline{\phi}_{xx}) - \phi^2,$$

and

$$T_2^t = v = \sqrt{\phi\overline{\phi} - (\frac{1}{2}(\phi + \overline{\phi})^2)^2},$$

$$T_2^x = \alpha v_x - 2uv - v_{xx} = \alpha\sqrt{\phi_x\bar{\phi}_x - \frac{1}{2}(\phi + \bar{\phi}_x)} - \sqrt{\phi_{xx}\bar{\phi}_{xx} - \frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx})} - \sqrt{(\bar{\phi}^2 - \phi\bar{\phi})^2}.$$

The conservation laws, for $\alpha = 0$ are the following six relations:

$$T_1^t = u = \frac{1}{2}(\phi + \bar{\phi}),$$

$$T_1^x = -u^2 + v^2 - u_{xx} = \frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx}) - \phi^2,$$

and the conservation law

$$T_2^t = v = \sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi + \bar{\phi})^2)^2},$$

$$T_2^x = -2uv - v_{xx} = -\frac{1}{2}(\phi + \bar{\phi}_x) - \sqrt{\phi_{xx}\bar{\phi}_{xx} - \frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx})} - \sqrt{(\bar{\phi}^2 - \phi\bar{\phi})^2}.$$

$$T_3^t = uv = \frac{1}{2}\sqrt{(\bar{\phi} - \phi\bar{\phi})^2},$$

$$\begin{aligned} T_3^x &= -2u^2v + \frac{2}{3}v^3 - uv_{xx} + v_x u_x - vu_{xx} \\ &= -(\phi + \bar{\phi})\sqrt{\phi\bar{\phi} - \left(\frac{1}{2}(\phi + \bar{\phi})\right)^2} + \frac{2}{3}\left(\sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi + \bar{\phi}))^2}\right)^3 \\ &\quad - \frac{1}{2}(\phi + \bar{\phi})\sqrt{\phi_{xx}\bar{\phi}_{xx} - \left(\frac{1}{2}(\phi + \bar{\phi})\right)^2} + \frac{1}{2}\sqrt{(\bar{\phi} - \phi\bar{\phi})^2} \\ &\quad - \frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx})^2\sqrt{\phi\bar{\phi} - \left(\frac{1}{2}(\phi + \bar{\phi})\right)^2}. \end{aligned}$$

$$T_4^t = \frac{1}{2}(u^2 - v^2) = \frac{1}{2}\left[\left(\frac{1}{2}(\phi + \bar{\phi})\right)^2 - \sqrt{\phi\bar{\phi} - (\frac{1}{2}(\phi\bar{\phi}))^2}\right],$$

$$\begin{aligned} T_4^x &= \frac{-2}{3}u^3 + 2v^2u - uu_{xx} + \frac{1}{2}u_x^2 + vv_{xx} - \frac{1}{2}v_x^2 \\ &= \frac{-2}{3}\left(\frac{1}{2}(\phi + \bar{\phi})\right)^3 + (\phi + \bar{\phi})(\phi\bar{\phi} - \left(\frac{1}{2}(\phi + \bar{\phi})\right)^2 - \frac{1}{2}(\phi + \bar{\phi})\left(\frac{1}{2}(\phi_{xx} + \bar{\phi}_{xx})\right) \end{aligned}$$

$$+\frac{1}{2}\left(\frac{1}{2}(\phi_x+\bar{\phi}_x)\right)^2+\left(\sqrt{\phi\bar{\phi}-(\frac{1}{2}(\phi+\bar{\phi}))^2}\right)\left(\sqrt{\phi_{xx}\bar{\phi}_{xx}-(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx}))^2}\right)$$

$$-\frac{1}{2}\left(\sqrt{\phi_x+\bar{\phi}_x-\left(\frac{1}{2}(\phi_x+\bar{\phi}_x)\right)^2}\right)^2,$$

$$T_5^t=utv+\frac{1}{2}vx=\frac{t}{2}\sqrt{(\phi-\phi\bar{\phi})^2}+\frac{1}{2}\sqrt{\phi\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\,x,$$

$$\begin{aligned}T_5^x&=-2u^2tv+\frac{2}{3}v^3t-utv_{xx}+tu_xv_x-u_{xx}tv-uxt+\frac{1}{2}v_x-\frac{1}{2}xv_{xx}\\&=-2t\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}+\frac{2t}{3}\left(\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\right)^2\\&-t\left(\frac{1}{2}(\phi+\bar{\phi})\right)\sqrt{\phi_{xx}+\bar{\phi}_{xx}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})\right)}+t\frac{1}{2}\sqrt{(\bar{\phi}_x-\phi_x\bar{\phi}_x)^2}\\&-t\left(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})\right)\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}-t\frac{1}{2}\sqrt{(\bar{\phi}-\phi\bar{\phi})^2}+\\&\frac{1}{2}\sqrt{\phi_x+\bar{\phi}_x-\left(\frac{1}{2}(\phi_x+\bar{\phi}_x)\right)^2}-\frac{x}{2}\sqrt{\phi_{xx}+\bar{\phi}_{xx}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})\right)^2},\end{aligned}$$

and finally,

$$T_6^t=t(u^2-v^2)+ux=t\left(\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2-\left(\phi\bar{\phi}-\frac{1}{2}(\phi+\bar{\phi})\right)^2\right)+\frac{1}{2}(\phi+\bar{\phi})x,$$

$$\begin{aligned}T_6^x&=\frac{-4}{3}u^3t+4^2tu-xu^2-2utu_{xx}+ux^2t+xv^2+2vtv_{xx}-vtx^2+u_x-xu_{xx}\\&=\frac{-4t}{3}\left(\frac{1}{2}(\phi+\bar{\phi})\right)^3+4t\frac{1}{2}(\phi+\bar{\phi})\left(\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\right)^2\\&-2t\left(\frac{1}{2}(\phi+\bar{\phi})\right)\left(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})\right)+x^2t\left(\frac{1}{2}(\phi+\bar{\phi})\right)\\&+2t\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}\sqrt{\phi_{xx}+\bar{\phi}_{xx}-\left(\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})\right)^2}\end{aligned}\tag{13}$$

$$-t\sqrt{\phi+\bar{\phi}-\frac{1}{2}(\phi+\bar{\phi})^2}\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2+\frac{1}{2}(\phi_x+\bar{\phi}_x)-x\frac{1}{2}(\phi_{xx}+\bar{\phi}_{xx})$$
$$-x\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2+x\sqrt{\phi+\bar{\phi}-\left(\frac{1}{2}(\phi+\bar{\phi})\right)^2}.$$

3.1 *Symmetry Reductions and Solutions*

In this section, we use the method of Lie invariants to firstly calculate invariant functions to reduce our system of equations. We find that the reduced equations are a system of third-order ordinary differential equations which cannot be solved further using symmetries or directly via software such as Maple or Mathematica. Instead, we find first integrals of the reduced system with the aim of solving via these expressions.

Suppose we consider the linear combination of symmetries $X_1^1+aX_2^1$, a constant, which gives the invariant functions

$$\begin{aligned}\zeta&=-at+x,\\ v(x,t)&=F_1(\zeta),\\ u(x,t)&=F_2(\zeta).\end{aligned}\tag{14}$$

Hence the reduced system of equations corresponding to (2), are

$$\begin{aligned}&-2F_1(\zeta)F_1'(\zeta)+(a+2F_2(\zeta))F_2'(\zeta)-\alpha F_2''(\zeta)+F_2'''(\zeta)=0,\\ &(a+2F_2(\zeta))F_1'(\zeta)+2F_1(\zeta)F_2'(\zeta)-\alpha F_1''(\zeta)+F_1'''(\zeta)=0.\end{aligned}\tag{15}$$

The above system is challenging to solve, so we find that the first integrals of (15) for $\alpha\neq 0$ are

$$I_1=-\alpha F_1'(\zeta)+aF_1(\zeta)+2F_1(\zeta)F_2(\zeta)+F_1''(\zeta),\tag{16}$$
$$I_2=-\alpha F_2'(\zeta)+aF_2(\zeta)-F_1(\zeta)^2+F_2(\zeta)^2+F_2''(\zeta).\tag{17}$$

Since I_1 and I_2 are of second-order, we attempt to use the first integrals to solve for the unknown functions.

In terms of a solution, we find that one exists in terms of Abel's ordinary differential equation of the second kind, viz.

$$I_2=-\alpha b(\gamma)+a\gamma+\gamma^2+b(\gamma)b'(\gamma).\tag{18}$$

for $F_1(\zeta)=0$ and for $F_2'(\zeta)=b(\gamma)$, $F_2(\zeta)=\gamma$.

For $\alpha = 0$, one may consider the symmetry combination $X_2^2 + aX_3^2$, that gives the invariant functions (14) and the same reduced system for (15).

The first integrals of (15) here are

$$I_1 = -\frac{1}{2}F_1'(\zeta)^2 + \frac{1}{2}F_2'(\zeta)^2 + F_1(\zeta)F_1''(\zeta) - F_2(\zeta)F_2''(\zeta)\frac{a}{2}F_1(\zeta)^2$$
$$+2F_2(\zeta)F_1(\zeta)^2 - \frac{2}{3}F_2(\zeta)^3 - \frac{a}{2}F_2(\zeta)^2,$$
$$I_2 = -F_2'(\zeta)F_1'(\zeta) + F_1''(\zeta)F_2(\zeta) + F_1(\zeta)F_2''(\zeta) - \frac{2}{3}F_1(\zeta)^3$$
$$+aF_1(\zeta)F_2(\zeta) + 2F_2(\zeta)^2F_1(\zeta),$$
$$I_3 = cF_1(\zeta) + 2F_1(\zeta)F_2(\zeta) + F_1''(\zeta),$$
$$I_4 = aF_2(\zeta) - F_1(\zeta)^2 + F_2(\zeta)^2 + F_2''(\zeta).$$

A special solution exists, where for $I_1 = I_4 = 0$, we have $F_1(\zeta) = 0$, and

$$F_2(\zeta) = -\frac{3a}{2}\left(\tan\left(\frac{C_1}{2}\sqrt{a} - \frac{\zeta}{2}\sqrt{a}\right)\right)^2 - \frac{3a}{2}. \tag{19}$$

Thus, $v(x,t) = 0$ and

$$\phi(x,t) = u(x,t) = -\frac{3a}{2}\left(\tan\left(\frac{C_1}{2}\sqrt{a} - \frac{-at+x}{2}\sqrt{a}\right)\right)^2 - \frac{3a}{2}, \tag{20}$$

or

$$\phi(x,t) = -\frac{3a}{2}\left(\sec\left(\frac{C_1}{2}\sqrt{a} - \frac{-at+x}{2}\sqrt{a}\right)\right)^2.$$

We plot the above solutions in Figure 1.

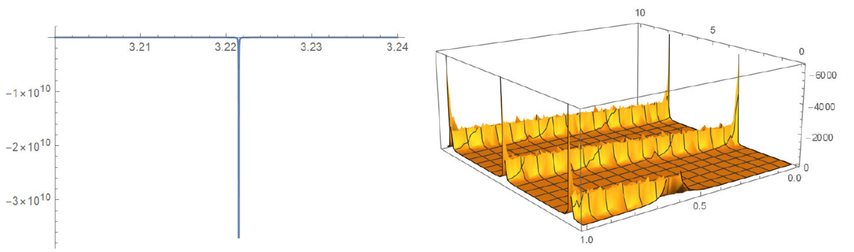

Figure 1

The evolution of (19) and 3D plot of (20) where the free parameters are chosen to be $C_1 = 1, a = 2$

4. The complex-valued mKdV equation

For this system as well, we note that there are two cases of interest: $\alpha \neq 0$ and $\alpha = 0$. The symmetries of (4), for $\alpha \neq 0$ are

$$X_1^3 = \partial_x, X_2^3 = \partial_t, X_3^3 = -v\partial_u + u\partial_v, \tag{21}$$

the commutators all vanish and for this case $\alpha \neq 0$, we find no multipliers and conserved vectors.

For $\alpha = 0$, in this case, the Lie point symmetries of (4) are

$$X_1^4 = v\partial_u - u\partial_t, X_2^4 = \partial_x, X_3^4 = \partial_t, X_4^4 = \frac{x}{3}\partial_x + t\partial_t - \frac{u}{3}\partial_u - \frac{v}{3}\partial_t, \tag{22}$$

with Lie brackets in Table 2.

Table 2

Lie brackets of (4) with $\alpha = 0$

$[,]$	X_1^4	X_2^4	X_3^4	X_4^4
X_1^4	0	0	0	0
X_2^4	0	0	0	$-\frac{X_2^4}{3}$
X_3^4	0	0	0	$-X_3^4$
X_4^4	0	$\frac{X_2^4}{3}$	X_3^4	0

The conservation laws are calculated to be only one for $\alpha = 0$.

$$T^t = \frac{1}{2}(u^2 + v^2),$$

$$T^x = \frac{3}{2}u^4 + 3u^2v^2 + \frac{3}{2}v^4 - uu_{xx} + \frac{1}{2}u_x^2 - vv_{xx} + \frac{1}{2}v_x^2,$$

and in the variable $\phi(x, t)$, the conservation law is

$$T^t = \frac{1}{2}|\phi|^2,$$

$$T^x = \frac{3}{2}(|\phi|^2)^2 + \frac{1}{2}\phi_x\bar{\phi}_x + \frac{1}{2}(\phi\bar{\phi}_{xx} + \bar{\phi}\phi_{xx}).z$$

4.1 *Symmetry Reductions and Solutions*

As in the above section, we utilize the symmetries $X_2^3 + aX_1^3$, to find the transformation (14) so that the system (4), with $\alpha \neq 0$ reduces to

$$\begin{aligned}(a-6F_1(\zeta)^2-6F_2(\zeta)^2)F_2'(\zeta)-\alpha F_2''(\zeta)+F_2'''(\zeta)&=0,\\(a-6F_1(\zeta)^2-6F_2(\zeta)^2)F_1'(\zeta)-\alpha F_1''(\zeta)+F_1'''(\zeta)&=0.\end{aligned} \tag{23}$$

The first integral of the system (23) is

$$I_1 = e^{-\alpha\zeta}(F_1'(\zeta)F_2''(\zeta)-F_1''(\zeta)F_2'(\zeta)), \tag{24}$$

which cannot be used to solve further.

The symmetry $X_3^4 + aX_2^4$, leads to the reduced equations (23), for $\alpha = 0$ where (14) is used once again. The system of reduced equations in this case, possesses the first integrals

$$I_1 = -F_1'(\zeta)F_2''(\zeta)+F_1''(\zeta)F_2'(\zeta) \tag{25}$$

$$\begin{aligned}I_2 &= F_1''(\zeta)F_1(\zeta)+F_2(\zeta)F_2''(\zeta)-\frac{1}{2}F_1'(\zeta)^2-\frac{1}{2}F_2'(\zeta)^2-\frac{3}{2}F_1(\zeta)^4\\&\quad-3F_2(\zeta)^2F_1(\zeta)^2+\frac{1}{2}aF_1(\zeta)^2-\frac{3}{2}F_2(\zeta)^4+\frac{1}{2}aF_2(\zeta)^2.\end{aligned} \tag{26}$$

For $I_1 = 0$, we have the solution $F_1(\zeta) = 0$ and the implicit solution

$$\pm\int^{F_2(\zeta)}\frac{1}{\sqrt{\gamma^4-\gamma^2 a+C_1\gamma-2I_2}}d\gamma-\zeta-C_2=0.$$

For $C_1 = I_2 = 0, a = 1$, the integral is evaluated to be

$$\pm\arctan\left(\frac{1}{\sqrt{F_2(\zeta)^2-1}}\right),$$

hence the explicit solution is

$$F_2(\zeta) = \pm\frac{1}{\tan(\zeta+C_2)}\sqrt{(\tan(\zeta+C_2))^2+1}. \tag{27}$$

Thus we have that $v(x,t)$ is zero and we may recover that

$$\phi(x,t) = u(x,t) = \pm\frac{1}{\tan(-t+x+C_2)}\sqrt{(\tan(-t+x+C_2))^2+1}. \tag{28}$$

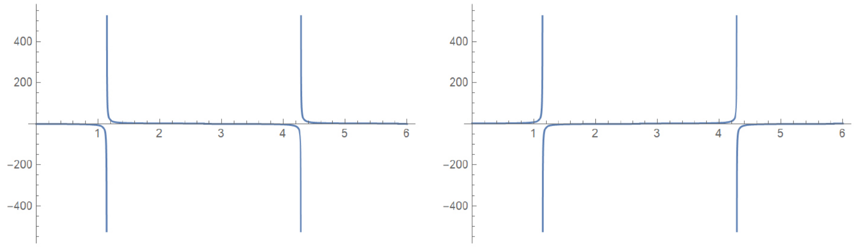

Figure 2

The function (27) with $C_2 = 1$, taking the positive and negative sign, respectively

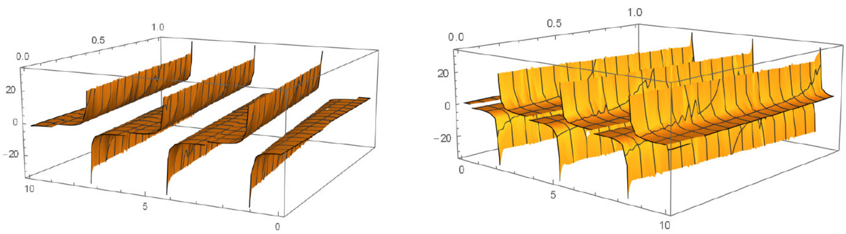

Figure 3

The function (28) with $C_2 = 1$, taking the positive and negative sign, respectively

This solution is easily simplified to be written in terms of $\csc(-t + x + C_2)$, so that one can see the singularity for $-t + x + C_2 = 0$. We plot the above solutions in Figure 2 and Figure 3.

5. Conclusion

The study of complex-valued KdV equations is far more involved than real-valued KdV models, but it is not surprising that complex-valued KdV equations share properties with real-valued KdV equations i.e., integrable properties, symmetries and conservation laws. In this study, we concentrated on travelling-wave reductions, but obtained nonlinear equations that were difficult to solve by quadrature. To this end, we investigated the conserved quantities of the reduced equations, which were of second-order and easier to deal with and found some solutions. On particular, the solutions were real and involved trigonometric functions. In one case, the solution depended on Abel's equation of the second kind. Our solutions involve trigonometric functions, but not the well-known solutions of the KdV equations that contain hyperbolic trigonometric functions. The latter gives rise to solitonic solutions, where our solutions are blow up solutions. In future work, we will extend this analysis to other higher-order partial differential equations.

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Received August, 2021

Revised February, 2022