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Approximation of functions in the $L_{P.w}$ ($\mathbb{X}$)-space and application

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Abstract

The purpose of this research is to study the approximation of functions in the space $L_{p.w}$ which is an expansion of the Lebesgue space L_p using the Fejer operator in terms of the average modulus of the second-order with the application of two of the theorems.

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1. Introduction

In the last decade of the 21 century, the approximation of functions was studied by many researchers, for example, Lupus (see [1]) and by Phillips (see [2]). Generalizations of the Bernstein polynomials based on the q-integers attracted a lot of interest and were studied widely by several authors (see [3]).

Recently, the approximation of unbounded functions in Locally-Global Weighted $L_{P.\delta.w}(X)$-Spaces and in Multiplier spaces $L_{P.\emptyset_n}(B).B=[0.2\pi]$, has been studied by some researchers, especially Dr. Sahib Al-Saadi (see [4] and [5]).

In this research, we will study one of the ways to approximate unbounded functions, but at the beginning, we will put the most important definitions.

Let $L_p(\mathbb{X})$ the space of all bounded measurable functions defined on $\mathbb{X}=[a.b]$ with the norm, $\|f\|_{L_p}=\|f\|_p=(\int_{\mathbb{X}} |f(x)|^p)^{\frac{1}{p}}<\infty \quad .1 \le p<\infty\cdot$

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Then for the real-valued function f defined on $\mathbb{X} = [-\pi.\pi]$, if there is a continuous function $w(x)$ such that, $\left(\int_{\mathbb{X}} (fw)(x)\right) < \infty$, then f is called weight integral function and $w(x)$ is called a weight for integral. The weight integral norm can be defined as follows:

$\|f\|_{L_{p.w}} = \left\{(\int_{\mathbb{X}} |(fw)(x)|^p \, dx)^{\frac{1}{p}} : x \in \mathbb{X}\right\}$, where $w(x)$ is the weight for integral with $\|f\|_{L_{p.w}} = \|f\|_{p.w}$.

Nots that for $f \in \boldsymbol{L_p}$ then $f \in L_{p.w}$ by taking $w(x) = 1$ thus $\boldsymbol{L_p} \subseteq \boldsymbol{L_{p.w}}$ But the converse is not true in general. From above.

Let $L_{p.w}(\mathbb{X}) = \{f : \|f\|_{p.w} < \infty\}$, $\mathbb{X} = [-\pi.\pi]$. be the all unbounded continuous functions f, $1 \le p < \infty$, which are associated with the above norm. Where $(fw)(x)$ is the real continuous functions on $\mathbb{X}$ and

$$\|(\boldsymbol{f}w)\|_{\boldsymbol{p}} = \|\boldsymbol{f}\|_{\boldsymbol{p}.w} \qquad \cdots (\mathbf{1})$$

2. Preliminary:

The following mathematical concepts are needed:

Definition 2.1: For $f \in L_{P.w}(\mathbb{X}). \mathbb{X} = [-\pi.\pi]. 1 \le P < \infty$ the Fourier series of the function f at any point x is given by, $f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, where:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)\cos kt\,dt \;\; and \;\; b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)\sin kt \; dt .$$

Definition 2.2: For $2\pi -$ periodic function $f \in L_{P.w}(\mathbb{X})$ let us define the n-th partial sum S_n of Fourier series as:

$$\mathbb{S}_{\boldsymbol{n}}(\boldsymbol{f}.\boldsymbol{x}) = \frac{\boldsymbol{a_0}}{\mathbf{2}} + \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{n}}(\boldsymbol{a_k} \cos \boldsymbol{kx} + \boldsymbol{b_k} \sin \boldsymbol{kx}) \qquad (\mathbf{2})$$

Proposition 2.3: Let $f \in L_{P.w}(\mathbb{X}) . \mathbb{X} = [-\pi.\pi]$. Then:

$$\mathbb{S}_{\boldsymbol{n}}(\boldsymbol{f}.\ \boldsymbol{x}) = \frac{\mathbf{1}}{\boldsymbol{\pi}}\int_{-\boldsymbol{\pi}}^{\boldsymbol{\pi}}(\boldsymbol{fw})(\boldsymbol{t})\,\frac{\sin(2\boldsymbol{n}+1)\frac{\boldsymbol{t}-\boldsymbol{x}}{2}}{2\sin\frac{\boldsymbol{t}-\boldsymbol{x}}{2}}\,\boldsymbol{dt} \qquad (\mathbf{3})$$

Proof: $\mathbb{S}_n(f.x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ where,
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)dt$. Then

$$\mathbb{S}_n(f.x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(fw)(t)dt$$

$$+\sum_{k=1}^{n}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)\cos kt\;\cos kx\;dt\; + \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)\sin kt\;\sin kx\;dt\right]$$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (fw)(t) \, [1 + 2 \sum_{k=1}^{n} (\cos kx \cdot \cos kt + \sin kx \cdot \sin kt)] \, dt =$
$\frac{1}{\pi} \int_{-\pi}^{\pi} (fw)(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)\right] \, dt$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} (fw)(t) \; \mathcal{D}_n(t-x) \, dt$, where $\mathcal{D}_n(t-x) = \frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)$

Called Dirichlete kernel for $(t-x)$. **Now** to prove $\mathcal{D}_n(t) = \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}$

$\sin(2n+1)\frac{t}{2} = \sin\frac{t}{2} + \sin\frac{3t}{2} - \sin\frac{t}{2} + \sin\frac{5t}{2} - \sin\frac{3t}{2} + \cdots + \sin\frac{2n+1}{2}t$
$-\sin(2\mathrm{n}-1)\frac{t}{2}$

$= \sin\frac{t}{2} + \sin\frac{t}{2} \cdot 2\cos t + 2\sin\frac{t}{2} \cdot \cos t + 2\sin\frac{t}{2} \cdot \cos 3t + \cdots +$
$2\sin\frac{t}{2}\cos nt = \sin\frac{t}{2}[1 + 2\sum_{k=1}^{n}\cos kt]$.Then $\sin\ (2n+1)\frac{t}{2} = \sin\ \frac{t}{2}\ [1 +$
$2\sum_{k=1}^{n}\ \cos kt\ ]$.

So that $\frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}} = \left[[1 + 2\sum_{k=1}^{n}\cos kt]\right]$.

Thus $\frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos kt.$

Therefore $\mathbb{S}_n(f.x) = \frac{1}{\pi}\int_{-\pi}^{\pi} fw(t) \frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}} \ dt \ *$

Definition 2.4: For $f \in L_{P.W}(\mathbb{X})$, let $\mathbb{S}_n(f.x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$, be the partial sum of Fourier series then we will define the Fejer operator as follows:

$$\boldsymbol{\sigma_m(f.x) = \frac{\mathbb{S}_0(f.x)+\mathbb{S}_1(f.x)+\cdots\mathbb{S}_m(f.x)}{m+1}} \tag{4}$$

Proposition 2.5: For$f \in L_{p.w}(\mathbb{X})$. Then

$$\boldsymbol{\sigma_m(f.x) = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t-x)\,F_m(t)dt} \tag{5}$$

Where $F_m(t) = \left[\frac{\sin(\mathrm{m}\frac{t}{2})}{\sin\frac{t}{2}}\right]^2$

Proof: Since $\sigma_m(f.x) = \frac{1}{m+1}[\mathbb{S}_0(f.x) + \mathbb{S}_1(f.x) + \cdots \mathbb{S}_m(f.x)]$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(fw)(t)\frac{[\mathcal{D}_0(t-x)+\mathcal{D}_1(t-x)+\cdots\mathcal{D}_m(t-x)]}{m+1} = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)F_m(t-x)\,dt,$
where $F_m(t) = \frac{1}{m+1}[\mathcal{D}_0(t) + \mathcal{D}_1(t) + \cdots \mathcal{D}_m(t)]$. And since$\mathcal{D}_m(t) = \left[\frac{\sin\left(\frac{2m+1}{2}\right)t}{2\sin\frac{t}{2}}\right]$, we get

$$F_m(t) = \frac{\sin\frac{t}{2}+\sin\frac{3t}{2}+\cdots+\sin(\frac{2m-1}{2})t}{2m\sin\frac{t}{2}} \cdot \frac{2m\sin\frac{t}{2}}{2m\sin\frac{t}{2}}$$
$$=\frac{(1-\cos t)+(\cos t-\cos 2t)+\cdots+(\cos(m-1)t-\cos mt)}{4m\sin^2\frac{t}{2}}$$

$$F_m(t) = \frac{1-\cos m\,t}{4m\sin^2\frac{t}{2}} = \frac{2\sin^2\frac{mt}{2}}{4m\sin^2\frac{t}{2}} = \frac{\sin^2\frac{mt}{2}}{2m\sin^2\frac{t}{2}}.$$

Thus

$$\sigma_m(f.x) = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)F_m(t-x)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}(fw)(t)\frac{\sin^2(\frac{m(t-x)}{2})}{2m\sin^2(\frac{t-x}{2})}\,dt \ *$$

Definition 2.6: For $f \in L_{P.w}(\mathbb{X}).\mathbb{X} = [-\pi.\pi]$, let

$\Delta_h^k(f.x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x - \frac{k\,h}{2} + i\,h\right): x \mp \frac{k\,h}{2} \in$
X the difference of f order k

$\Omega^k\,(f;\,\delta)_{P.w} = Sup_{0<P\le\delta}\left\|\Delta_h^k(f.x)\right\|_{P.w}.\delta \ge 0$. Which is said the usually weighted modulus of smoothness of f and $\mathcal{T}^k(f;\delta)_{p.w} = \|\Omega^k(f.\delta)\|_{p.w}$ is the weighted averaged modulus of smoothness of f order .

Definition 2.7 [6]: Let $f \in L_{P.w}(\mathbb{X})$. Then the degree of best approximation of function f concerning trigonometric polynomials $T_n \in \Pi_n$ where Π_n the set of all trigonometric polynomials is given by:

$$\boldsymbol{E_n(f)_{P.w} = inf_{T_n\in\Pi_n}\{\|f - T_n\|_{P.w}\}} \tag{6}$$

3. The Main Results

Before we state our main results, we need the following lemma and notes

Lemma 3.1: *For* $f \in L_{p.w}(\mathbb{X})$. *Then*

$$\boldsymbol{\Omega^k(f;\delta)_{p.w} \le \mathcal{T}^k(f;\,\delta)_{p.w}} \tag{7}$$

Proof: Let $f \in L_{p.w}(\mathbb{X})$ then $(fw)(x)$ is an integral function on $\mathbb{X}$ this means that $\|(fw)\|_p = \{\int_{\mathbb{X}} |(fw)(x)|^p\,dx\}^{\frac{1}{p}}$. **Now** By **[6]** and **(1)** we have $\Omega^k\big((fw)(x);\ \delta\big)_p = \Omega^k(f;\,\delta)_{p.w} \le \mathcal{T}^k\big((fw)(x);\delta\big)_p = \mathcal{T}^k(f;\delta)_{p.w}$. Then $\Omega^k(f;\delta)_{p.w} \le \mathcal{T}^k(f;\delta)_{p.w}$ $*$

Lemma 3.2: For $f \in L_{P.w}(\mathbb{X})$. Where C is a constant, we have.

$$\boldsymbol{\|\sigma_m(f)\|_{p.w} \le C\|f\|_{p.w}} \tag{8}$$

Proof: $\|\sigma_m(f)\|_{p.w} = sup\{\int_{-\pi}^{\pi}[\frac{1}{2m\pi}\int_{-\pi}^{\pi}(fw)(t-x)F_m(t)dt]^p\, dx\}^{\frac{1}{p}} \leq$ $\sup\{\int_{-\pi}^{\pi}[\frac{1}{2m\pi}\int_{-\pi}^{\pi}|(fw)(t-x)|^p dx]^{\frac{1}{p}} \cdot\ F_m(t)\, dt\} \leq \frac{1}{2m\pi}\ sup\{[\int_{-\pi}^{\pi}|(fw)(t-x)|^p\, dx]^{\frac{1}{p}} \cdot \int_{-\pi}^{\pi} F_m(t)dt\ \}$

$\leq sup\{\int_{-\pi}^{\pi}|(fw)(t-x)|^p dx\}^{\frac{1}{p}} \cdot \frac{1}{2m\pi}\int_{-\pi}^{\pi} F_m\,(t)dt \leq \|f\|_{p.w}C \cdot$

Then $\|\sigma_m(f)\|_{p.w} \leq C\|f\|_{p.w}$ $*$

In this section, we will get the approximation for $f \in L_{p.w}(\ \mathbb{X}\ )$ by using $\sigma_m(\,f.x\,)$

Theorem 3.4: For $f \in L_{p.w}(\mathbb{X})$. we get that

$$E_n(f)_{p.w} \leq C(k)\Omega^k\left(f;\tfrac{1}{m}\right)_{p.w} \tag{9}$$

Proof: Let Π_m be the set of all trigonometric polynomials. $f \in L_{p.w}(\mathbb{X})$.

$\|fw\|_p = \left[\int_{\mathbb{X}} |fw(x)|^p dx\right]^{\frac{1}{p}}$, **exist** Then by (Sendov Theorem, see **[7]**) there is a polynomial $T_m^* \in \Pi_m$ such that $\|(fw) - T_m^*\|_p \leq C_1(k)\Omega^k\left((fw);\frac{1}{m}\right)_p$. Then by (1), we get $\|f - T_m^*\|_{\,p.w} \leq C_1(k)\Omega^k\left(f;\frac{1}{m}\right)_{p.w}$.

$E_m(f)_{\ p.w} = \inf_{T_m \in \Pi_m}\{\|f - T_m\|_{\ p.w}\} = \|f - T_m^*\|_{p.w} \leq$ $C(k)\Omega^k\left(f;\frac{1}{m}\right)_{p.w}$

Thus $E_m(f)_{\,P.w} \leq C(k)\Omega^k\left(f;\frac{1}{m}\right)_{p.w}$, where T_m^* is the best approximation of f $*$

Theorem 3.5: *For* $f \in L_{P.w}(\mathbb{X}).\mathbb{X} = [-\pi.\pi].n \in N.$ *we get*

$$\|\delta_m(f) - f\|_{\,P.w} \leq C\,\mathcal{T}^k\left(f;\tfrac{1}{m}\right)_{p.w} \tag{10}$$

Proof: Let $T_m^* \in \Pi_m$ be the best approximation of f and $\delta_m(T_m^*) \to T_m^*$, when $m \to \infty$ we have:

$\|\delta_m(f) - f\|_{P.w} = \|\delta_m(f) - f - \delta_m(T_m^*) + \delta_m(T_m^*) - T_m^* + T_m^*\|_{P.w}$

$\leq \|\delta_m(f) - \delta_m(T_m^*)\|_{P.w} + \|\delta_m(T_m^*) - T_m^*\|_{P.w} + \|f - T_m^*\|_{P.w}$

$\leq \|\delta_m(f - T_m^*)\|_{P.w} + \|\delta_n(T_m^*) - T_m^*\|_{P.w} + \|f - T_m^*\|_{P.w}$

By linearly of δ_m . Then using (8) we get

$\|\delta_m\,(f) - f\,)\|_{P.w} \leq C_1\|\,f - T_m^*\,\|_{P.w} + \|\delta_m\,(T_m^*) -\ T_m^*\|_{P.w} +$ $\|\,f - T_m^*\,\|_{P.w}$.

And since $\lim_{m\to\infty}\| \delta_m(T_m^*) - T_m^*\|_{P.w} = 0$ we have $\| \delta_m(f) - f\|_{P.w} \le C_2\| f - T_m^*\|_{P.w}$. And by (7) and (9) we get

$\| \delta_m(f) - f\|_{P,w} \le C_2\| f - T_m^*\|_{P.w} = E_m(f)_{p.w} \le C_3\Omega^k\left(f, \frac{1}{m}\right)_{p.w} \le C_3\,\mathcal{T}^k\left(f, \frac{1}{m}\right)_{p.w}$. Then $\|\delta_m(f) - f\|_{P.w} \le C_3\,\mathcal{T}^k\left(f; \frac{1}{m}\right)_{p.w}$ *

Theorem 3.6: For $f \in L_{P.w}(\mathbb{X}).\mathbb{X} = [-\pi.\pi].1 \le P < \infty$ *and* $m \in N$

We have

$$\lim_{m\to\infty}\|\delta_m(f) - f\|_{P.w} = 0 \tag{11}$$

Proof: by using $(\mathbf{10})$ we get $\|\delta_m(f) - f\|_{P.w} \le C\mathcal{T}^k\left(f; \frac{1}{m}\right)_{P.w}$.

Then $\lim_{m\to\infty}\|\delta_m(f) - f\|_{P.w} \le \lim_{m\to\infty} C\,\mathcal{T}^k\left(f; \frac{1}{m}\right)_{P.w}$ and since $\lim\limits_{m\to\infty}\frac{1}{m} = 0$, we have

$\lim_{m\to\infty}\|\delta_m(f) - f\|_{P.w} \le \lim_{m\to\infty} C\,\mathcal{T}^k\left(f; \frac{1}{m}\right)_{P.w} = C\mathcal{T}^k(f;0)_{P.w} = 0.$

This means that $\lim_{m\to\infty}\|\delta_m(f) - f\|_{P.w} = 0$ *

4. Application

For $f \in L_P(\mathbb{X}).1 \le p < \infty$.$\mathbb{X} = [-\pi.\pi]$, where $L_P(\mathbb{X}) \subseteq L_{P.w}(\mathbb{X})$, and for $w(x) = \sqrt{\pi^2 x - x^3}$ with $x \in [0.1]$ be the weight for integral let use the following function $f: \mathbb{X} \to \mathbb{R}$: $f(x) = \sqrt{\frac{\pi^2 - x^2}{x}}$ which is an unbounded function to an application $(\mathbf{10})$ and$(\mathbf{11})$. Let

$$(fw)(x) = \left(\sqrt{\frac{\pi^2 - x^2}{x}}\,\sqrt{\pi^2 x - x^3}\right) = \left(\frac{\sqrt{\pi^2 - x^2}}{\sqrt{x}}\,\sqrt{x}\,\sqrt{\pi^2 - x^2}\right) = \pi^2 - x^2$$

For $f \in L_{P.w}(\mathbb{X})$. we have $\mathbb{S}_n(f.x)_{p.w} = \frac{1}{\pi}\int_{\mathbb{X}} (fw)(t - x)\mathcal{D}_n(t)\,dt$ for all $x \in \mathbb{X}$ where

$\mathcal{D}_n(t) = \frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos kt$ we get. **For $\boldsymbol{n = 1}$** we have $\mathcal{D}_1(t) = \frac{1}{2} + \cos t$ thus

$\mathbb{S}_1(f.x)_{p.w} = \frac{1}{\pi}\int_{\mathbb{X}} (fw)(t - x)\mathcal{D}_1(t)\,dt = \frac{1}{\pi}\int_{\mathbb{X}} (\pi^2 - (t - x)^2)(\frac{1}{2} + \cos t)dt$

$= \frac{1}{\pi}\int_{-\pi}^{0}(\pi^2 - (t - x)^2)(\frac{1}{2} + \cos t)dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - (t - x)^2)(\frac{1}{2} + \cos t)dt$

By using integration by part we have $\mathbb{S}_1(f.x) = -x^2 + \frac{2\pi^2}{3} + 4$

For $\boldsymbol{n=2}$ we have $\mathcal{D}_2(t)=\frac{1}{2}+\cos t+\cos 2t$ Again by using integration by part we have

$\mathbb{S}_2(f.x)=\frac{1}{\pi}\int_{\mathbb{X}}(fw)(t-x)\mathcal{D}_2(t)\,dt=\frac{1}{\pi}\int_{\mathbb{X}}(\pi^2-(t-x)^2)(\frac{1}{2}+\cos t+\cos 2t)dt$

$=-x^2+\frac{2\pi^2}{3}+3$ Thus $\mathbb{S}_2(f.x)=-x^2+\frac{2\pi^2}{3}+3,$

$\sigma_m(f.x)_{p.w}=\frac{1}{m+1}\sum_{n=1}^{m+1}\mathbb{S}_n(f.x)$, and for $m=1$, $\sigma_1(f.x)_{p.w}=\frac{1}{2}[\mathbb{S}_1+\mathbb{S}_2]=\frac{1}{2}\left[(-x^2+\frac{2\pi^2}{3}+4)+(-x^2+\frac{2\pi^2}{3}+3\ )\right]=-x^2+\frac{2\pi^2}{3}+3.5$

$\delta_1(f.x)=-x^2+\frac{2\pi^2}{3}+3.5$. Now for $p=1$ we have $\|\sigma_1(f)-f\|_{1.w}=\int_{\mathbb{X}}\left|-x^2+\frac{2\pi^2}{3}+3.5-(\pi^2-x^2)\right|dx$

$=\int_{-\pi}^{0}\left|3.5-\frac{\pi^2}{3}\right|dx+\int_{0}^{\pi}\left|3.5-\frac{\pi^2}{3}\right|dx=1.3$ Then $\|\sigma_1(f)-f\|_{1.w}=1.3$

Then the difference between $\boldsymbol{\sigma_1(f.x)_{1.w}}$ and $\boldsymbol{f(x)}\in\boldsymbol{L_{1.w}}(\mathbb{X})$ equal $(\mathbf{1.3})$

For $=\mathbf{3}$, we have $\mathbb{S}_3(f.x)_{1.w}=\frac{1}{\pi}\int_{\mathbb{X}}(fw)(t-x)\mathcal{D}_3(t)\,dt$. Where

$\mathcal{D}_3(t)=\frac{1}{2}+\cos t+\cos 2t+\cos 3t$ Also By using integration by part we have

$\mathbb{S}_3(f.x)=\frac{1}{\pi}\int_{\mathbb{X}}(\pi^2-(t-x)^2)(\frac{1}{2}+\cos t+\cos 2t+\cos 3t\ )dt$

$=-x^2+\frac{2\pi^2}{3}+\frac{62}{18}$ So that $\mathbb{S}_3(f.x)=-x^2+\frac{2\pi^2}{3}+3.4$. **For $\boldsymbol{n=4}$** we have

$\mathbb{S}_4(f.x)_{1.w}=\frac{1}{\pi}\int_{\mathbb{X}}(fw)(t-x)\mathcal{D}_4(t)\,dt$ Where

$\mathcal{D}_4(t)=\frac{1}{2}+\cos t+\cos 2t+\cos 3t+\cos 4t$ Then by using integration by part again $\mathbb{S}_4(f.x)_{1.w}=\frac{1}{\pi}\int_{\mathbb{X}}(\pi^2-(t-x)^2)(\frac{1}{2}+\cos t+\cos 2t+\cos 3t+\cos 4t\ )dt=-x^2+\frac{2\pi^2}{3}+\frac{115}{36}$ So that $\mathbb{S}_4(f.x)_{1.w}=-x^2+\frac{2\pi^2}{3}+3.194$, and **for $\boldsymbol{m=3}$**

$\sigma_3(f.x)_{1.w}=\frac{1}{4}[\mathbb{S}_1+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4]=\frac{1}{4}\left[-x^2+\frac{2\pi^2}{3}+4+(-x^2)+\frac{2\pi^2}{3}+3+(-x^2)+\frac{2\pi^2}{3}+3.4+(-x^2)+\frac{2\pi^2}{3}+3.194\ \right]=-x^2+\frac{2\pi^2}{3}+3.4$. Thus $\sigma_3(f)=-x^2+\frac{2\pi^2}{3}+3.4$. Thus

$\|\sigma_3(f) - f\|_{1.w} = \int_{\mathbb{X}} \left| -x^2 + \frac{2\pi^2}{3} + 3.4 - (\pi^2 - x^2)\right| dx =$
$\int_{-\pi}^{0} \left|3.4 - \frac{\pi^2}{3}\right| dx + \int_0^{\pi} \left|3.4 - \frac{\pi^2}{3}\right| dx = 0.675.$

Then the difference between $\boldsymbol{\delta_3(f(x).x)}$ and $\boldsymbol{f(x) \in L_{1.w}(\mathbb{X})}$ equal $\boldsymbol{(0.675)}$.

Now **for $\boldsymbol{n = 5}$** we have $\mathbb{S}_5(f.x)_{1.w} = \frac{1}{\pi}\int_{\mathbb{X}} (fw)(t-x)\mathcal{D}_5(t)\, dt$

$\mathcal{D}_5(t) = \frac{1}{2} + \cos t + \cos 2t + \cos 3t + \cos 4t + \cos 5t$ Then

$\mathbb{S}_5(f.x)_{1.w} = \frac{1}{\pi}\int_{\mathbb{X}} (\pi^2 - (t-x)^2)(\frac{1}{2} + \cos t + \cos 2t + \cos 3t +$ $\cos 4t + \cos 5t\,)dt = -x^2 + \frac{2\pi^2}{3} + \frac{3019}{900}$ So that $\mathbb{S}_5(f.x) = -x^2 + \frac{2\pi^2}{3} +$ 3.354.

In general $\boldsymbol{\mathbb{S}_n(f.x)_{1.w} = -x^2 + \frac{2\pi^2}{3} + \sum_{k=1}^{n} \frac{4(-1)^{k+1}}{k^2}}$

Then $\sigma_m(f.x)_{1.w} = \frac{1}{m+1}\sum_{n=1}^{m+1} \mathbb{S}_n(f.x) = \frac{1}{m+1}[\mathbb{S}_1 + \mathbb{S}_2 \cdots S_{m+1}] =$ $-x^2 + \frac{2\pi^2}{3} + \frac{1}{m+1}\sum_{n=1}^{m+1}\sum_{k=1}^{n}\frac{4(-1)^{k+1}}{k^2}$

In general when $m \to \infty$ the amount $\frac{1}{m+1}\sum_{n=1}^{m+1}\sum_{k=1}^{n}\frac{4(-1)^{k+1}}{k^2} \to$ $3.292517 \cong \frac{\pi^2}{3}$ we get $\boldsymbol{\|\sigma_m(f) - f\|_{1.w} = \int_{\mathbb{X}} \left| (-x^2 + \frac{2\pi^2}{3} + \frac{\pi^2}{3}) - (\pi^2 - x^2)\right| dx = 0}$

Then the difference between $\boldsymbol{\sigma_m(f)_{1.w}}$ and $\boldsymbol{f(x) \in L_{1.w}(\mathbb{X})}$ equal zero, when $\boldsymbol{m \to \infty}$. Then

$\boldsymbol{\lim_{n\to\infty}\|\delta_m(f) - f\|_{1.w} = 0}$ Which means that $\sigma_m(f)_{1.w}$ is the best approximation of f. And in general, we have $\boldsymbol{\|\sigma_m(f) - f\|_{p.w} \le C(k)\, \mathcal{T}^2\left(f; \frac{1}{m}\right)_{p.w}}$ *

5. Conclusion

Through this research, a wider space was found than the Lebesgue space, to approximate the unbounded functions of the Fejer operator using the second-degree of the weighted averaged modulus of smoothness, which led to finding the best approximation of the function$f$, such that the difference between the function f and its approximation $\delta_m(f)$ is equal to zero, as this has been proven in practice through the application.

References

[1] A. Lupus, A q-analogue of the Bernstein operator, seminar on numerical and Statistical Calculus, University of Cluj-Napoca, No. 9, pp. 85-92 (1987).

[2] G. M. Philips, Bernstein polynomials based on the q-integers, *Annals of Numerical Mathematics*, Vol. 4, No. 1-4, pp. 511-518 (1997).

[3] S. Ostrovska, The first decade of the q-Bernstein polynomials: results and perspectives, *Journal of Mathematical Analysis and Approximation Theory*, Vol. 2, No.1, pp. 35-51 (2007).

[4] Ali H. Zaboon and Sahib K. Jasim, Approximation of Unbounded Functions by Trigonometric Polynomials in Locally-Global Weighted, L_(P.δ.w) (X)-Spaces, *International Journal of Mathematical Archive*-5(4), India (2014).

[5] Ali H. Zaboon and Sahib K. Jasim, Multiplier Approximation of Functions by q-Bernstein Kantorovich Operator, *Journal of Physics: Conference Series*, 1897 012055 (2021).

[6] Z. Ditzian, V. Totik, Moduli of smoothness, Springer-Verlag, New York (1987).

[7] Carothers N. L., Approximation Theory, Dept. of Math. and Statistics Bowling Green, State University, USA (1998).

[8] Longquan Yong. Smooth Newton method to absolute value equation based on lower uniform smoothing approximation function, *Journal of Interdisciplinary Mathematics*, 19:1, 191-205 (2016), DOI: 10.1080/09720502.2015.1104937.

[9] Seyyede Robab Fayyaz Behrooz & Majid Amirfakhrian. Numerical solution of algebraic fuzzy complex equations with complex fuzzy coefficients, *Journal of Interdisciplinary Mathematics*, 23:8, 1553-1568 (2020), DOI: 10.1080/09720502.2020.1776935.

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