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Upper and lower blow-up rate estimates of a semilinear heat equation with a nonlinear boundary condition

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Abstract

In this paper, we consider the blow-up solutions of a semi-linear heat equation with a Neumann boundary condition, where the nonlinear terms, appear in the differential equation and in the boundary conditions, are of exponential types. We study the effect of the nonlinear terms on the last blow-up behavior. Precisely, the upper (lower) blow-up rate estimates are established by using integral equation method and maximum principle techniques. The results show that the presentence of the reaction and boundary terms has important effects on the upper (lower) blow-up rate estimates forms.

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1. Introduction

We consider the following problem:

$$\left.\begin{aligned} &u_t = \Delta u + \lambda e^{pu}, && (x,t) \in B_R \times (0,T),\\ &\frac{\partial u}{\partial \eta} = e^{qu}, && (x,t) \in \partial B_R \times (0,T)\\ &u(x,0) = u_0(x), && x \in B_R, \end{aligned}\right\} \tag{1}$$

where $p, q, \lambda > 0$; η is the outward normal; B_R is a ball in R^n; u_0 is non-negative, radially symmetric, non-decreasing, smooth function and satisfying the conditions:

$$\frac{\partial u_0}{\partial \eta} = e^{qu_0}, \qquad x \in \partial\Omega, \tag{2}$$

$$\Delta u_0 + \lambda e^{pu_0} \geq 0, \quad u_{0r}(|x|) \geq 0, \quad x \in \overline{\Omega}_R. \tag{3}$$

Over the last years, many authors have studied the blow-up phenomena in time dependent problems [1-7], such as, the semilinear heat equation with a Neumann boundary condition, which takes the form:

$$\left.\begin{aligned} &u_t = \Delta u + \lambda f(u), && (x,t) \in \Omega \times (0,T),\\ &\frac{\partial u}{\partial \eta} = g(u), && (x,t) \in \partial\Omega \times (0,T),\\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{4}$$

This problem has been considered by some authors, [8-11]. The main question of these works was "how do the nonlinear reaction and boundary terms affect the blow-up properties?"

In [8], a special case of problem (4) has been studied, where $\lambda < 0$; $n = 1$ or $\Omega = B_R$, and

$$f(u) = u^p, \quad g(u) = u^q, \quad p, q > 1. \tag{5}$$

It is proved that the value $p = 2q - 1$ is a critical for blow-up in the two cases:

1. for $p < 2q - 1$ (or $p = 2q - 1$ and $-\lambda < q$), there are some blow-up solutions and the blow-up occurs in a finite time only on the boundary.
2. for $p > 2q - 1$ (or $p = 2q - 1$ and $-\lambda > q$), every solution exists globally and it is globally bounded.

Moreover, for the case (2), with $n=1$, $\Omega=[0,1]$, J. D. Rossi, [9], has shown that the upper (lower) blow-up rate estimates are as follwos:

$$c \le \max_{[0,1]} u(\cdot,t)(T-t)^{\frac{1}{2(q-1)}} \le C, \quad 0<t<T.$$

where C, c are positive constants.

Later, another special case has been studied [9], in case of $\lambda=1$; f, g is taken as in (5); Ω is a bounded domain or the interval [0, 1]. It has been shown that if and only if $\max\{p,q\}\le 1$, then every solution is global. While, if $\max\{p,q\}>1$, then every solution has to blow up, only on the boundary, in a finite time. In addition, for $n=1, \Omega=[0,1]$, the forms of upper and lower blow-up rate estimates has been established as follows [9,10]:

$$c \le \max_{[0,1]} u(\cdot,t)(T-t)^{\alpha} \le C, \quad 0<t<T,\ C,c>0$$

where
$$\alpha = \left\{ \begin{array}{ll} \dfrac{1}{p-1}, & \text{for } p \ge 2q-1 \\ \dfrac{1}{2(q-1)} & \text{for } p < 2q-1 \end{array} \right\}$$

It is clear that, if then the nonlinear boundary term determines the blow-up rate estimates, whereas, in case of $p>2q-1$, the nonlinear term, which appears in the differential equation, is dominated and it shapes the blow-up rate estimate.

Another case was considered later, [11]:

$$\left. \begin{array}{ll} u_t = \Delta u - a e^{pu}, & (x,t)\in\Omega\times(0,T), \\ \dfrac{\partial u}{\partial \eta} = e^{qu}, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & x\in\Omega, \end{array} \right\} \tag{6}$$

where $a,p,q>0$; u_0 satisfies (2) and (3); Ω is a bounded domain.

It has been shown that the critical exponent for problem (6) is given as follows:

1. If $2q < p$, for all initial function, solutions are globally bounded.
2. If $2q > p$, for large initial functions, solutions blow up in finite times.
3. If $2q = p$, for large initial functions, some solutions may blow up in finite times.

In addition, if $\Omega = B_R$, then every solution blows up only on the boundary.

Moreover, the upper (lower) blow-up rate estimates are as follows:

$$\log C_1 - \frac{1}{2q}\log(T-t) \le \max_{\bar{B}} u(\cdot,t) \le \log C_2 - \frac{1}{2q}\log(T-t),\ 0 < t < T.$$

This indicates that the blow-up rate estimates of problem (6) takes the same forms as those established for problem (6), in case of $a = 0$, [12].

In this paper, we consider the last blow-up profile of problem (1). Namely, the upper and lower blow-up rate estimates are derived.

This paper is divided into five sections. In section two we discuss the local existence and uniqueness to problem (1)-(3). Moreover, some basic properties of the solutions of this problem are given. In section three, we derive the lower blow-up rate estimate for problem (1)-(3) using two techniques: maximum principle and integral equation. Section four is devoted to establish the upper blow-up rate estimate form for problem (1)-(3). In the last section, based on the proved results, some conclusions are pointed out.

2. Preliminaries

The local existence and uniqueness of classical solutions for problem (1) can be guaranteed by the fact that the problem is uniformly parabolic, and the nonlinear functions, appeared in the equations and in the boundary conditions, are smooth functions [13]. In addition, u_0 satisfies the compatibility condition (2). On the other hand, we can show that every nontrivial solution to this problem has to blow up in a finite time. Moreover, based on the known blow-up properties of problem (1), with $\lambda = 0$, [13], we can show that ∂B_R is a subset of blow-up set by using the comparison principle [14].

For simplicity, we may denote $u(x, t) = u(r, t)$.

The next lemma states some properties to the solutions of problem (1), and it can be proved easily using maximum principle techniques [1].

Lemma 1 : *Let u be a solution to problem (1)-(3). Then*

1. $u > 0$ *and it is radial for* $(x,t) \in \overline{B}_R \times (0,T)$.
2. $u_r \ge 0$, *for* $(r,t) \in [0,R] \times [0,T)$.
3. $u_t > 0$, *for* $(x,t) \in \overline{B}_R \times (0,T)$.

From Lemma (1), $u_r \geq 0,$ for $(x,t) \in [0,R] \times (0,T),$ thus:

$$\max_{\overline{B}_R} u(\cdot,t) = u(R,t), \quad t \in (0,T).$$

Therefore, we can derive the upper (lower) blow-up rate estimates only for $u(R,t)$.

3. Lower Blow-Up Rate Estimate

Next, the lower blow-up rate estimate to problem (1)-(3) is derived.

Theorem 1 : *Let u(x, t) be a solution of problem (1)-(3), which blows-up at T > 0. Then there exists c > 0 such that:*

$$\log c - \frac{1}{2\alpha} \log(T-t) \leq u(R,t),\ t \in (0,T),$$

where $\alpha = \max\{p, q\}$.

Proof : Define the function:

$$M(t) = \max_{\overline{B}_R} u(\cdot,t) = u(R,t), \quad \text{for} \quad t \in [0,T).$$

By Lemma(1), $u_t > 0,$ for $t \in (0,T),\ x \in \overline{B}_R.$ Thus, $M(t)$ is increasing in $(0,T)$

With respect to *u,* the integral equation of problem (1) has the from [11]:

$$u(x,t) = \int_{B_R} \Gamma(x-y,t-z) u(y,z) dy + \lambda \int_z^t \int_{B_R} \Gamma(x-y,t-\tau) e^{pu(y,\tau)} dy d\tau$$
$$+ \int_z^t \int_{S_R} \Gamma(x-y,t-\tau) e^{qu(y,\tau)} ds_y d\tau - \int_z^t \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau) ds_y d\tau, \qquad (7)$$

For $0 < z < t < T, x \in B_R,$ and Γ denotes the fundamental solution of heat equation, which has the following from:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{(n/2)}} \exp\left[-\frac{|x|^2}{4t}\right]. \qquad (8)$$

Since $u(y,t) \leq u(R,t),$ for $y \in \overline{B}_R;$ equation (7) becomes

$$u(x,t) \leq u(R,z)\int_{B_R} \Gamma(x-y,t-z)\,dy + \lambda\int_z^t e^{pu(R,\tau)}\int_{B_R} \Gamma(x-y,t-\tau)dyd\tau.$$
$$+\int_z^t e^{qu(R,\tau)}\int_{S_R} \Gamma(x-y,t-\tau)\,ds_y d\tau + \int_z^t u(R,\tau)\int_{S_R}\left|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\right|ds_y d\tau.$$

Based on the last inequality, and since u is a continuous function on $\overline{B}_R$, we obtain

$$M(t) \leq M(z)\int_{B_R} \Gamma(x-y,t-z)\,dy + \lambda e^{pM(t)}\int_z^t\int_{B_R} \Gamma(x-y,t-\tau)dyd\tau$$
$$+e^{qM(t)}\int_z^t\int_{S_R} \Gamma(x-y,t-\tau)ds_y d\tau + M(t)\int_z^t\int_{S_R}\left|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\right|ds_y d\tau. \tag{9}$$

According to [14, 15], for $0 < t_1 < t_2, x, y \in R^n, \Gamma$ satisfies

$$\int_{B_R} \Gamma(x-y,t_2-t_1)dy \leq 1.$$

Moreover, there are $k_1, k_2 > 0$ such that

$$\Gamma(x-y,t_2-t_1) \leq \frac{k_1}{(t_2-t_1)^{\mu_0}}\cdot\frac{1}{|x-y|^{n-2+\mu_0}}, \quad 0<\mu_0<1,$$
$$\left|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t_2-t_1)\right| \leq \frac{k_2}{(t_2-t_1)^{\mu}}\cdot\frac{1}{|x-y|^{n+1-2\mu-\sigma}}, \quad \sigma\in(0,1), \mu\in\left(1-\frac{\sigma}{2},1\right).$$

With choosing $\mu_0 = 1/2$, [15], there are $d_1, d_2 > 0$ such that:

$$\int_{S_R}\frac{ds_y}{|x-y|^{n-2+\mu_0}} \leq d_1, \qquad \int_{S_R}\frac{ds_y}{|x-y|^{n+1-2\mu-\sigma}} \leq d_2.$$

It follows that the inequality (9) becomes

$$M(t) \leq M(z) + \lambda e^{pM(t)}(t-z) + C_1 e^{qM(t)}\sqrt{t-z} + C_2 M(t)(t-z)^{1-\mu},$$

where C_1, C_2 are two positive constants.

Now, since $t-z \leq T-z$, we have

$$M(t) \leq M(z) + \lambda e^{pM(t)}\sqrt{T-z} + C_1 e^{qM(t)}\sqrt{T-z} + C_2 M(t)(T-z)^{1-\mu}, \tag{10}$$

for $(T-z) \leq 1$.

It is clear that, $\frac{M(t)}{e^{\alpha M(t)}} \to 0$, when $t \to T$.

Thus

$$\frac{M(t)}{e^{\alpha M(t)}} \leq (T-z)^{\frac{1}{2}-(1-\mu)}, \text{ for } t \text{ close to } T.$$

Thus, the inequality (10) gives

$$M(t) \leq M(z) + \lambda e^{pM(t)}\sqrt{T-z} + C_1 e^{qM(t)}\sqrt{T-z} + C_2 e^{\alpha M(t)}\sqrt{T-z},$$

Which leads to there exists a constant C^* such that

$$M(t) \leq M(z) + C^* e^{\alpha M(t)}\sqrt{T-z}, \quad z < t < T, \text{ for } t \text{ close to } T$$

We can choose $< t < T$, where z is any value close to T, such that

$$M(t) - M(z) = C_0 > 0,$$

It follows that

$$C_0 \leq C^* e^{\alpha M(z) + \alpha C_0}\sqrt{T-z}.$$

Thus

$$\frac{C_0}{C^* e^{(\alpha C_0)}\sqrt{T-z}} \leq e^{\alpha u(R,z)}.$$

Therefore, there is $c > 0$ such that

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

A similar result is proved in the next theorem, by adding some other assumptions on q and u_0. As we shall see that the proof of next theorem depends on the maximum principle and we there is no need to use the integral equation.

Theorem 2 : *Let u be a solution to problem (1)-(3), with* $q \geq 1$, *which blow-up at T. Moreover, assume that the initial function satisfies the following condition:*

$$u_{0r}(r) - \frac{r}{R}e^{u_0(r)} \geq 0, \quad r \in [0,R]. \tag{11}$$

Then there exists c > 0 such that

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T),$$

where $\alpha = \max\{p,q\}$.

Proof : The new function J is defined as:

$$J(x,t)=u_r(r,t)-\frac{r}{R}e^{u(r,t)}, \quad x\in B_R\times(0,T).$$

By doing some calculations, we obtain

$$J_t=u_{rt}-\frac{r}{R}e^u\left[u_{rr}+\frac{n-1}{r}u_r+\lambda pe^{pu}\right],$$

$$J_r=u_{rr}-\frac{r}{R}e^u u_r-\frac{1}{R}e^u,$$

$$J_{rr}=\left[u_{rt}-\frac{n-1}{r}u_{rr}+\frac{n-1}{r^2}u_r-\lambda pe^{pu}u_r\right]$$
$$-\frac{r}{R}\left[e^u u_{rr}+e^u u_r^2\right]-\frac{2}{R}e^u u_r.$$

It follows that

$$J_t-J_{rr}-\frac{n-1}{r}J_r=-\frac{n-1}{r^2}\left[u_r-\frac{r}{R}e^u\right]+\lambda pe^{pu}\left[u_r-\frac{r}{R}e^u\right]+\frac{r}{R}e^u u_r^2+\frac{2}{R}e^u u_r.$$

Thus

$$J_t-\Delta J-bJ=\frac{r}{R}e^u u_r^2+\frac{2}{R}e^u u_r\geq 0,$$

$(x,t)\in B_R\times(0,T)\cap\{r>0\}$, and $b=\left[\lambda pe^{pu}-\frac{n-1}{r^2}\right]$.

From condition (11), we get $J(x,0)\geq 0, \quad x\in B_R$.

Moreover, we have

$$J(0,t)=u_r(0,t)\geq 0; \quad J(R,t)=0, \quad t\in(0,T).$$

Since

$$\sup_{(0,R)\times(0,t]} b<\infty, \quad \text{for} \quad t<T,$$

by maximum principle [1], we obtain

$$J(x,t)\geq 0, \quad (x,t)\in B_R\times(0,T), \text{ and } \left.\frac{\partial J}{\partial\eta}\right|_{\partial B_R}\leq 0.$$

It follows that

$$\left.\left(u_{rr} - \frac{r}{R}e^u u_r - \frac{1}{R}e^u\right)\right|_{\partial B_R} \leq 0.$$

Thus

$$u_t \leq \left.\left(\frac{n-1}{r}u_r + \lambda p e^{pu} + e^u u_r + \frac{1}{R}e^u\right)\right|_{\partial B_R}.$$

This implies that

$$u_t(R,t) \leq \frac{n-1}{R}e^{qu(R,t)} + \lambda p e^{pu(R,t)} + e^{(1+q)u(R,t)} + \frac{2}{R}e^{u(R,t)}, \quad t \in (0,T).$$

So that, there exist $C > 0$, such that

$$u_t(R,t) \leq Ce^{2\alpha u(R,t)}, \quad t \in (0,T).$$

Since u blows up at, by integrating the last inequality from t to T we obtain:

$$\frac{c}{(T-t)^{\frac{1}{2}}} \leq e^{\alpha u(R,t)}, \quad t \in (0,T)$$

This leads to $\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T).$

4. Upper Blow-Up Rate Estimate

Next, the upper blow-up rate estimate to problem (1)-(3) will be established.

Theorem 3 : *Let* $u(x,t)$ *be a solution of problem (1)-(3), which blows-up at T > 0. Moreover, we assume that*

$$\Delta u_0 + \lambda e^{pu_0} \geq a > 0, \qquad \text{in } \overline{B}_R. \tag{12}$$

Then there exists C > 0 such that

$$u(R,t) \leq \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T). \tag{13}$$

Proof : The new function J is defined as:

$$J(x,t) = u_t(r,t) - \varepsilon u_r(r,t), \quad (x,t) \in B_R \times (0,T), \ \varepsilon > 0$$

By the assumption (12) and since $u_0(r)$ is bounded in B_R, we obtain

$$J(x,0) = \Delta u_0(r) + f(u_0(r)) - \varepsilon u_{0r}(r) \geq 0, \quad x \in \overline{B}_R,$$

for some $\varepsilon > 0$

By doing some computations, we can show that

$$J_t = u_{rrt} + \frac{n-1}{r} u_{rt} + \lambda p e^{pu} u_t - \varepsilon u_{rt},$$
$$J_r = u_{tr} - \varepsilon u_{rr},$$
$$J_{rr} = u_{trr} - \varepsilon u_{tr} + \varepsilon \frac{n-1}{r} u_{rr} - \varepsilon \frac{(n-1)}{r^2} u_r + \varepsilon \lambda p e^{pu} u_r.$$

It follows that

$$J_t - J_{rr} - \frac{n-1}{r} J_r - \lambda p e^{pu} J = \varepsilon \frac{(n-1)}{r^2} u_r \geq 0,$$

This means

$$J_t - \Delta J - \lambda p e^{pu} J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

In addition,

$$\left.\frac{\partial J}{\partial \eta}\right|_{x \in \partial B_R} = u_{rt}(R,t) - \varepsilon u_{rr}(R,t)$$
$$= q e^{qu(R,t)} u_t - \varepsilon \left[u_t(R,t) - \frac{n-1}{r} u_r(R,t) - \lambda e^{pu(R,t)} \right]$$
$$\geq \left[q e^{qu(R,t)} - \varepsilon \right] u_t(R,t)$$

By Lemma 1, $u_t > 0$ for $(x,t) \in \overline{B}_R \times (0,T)$, so that

$$\frac{\partial J}{\partial \eta} \geq 0, \quad \text{on} \quad \partial B_R \times (0,T),$$

for $\varepsilon \leq q e^{\{q u_0(R)\}}$.

By maximum principle [14, 15] and above, and since e^{pu} is bounded on $B_R \times (0,t]$ for $t < T$, we obtain

$$J \geq 0, \quad (x,t) \in \overline{B}_R \times (0,T).$$

Particularly, $J(x,t) \geq 0$ for $x \in \partial B_R$, which leads to

$$u_t(R,t) \geq \varepsilon u_r(R,t) = \varepsilon e^{qu(R,t)}, \quad t \in (0,T).$$

Since u blows up at R, by integrating the last inequality from t to T and , we obtain

$$e^{qu(R,t)} \le \frac{1}{q\varepsilon(T-t)}, \quad t \in (0,T),$$

or $$u(R,t) \le \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T).$$

5. Conclusions

In this article, we have derived the upper (lower) blow-up rate estimates to problem (1)-(3). From Theorems 1, 2 and 3, we can state some conclusions:

- In case of $q > p$, the nonlinear term, which appears in the boundary condition, has an important role in determining the lower blow-up rate estimate, which has the from:

  $$\log c - \frac{1}{2q}\log(T-t) \le u(R,t), \quad t \in (0,T).$$

  In addition, this estimate has the same form as the lower blow-up rate estimate of problem (1), with $\lambda = 0$, derived in [9]. While, in case of $p > q$, the nonlinear term, appears in the differential, equation take an important role, and the lower blow-up rate estimate is given as:

  $$\log c - \frac{1}{2p}\log(T-t) \le u(R,t), \quad t \in (0,T).$$

- The upper blow-up rate estimate of problem (1) depends only on the nonlinear term, which appears in the boundary condition, and takes the form:

  $$u(R,t) \le \log c - \frac{1}{q}\log(T-t).$$

  On the other hand, it is known that the upper blow-up rate estimate of problem (1), with $\lambda = 0$, [9], is as follows:

  $$u(R,t) \le \log c - \frac{1}{2q}\log(T-t).$$

This indicates that the nonlinear term, which appears in the differential equation, has an important role in shaping the form of the upper blow-up rate estimate.

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