

On the Contraharmonic-mean Newton's method(CHN)

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Abstract

The objective of this study is to investigate the Contraharmonic-mean Newton's method (CHN) for solving equations in Banach space. Local analysis has been performed by Taylor's expansion utilizing high order derivatives and also using only the first derivatives appearing on the method. We have checked the theoretical results by comparing the CHN method with the Newton's and related third order methods by Weerakoon et al. utilizing some test functions. The theoretical and numerical results are examined by the dynamical analysis of a selected test function.

Subject Classification: 65H10, 65J15, 47J25, 65G99.

Keywords: Newton's method, Semilocal convergence, Banach space, Fractal, Efficiency index.

1. Introduction

Consider $F : D \subset X \rightarrow X$, where D is convex and open, X is a Banach space and F is Fréchet differentiable. We develop the convergence of Contraharmonic-mean method (CHN) for $x_0 \in D$. It is given as

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$$\begin{cases} y_k = x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} = y_k + F'(x_k)^{-1}F'(y_k)(F'(y_k) - F'(x_k))(F'(x_k)^2 + F'(y_k)^2)^{-1}F(x_k), \end{cases} \quad (1)$$

for $k = 0, 1, 2, \dots$ here $F'(x)$ is the derivative of operator F . We use the method (1) to find the solution $x^*(x_*)$ of equation

$$F(x) = 0. \quad (2)$$

Newton's method has been used to solve such equations. But it is only of order two under some conditions [1-20]. It is defined as follows

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \dots \quad (3)$$

Papers have appeared on the variant or modification of Newton's method in real ([7], [12], [13]), [14], [15]) as well as in Banach space ([2], [3], [4], [6]). Newton's method is also useful in some other aspect of study e.g. [5], [8]. Generally, we use two types of convergence analysis for the solutions of nonlinear equation (2). First is local convergence analysis in which firstly we assume that there exists a particular solution, around this solution there will be a neighborhood and starting with any vector in this neighborhood lead to a sequence which converges to the root under some suitable conditions [19]. Second is semilocal convergence analysis, it does not need the existence of a solution, rather then demands the same conditions around the initial vector ([16], [17], [18], [20]). Kantorovich [10] has provided the semilocal convergence of (3) in Banach space [9]. He used the technique of majorizing sequences. Majorizing sequences have been used by many researchers for the variants of Newton's method ([3], [6]). In [11], Rall introduced a different technique for the semilocal convergence of these methods, which is based on recurrence relations. Parida and Gupta in [16], Madhu in [9] have used this idea to prove the semilocal convergence for several Newton-like methods of different orders.

The earlier study of the "Contra-harmonic-mean" Newton method for the solution of (2) in real space uses the higher order derivatives while the method involves only the first-order derivative, hence it limits the applications of the method (1). In this study first we have performed the local convergence analysis of the CHN method (1) around the local root $x^*(x_*)$ which requires third-order derivative of F . Then, we develop the local convergence analysis by only one Fréchet derivative. The second local convergence analysis is important too, especially if F has no third order Fréchet derivative. Indeed set $X = Y = \mathbb{R}$, $D = [-1/2, 3/2]$. Consider H on D as

$$H(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0 \end{cases}$$

We get, $H'''(t) = 6\ln t^2 + 60t^2 - 24t + 22$. So the conclusion of the Theorem 1 (see section 2) may not hold, since H''' is not continuous. But the conclusion of the Theorem 2 holds (see section 3) with $W_0(t) = W(t) = 97t$, and $W_1(t) = 2$.

The convergence order in second case is found using the following formula

$$\mu(\text{COC}) = \frac{\ln(\|x_{j+1} - x_*\| / \|x_j - x_*\|)}{\ln(\|x_j - x_*\| / \|x_{j-1} - x_*\|)}, \quad (4)$$

or

$$\mu_1(\text{ACOC}) = \frac{\ln(\|x_{j+1} - x_j\| / \|x_j - x_{j-1}\|)}{\ln(\|x_j - x_{j-1}\| / \|x_{j-1} - x_{j-2}\|)}. \quad (5)$$

These do not require the F''' or $x^*(x_*)$ (in the formula 5).

Numerical results consist of the comparative study of the proposed method along the method of Weerakoon et al. (see [19]) by using the some test functions. One important characteristic is the comparability of the dynamics of the CHN method with the method of Weerakoon et al. [19].

2. Local Convergence Analysis 1

Theorem 1 : Let $I \subset \Re^i$ stand for a convex set and $F : I \rightarrow \Re^i$ with

- (1) $x^* \in I$ solves equation
- (2) so that $F'(x^*)^{-1}$ is well defined.
- (3) F is third-order Fréchet differential in I at some neighbourhood S of zero x^* . Then, the method (1) reaches convergence order three.

Proof : Set $e_k = x_k - x^*$ with $A_k = \left(\frac{1}{k!}\right) F'(x^*)^{-1} F^{(k)}(x^*)$. By Taylor series of F about x^* , we get

$$F(x_k) = F'(x^*)[e_k + A_2 e_k^2 + A_3 e_k^3 + O(e_k^4)], \quad (6)$$

$$F'(x_k) = F'(x^*)[1 + 2A_2 e_k + 3A_3 e_k^2 + 4A_4 e_k^3 + O(e_k^4)]. \quad (7)$$

So, we can write

$$F'(x_k)^{-1}F(x_k) = e_k - A_2 e_k^2 + (2A_2^2 - 2A_3)e_k^3 + O(e_k^4).$$

Then by (2), we obtain

$$y_k = x^* + A_2 e_k^2 + (2A_3 - 2A_2^2)e_k^3 + (4A_2^3 - 7A_2A_3 + 3A_4)e_k^4 + O(e_k^5). \quad (8)$$

Hence, we have

$$F(y_k) = F'(x^*)[A_2 e_k^2 - 2(A_2^2 - A_3)e_k^3 + (5A_2^3 - 7A_2A_3 + 3A_4)e_k^4 + O(e_k^5)]. \quad (9)$$

$$F'(y_k) = F'(x^*)[1 + 2A_2^2 e_k^2 + 4A_2(A_3 - A_2^2)e_k^3 + O(e_k^4)]. \quad (10)$$

We have again

$$F'(x_k)^2 = F'(x^*)^2[1 + 4A_2 e_k + (4A_2^2 + 6A_3)e_k^2 + (8A_4 + 12A_2A_3)e_k^3 + O(e_k^4)]. \quad (11)$$

$$F'(y_{(k)})^2 = F'(x^*)^2[1 + 4A_2^2 e_k^2 + (8A_2A_3 - 8A_2^3)e_k^3 + O(e_k^4)]. \quad (12)$$

Therefore by the CHN method we have

$$e_{k+1} = (2A_2^2 + \frac{A_3}{2})e_k^3 + O(e_k^4). \quad (13)$$

Hence CHN method (1) has local convergence of third-order to the root of F.

3. Local Convergence Analysis 2

Analysis uses some real functions and parameters. Set $A = [0, \infty)$. Suppose:

(i) equation

$$W_0(t) - 1 = 0, \quad (14)$$

has smallest root $\rho_0 \in (0, \infty)$, where $W_0 : A \rightarrow A$ stands for a nondecreasing and continuous function. Set $A_0 = [0, \rho_0)$.

(ii) equation

$$\psi_1(t) = 0, \quad (15)$$

where, $\psi_1(t) = \phi_1(t) - 1$, $\phi_1(t) = \frac{\int_0^1 W((1-\theta)t) d\theta}{1 - W_0(t)}$, $t \in [0, \rho_0]$ has a smallest root $r_1 \in [0, \rho_0)$, where $W : A_0 \rightarrow A_0$ is continuous and nondecreasing.

(iii) equation

$$P(t) - 1 = 0, \quad (16)$$

where $P(t) = \frac{1}{2}(v(t) + v(\phi_1(t)t))$ has a smallest root $\rho_p \in (0, \infty)$. Set $\rho = \min\{\rho_0, \rho_p\}$ and $A_{00} = [0, \rho)$

(iv) equation

$$\psi_2(t) = 0, \quad (17)$$

has a smallest root $r_2 \in (0, \rho)$, where $\psi_2(t) = \phi_2(t) - 1$ and

$$\phi_2(t) = \phi_1(t) + \frac{W_1(\phi_1(t)t)(W_0(\phi_1(t)t) + W_0(t))a \int_0^1 W_1(\theta) d\theta}{2(1 - W_0(t))(1 - P(t))},$$

where $W_1 : A_{00} \rightarrow A_{00}$ is continuous and nondecreasing.

Next,

$$r = \min\{r_1, r_2\}. \quad (18)$$

is shown to be convergence radius for CHN. By these definitions for all $t \in [0, r)$

$$0 \leq W_0(t) < 1 \quad (19)$$

$$0 \leq P(t) < 1 \quad (20)$$

and

$$0 \leq \phi_i(t) < 1, \quad i = 1, 2 \text{ hold.} \quad (21)$$

We utilize conditions (A) in our analysis provided the functions W_0, W, W_1 and P are as given previously:

(a₁) $F : D \rightarrow X$ has a simple solution x_* satisfying

$$\|F(x_*)^{-1}\| \leq a.$$

(a₂) For all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq W_0(\|x - x_*\|).$$

Define $D_0 = D \cap B(x_*, \rho_0)$.

(a₃) For all $x, y \in D_0$

$$\|F'(x) - F'(x_*)\| \leq W(\|x - x_*\|),$$

$$\|F'(x_*)^{-1}F'(x)\| \leq W_1(\|x - x_*\|),$$

and

$$\|F'(x_*)^{-2}(F'(x)^2 - F'(x_*)^2)\| \leq V(\|x - x_*\|).$$

(a₄) $B(x_*, r) \subset D$ and

(a₅) There exists $\bar{r} \geq r$ satisfying $\int_0^1 W_0(\theta\bar{r})d\theta < 1$.

Set

$$D_1 = D \cap \bar{B}(x_*, \bar{r}). \quad (22)$$

Based on the conditions (A) and the developed notation, we prove the second local convergence result for method (1).

Theorem 2 : *Under the conditions (A), pick $x_0 \in B(x_*, r) - \{x_*\}$. Then, the sequence generated by the method CHN starting at x_0 is defined well, remains in $B(x_*, r) - \{x_*\}$ and converges to x_* . Moreover, x_* is the only solution of equation $F(x) = 0$, in the set D_1 .*

Proof : Choose $u \in B(x_*, r)$. Then, by (a₂) and (16) we obtain

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \leq v \|u - x_*\| \leq W_0(r) < 1.$$

Using the Banach lemma on invertible operators [10] and the preceding inequality, we have

$$\|F'(u)^{-1}F'(x_*)\| \leq \frac{1}{1 - W_0\|u - x_*\|}. \quad (23)$$

In particular, if $u = x_0$, $F'(x_0)^{-1}$ exists. Since $x_0 \in B(x_*, r)$, then y_0 exists and

$$\begin{aligned} y_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) = \\ &= F'(x_0)^{-1}F'(x_*) \int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_*))(x_0 - x_*)d\theta. \end{aligned} \quad (24)$$

We set $\|x_n - x_*\| = S_n$. So, by condition (a₃) and (16), (18), (19), (21)

$$\begin{aligned}
\|y_0 - x_*\| &\leq \frac{\int_0^1 W((1-\theta))S_0 d\theta S_0}{1 - W_0(S_0)} \\
&\leq \phi_1(S_0)S_0 \\
&\leq S_0 \\
&< r,
\end{aligned} \tag{25}$$

Hence, $y_0 \in B(x_*, \rho)$.

Next, we show linear operator $F'(x_0)^2 + F'(y_0)^2$ is invertible, which will well define x_1 . By (a_3) and (16), (23), we have

$$\begin{aligned}
&\|(2F'(x_*)^2)^{-1}(F'(x_0)^2 + F'(y_0)^2 - 2F'(x_*)^2)\| \\
&\leq \frac{1}{2}(\|(F'(x_*)^2)^{-1}(F'(x_0)^2 - F'(x_*)^2)\| + \|(F'(x_*)^2)^{-1}(F'(y_0)^2 - F'(x_*)^2)\|) \\
&\leq \frac{1}{2}[VS_0 + V\|y_0 - x_*\|] \\
&\leq \frac{1}{2}[VS_0 + V(\phi_1(S_0)S_0)] \\
&= P(S_0) \\
&< 1,
\end{aligned} \tag{26}$$

so

$$\|(F'(x_0)^2 + F'(y_0)^2)^{-1}F'(x_*)^2\| \leq \frac{1}{2(1 - P(S_0))}. \tag{27}$$

By second substep of the method CHN, we have

$$\begin{aligned}
&x_1 - x_* \\
&= x_0 - x_* - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}F(x_0) - (F'(x_0) + F'(y_0))(F'(x_0)^2 \\
&\quad + F'(y_0)^2)^{-1}F(x_0) \\
&= y_0 - x_* + [F'(x_0)^{-1} - (F'(x_0) + F'(y_0))(F'(x_0)^2 + F'(y_0)^2)^{-1}]F(x_0) \\
&= y_0 - x_* + F'(x_0)^{-1}F'(y_0)(F'(y_0) - F'(x_0))(F'(x_0)^2 + F'(y_0)^2)^{-1}F(x_0) \tag{28}
\end{aligned}$$

So by $(a_1), (a_3), (16), (23), (26)$ we obtain

$$\begin{aligned}
&\|x_1 - x_*\| \\
&\leq \|y_0 - x_*\| + \|F'(x_0)^{-1}F'(x_*)\| \|F'(x_*)^{-1}F'(y_0)\| \|[F'(y_0) - F'(x_*)^{-1}]\| + \\
&\quad \|F'(x_0) - F'(x_*)\| \|(F'(x_0)^2 + F'(y_0)^2)^{-1}F'(x_*)^2\| \|F'(x_*)^{-1}\| \|F'(x_*)^{-1}F(x_0)\|
\end{aligned}$$

$$\begin{aligned}
&\leq [\phi_1(S_0) \frac{W_1(\|y_0 - x_*\|)(W_0(\|y_0 - x_*\|) + W_0(S_0))a \int_0^1 W_1(\theta S_0) d\theta}{2(1 - W_0(S_0))(1 - P(S_0))}] \|y_0 - x_*\| \\
&\leq \phi_2(S_0) S_0 \\
&\leq S_0,
\end{aligned} \tag{29}$$

so $x_1 \in B(x_*, r)$.

Switching x_0, y_0, x_1 with x_k, y_k, x_{k+1} in the preceding estimations, we get

$$\|x_{k+1} - x_*\| \leq c \|x_k - x_*\|, \tag{30}$$

where, $c = \phi_2(S_0) \in [0, 1)$ leading to $\lim_{m \rightarrow \infty} x_k = x_*$, and $x_{k+1} \in B(x_*, r)$.

Then consider $T = \int_0^1 F'(x_* + \theta(u - x_*)) d\theta$ for $u \in D_1$ with $F(u) = 0$. Using (a_2) and (a_5) we get

$$\begin{aligned}
\|F'(x_*)^{-1}(T - F'(x_*))\| &\leq \int_0^1 W_0(\theta \|u - x_*\|) d\theta \\
&\leq \int_0^1 W_0(\theta r) d\theta \\
&< 1.
\end{aligned} \tag{31}$$

Finally by the invertability of T and the estimation

$$0 = F(u) - F(x_*) = T(u - x_*),$$

we get

$$u = x_*.$$

□

4. Numerical Conclusion

In this section, We have done the comparative study of the CHN method along with the classical Newton's method (3), and third order Weerakoon et al. method [19] whose iterative expression is given by

$$\begin{cases} y_k = x_k - F'(x_k)^{-1} F(x_k), \\ x_{k+1} = x_k - (F'(x_k) + F'(y_k))^{-1} 2F(x_k), \quad k = 0, 1, 2, \dots \end{cases} \tag{32}$$

Example 1 : Let $D = (-1, 1)$, consider $F : D \rightarrow \mathbb{R}$ be given as

$$F(x) = e^x - 1.$$

Hence, we get

$$F'(x) = e^x.$$

MATLAB 2007 is used for the calculations with the stopping criterion $|x_{k+1} - x^*| + |f(x_{k+1})| < 10^{-14}$.

The initial approximation is 1.0 and approximate solution is 0. The numerical solution of example 1 by 2nd order Newton's method (2), 3th order method of Weerakoon et al. [19] and 3th order CHN method (1) is presented in Table 1. These results show that starting with the point 1.0, the CHN method is well competing to the other method in converging to root 0.

Table 1
Display of competing methods example 1

Method	N	x	$f(x)$
Newton Method	1	1.00000000	1.71828182845905
	2	0.36787944117144	0.44466786100977
	3	0.06008006872679	0.06192156984951
	4	0.00176919944264	0.00177076539934
	5	$1.564110789898428 \times 10^{-06}$	$1.564112013019425 \times 10^{-06}$
	6	$1.223321565989411 \times 10^{-12}$	$1.223243728531998 \times 10^{-12}$
	7	$7.783745890945912 \times 10^{-17}$	0.00000000
Weerakoon Method	1	1.00000000	1.71828182845905
	2	0.17448830438386	0.19063681707241
	3	0.00158415756260	0.00158541300305
	4	$1.323865209948080 \times 10^{-09}$	$1.323865239655220 \times 10^{-09}$
	5	$-2.883082976183813 \times 10^{-17}$	0.00000000
CHN Method	1	1.00000000	1.71828182845905
	2	0.24514254757422	0.27780344800498
	3	0.00698738950784	0.00701185827164
	4	$1.979112795331278 \times 10^{-07}$	$1.979112991268295 \times 10^{-07}$
	5	$-9.258546569522333 \times 10^{-18}$	0.00000000

Example 2 : Let $D = X = Y = \mathbb{R}^2$. Consider an operator $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as

$$F(x) = (-s^2 + 1/3, -t^2 + 1/3), \forall x = (s, t) \in \mathbb{R}^2.$$

Table 2
Display of competing methods example 2

Method	N	x	Y	$f(x,y)$	$g(x,y)$
Newton Method	1	0.500000	0.500000	0.0833333	0.0833333
	2	0.583333	0.583333	-0.00694444	-0.00694444
	3	0.577381	0.577381	-0.0000354308	-0.0000354308
	4	0.57735	0.57735	-9.41408×10^{-10}	-9.41408×10^{-10}
	5	0.57735	0.57735	0.000000	0.000000
Weerakoon Method	1	0.500000	0.500000	0.0833333	0.0833333
	2	0.576923	0.576923	0.000493097	0.000493097
	3	0.57735	0.57735	6.75901×10^{-11}	6.75901×10^{-11}
	4	0.57735	0.57735	0.000000	0.000000
CHN Method	1	0.500000	0.500000	0.0833333	0.0833333
	2	0.576471	0.576471	0.00101499	0.00101499
	3	0.57735	0.57735	1.18221×10^{-9}	1.18221×10^{-9}
	4	0.57735	0.57735	0.000000	0.000000

The starting vector is $[0.5, 0.5]$ and approximate solution is $[0.57735, 0.57735]$. The numerical solution of example 2 by methods are given in Table 2. Hence, the CHN method is converging to root in very well manner.

5. Dynamics of Methods

We have studied fractal patterns and dynamics of the function $F(x) = (e^x - 1)$ by using different iterative methods ([4], [1]). For this purpose, we have taken a square $R \times R = [-5.0, 5.0] \times [-5.0, 5.0]$ of 500×500 points of tolerance- $|F(x_k)| < 5 \times 10^{-2}$ with maximum of 21 iterations. we have described the basins of attraction with a fixed color for Newton's method, method of Weerakoon et al. [19] and CHN method to find the complex roots (Fig. 1-3).

1. The basins of attraction for all the methods look almost similar and have fractal Julia set.
2. The Julia set with the red colour show the failure of the method. The Newton's method have the largest area of Julia set with red color while the CHN method contains smallest red color area.
3. Again the Fatou set with blue color shows the basins of the methods. The blue color area shows that all the methods contain almost equal Fatou set.

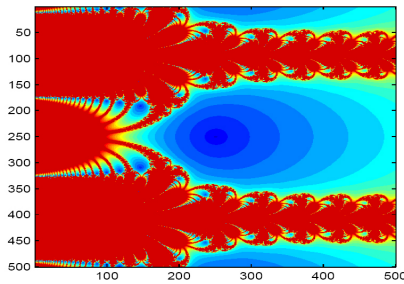


Fig. 1

Basin of attraction for $e^x - 1$ by
Newton method

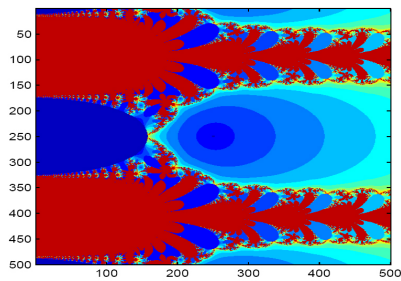


Fig. 2

Basin of attraction for $e^x - 1$ by
Weerakoon method.

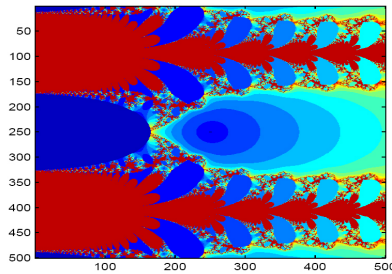


Fig. 3

Basin of attraction for $e^x - 1$ by Proposed method.

6. Conclusion

We discussed a third-order Newton-like method with two local convergence analysis first. Local convergence analysis I demands the third-order derivative but the second version needs only the first-order derivative. Theoretical results are checked by the numerical examples and numerical results are examined with the basins of attraction for a selected example. All the results (Theoretical, numerical, dynamical) are fruitful for the further study of Newton-like methods.

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Received June, 2021

Revised November, 2021