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New problem of Lane Emden type via fractional calculus

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Abstract

We study a nonlinear singular differential equation of Lane Emden type. The problem used the Caputo derivatives and Riemann-Liouville integrals. We first prove an existence and uniqueness result. Another main result is established. At the end, the discussion of two examples to show the applicability of the main results is given. This paper has some relationship with some other papers that have already been published by the same authors.

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1. Introduction

The differential equations of fractional order is of great importance and many researchers have interested in their applications. One can read the references [5, 7, 11, 18] for more details. The differential equations

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with singularities have also importance in applied sciences, see the papers for instance [1, 2, 3, 14] for details. In this paper we study a part of the standard Lane-Emden equation which has a considerable importance in astrophysics, see [8, 16, 17] and the reference therein. The standard Lane Emden equation is:

$$V''(t)+\frac{a}{t}V'(t)+f(t,V(t))=g(t),\ t\in(0,1],\ a\geq 0,$$

with

$$V(0)=b_1,\ V'(0)=b_2,$$

where b_1, b_2 are constants, f and g are continuous functions (see [17]).

In [12], the authors have studied the following problem:

$$\begin{cases} D^{\alpha}X(t)+\dfrac{k}{t^{\alpha-\beta}}D^{\beta}X(t)+f(t,X(t))=g(t),t\in(0,1], \\ \qquad k\geq 0,1<\alpha\leq 2,0<\beta\leq 1, \end{cases}$$

with

$$X(0)=X_0,X'(0)=X_1,$$

where u_0 and u_1 are constants, f and g are continuous functions and the derivatives are of Riemann-Liouville. The collocation method has been used to present some numerical solutions for the above proposed problem.

In [8, 9], the author has studied the problem given by:

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\dfrac{a}{t}\right)V(t)+f(t,V(t))=g(t), \\ \quad 0<\alpha,\beta\leq 1,0<t\leq 1,a\geq 0, \end{cases}$$

and he has considered that $V(0)=V(1)=V(r)=0,0<r<1;$ the existence and stability analysis have been studied.

In [4], the two authors have proved the existence of solutions and the Ulam stabilities for the non homogeneous system:

$$\begin{cases} D^{\beta_1}(D^{\alpha_1}+b_1g_1(t))x_1(t)+f_1(t,x_1(t),x_2(t)) \\ \qquad =\omega_1S_1(t,x_1(t),x_2(t)),0<t<1, \\ D^{\beta_2}(D^{\alpha_2}+b_2g_2(t))x_2(t)+f_2(t,x_1(t),x_2(t)) \\ \qquad =\omega_2S_2(t,x_1(t),x_2(t)),0<t<1, \\ \quad x_k(0)=0,D^{\alpha}x_k(1)+b_kg_k(1)x_k(1)=0, \end{cases}$$

such that $0 < \beta_k < 1, 0 < \alpha_k < 1, b_k \geq 0, 0 < \omega_k < \infty, k = 1,2$ and the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo. The functions $f_k : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $S_k : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $g_k :]0,1] \to [0,+\infty)$ is continuous and singular at $t = 0$.

In [6], the three authors have treated the following problem

$$\begin{cases} D^{\alpha_1}\left(D^{\alpha_2}\ldots\left(D^{\alpha_n}\left(D^{\beta}+\dfrac{k}{t^{\lambda}}\right)\right)\ldots\right)S(t)+f(t,S(t),D^{\delta}S(t))+g(t,S(t),I^{\rho}S(t)) \\ +h(t,S(t))=l(t), t\in]0,1[, \\ S(0)=0, \\ S(1)=\theta, \\ D^{\alpha_n}(D^{\beta}S(0))=0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}S(0)))=0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4}\ldots(D^{\alpha_n}(D^{\beta}S(0)))\ldots)=0, \\ D^{\alpha_2}(D^{\alpha_3}\ldots(D^{\alpha_n}(D^{\beta}S(1)+\phi_{k,\lambda}(1)S(1)))\ldots)=0, \\ k>0, \end{cases}$$

such that $J := [0,1]$, $0 \leq \beta \leq 1$, $0 \leq \alpha_i < 1$; $i = 1,2,\ldots,n$, $\delta < min(\beta, \alpha_i)$, $\phi_{k,\lambda}(t) = \frac{k}{t^{\lambda}}$, the derivatives are in the sense of Caputo, I^{ρ} denotes the Riemann-Liouville integral of order ρ. Some results on the existence f solutions have been proved by the authors.

In our present paper, we pass to generalize problem of [6] and we study the following problem:

$$\begin{cases} D^{\alpha_1}\left(D^{\alpha_2}\ldots\left(D^{\alpha_n}\left(D^{\beta}+\dfrac{k}{t^{\mu}}\right)\right)\ldots\right)u(t)+A(t)f_1(u(t),D^{\delta}u(t)) \\ +B(t)f_2(u(t),I^{\rho}u(t))+f_3(t,u(t))=f_4(t), t\in]0,1[, \\ u(0)=0, \\ u(1)=\theta, \\ D^{\beta}u(0)=0, \\ D^{\alpha_n}(D^{\beta}u(0))=0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0)))=0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4}\ldots(D^{\alpha_n}(D^{\beta}u(0)))\ldots)=0, \end{cases} \tag{1}$$

$$\begin{cases} D^{\alpha_2}(D^{\alpha_3}...(D^{\alpha_n}(D^{\beta}u(1)+\phi_{k,\mu}(1)u(1)))...)=0, \\ k>0, 0<\mu<1 \end{cases}$$

where $1<\beta<2$, $0<\alpha_i<1$; $i=1,2,...,n$, $0<\delta\leq min(\alpha_i)$, $\phi_{k,\mu}(t)=\frac{k}{t^{\mu}}$, $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are two given functions, also $f_3 : J\times\mathbb{R}\to\mathbb{R}$ is a given function and f_4, A and B are functions defined on J.

2. On Fractional Caputo Approach

We refer the reader to [6, 10, 13, 15] for more information on fractional calculus preliminaries.

Definition 1 : Let φ be a continues function on [0, 1] and a real $\alpha>0$. The Riemann-Liouville integral is:

$$I^{\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\varphi(s)}{(t-s)^{1-\alpha}}ds. \tag{2}$$

Definition 2 : The Caputo derivative for a function $\varphi\in C^n([0,1],\mathbb{R})$ is:

$$D^{\alpha}\varphi(t)=I^{n-\alpha}\frac{d^n}{dt^n}(\varphi(t)), \tag{3}$$

and then we get

$$D^{\alpha}\varphi(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\varphi^{(n)}(s)}{(t-s)^{1+\alpha-n}}ds, \tag{4}$$

where $n=[\alpha]+1$.

Lemma 3 : [10] *Iff $n\in\mathbb{N}^*\setminus\{1\}$, and $n-1<\alpha<n$, then the solutions of the equation $D^{\alpha}u(t)=0$ are*

$$u(t)=\sum_{i=0}^{n-1}c_it^i,\ (c_i)_{i=0,1,2,..,n-1}\in\mathbb{R}. \tag{5}$$

Lemma 4 : [10] *In the case of $n\in\mathbb{N}^*\setminus\{1\}$ and $n-1<\alpha<n$, we have*

$$I^{\alpha}D^{\alpha}u(t)=u(t)+\sum_{i=0}^{n-1}c_it^i,\ (c_i)_{i=0,1,2,..,n-1}\in\mathbb{R}. \tag{6}$$

We prove the result:

Lemma 5 : *If* $h \in C([0,1])$, *then the solution of*

$$\begin{cases} D^{\alpha_1}\left(D^{\alpha_2}\ldots\left(D^{\alpha_n}\left(D^{\beta}+\frac{k}{t^{\mu}}\right)\right)\ldots\right)u(t)=h(t), t\in]0,1[, \\ u(0)=0, \\ u(1)=\theta, \\ D^{\beta}u(0)=0, \\ D^{\alpha_n}(D^{\beta}u(0))=0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0)))=0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4}\ldots(D^{\alpha_n}(D^{\beta}u(0)))\ldots)=0, \\ D^{\alpha_2}(D^{\alpha_3}\ldots(D^{\alpha_n}(D^{\beta}u(1)+\phi_{k,\lambda}(1)u(1)))\ldots)=0, \\ k>0, 0<\alpha_i<1, 1<\beta<2; i=1,2,\ldots,n \end{cases} \tag{7}$$

can be given by:

$$\begin{aligned} u(t)= & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)}\int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} h(\tau)d\tau - \frac{k}{s^{\mu}}u(s)\right)ds \\ & -\frac{t^{\sum_{i=2}^n \alpha_i+\beta}+t}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^n \alpha_i+\beta+1)}\int_0^1 (1-s)^{\alpha_1-1}h(s)ds \\ & +\left[\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\frac{1}{\Gamma(\sum_{i=1}^n \alpha_i)}\int_0^s (s-\tau)^{\sum_{i=1}^n \alpha_i - 1} h(\tau)d\tau - \frac{k}{s^{\mu}}u(s)\right)ds-\theta\right]t. \end{aligned} \tag{8}$$

Proof : Thanks to Lemma 4, we see that

$$D^{\alpha_2}\ldots(D^{\alpha_n}(D^{\beta}+\frac{k}{t^{\mu}}))\ldots)u(t)=I^{\alpha_1}h(t)-c_0,$$

$$D^{\alpha_3}\ldots(D^{\alpha_n}(D^{\beta}+\frac{k}{t^{\mu}}))\ldots)u(t)=I^{\alpha_1+\alpha_2}h(t)-I^{\alpha_2}c_0-c_1,$$

.
.
.

$$D^{\alpha_n}(D^{\beta}+\frac{k}{t^{\mu}})u(t)=I^{\sum_{i=1}^{n-1}\alpha_i}h(t)-I^{\sum_{i=2}^{n-1}\alpha_i}c_0-I^{\sum_{i=3}^{n-1}\alpha_i}c_1-...-I^{\alpha_{n-1}}c_{n-3}-c_{n-2},$$

$$(D^{\beta}+\frac{k}{t^{\mu}})u(t)=I^{\sum_{i=1}^{n-1}\alpha_i}h(t)-I^{\sum_{i=2}^{n-1}\alpha_i}c_0-I^{\sum_{i=3}^{n-1}\alpha_i}c_1-...-I^{\alpha_{n-1}}c_{n-3}-I^{\alpha_n}c_{n-2}-c_{n-1}.$$

So, we obtain

$$u(t)=I^{\beta}\left(I^{\sum_{i=1}^{n}\alpha_i}h(s)-\frac{k}{s^{\mu}}u(s)\right)(t)-\frac{c_0}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}t^{\sum_{i=2}^{n}\alpha_i+\beta}$$
$$-\frac{c_1}{\Gamma(\sum_{i=3}^{n}\alpha_i+\beta+1)}t^{\sum_{i=3}^{n}\alpha_i+\beta}-\frac{c_2}{\Gamma(\sum_{i=4}^{n}\alpha_i+\beta+1)}t^{\sum_{i=4}^{n}\alpha_i+\beta}-...-\frac{c_{n-2}}{\Gamma(\alpha_n+\beta+1)}t^{\alpha_n+\beta}$$
$$-\frac{c_{n-1}}{\Gamma(\beta+1)}t^{\beta}-c_nt-c_{n+1}. \tag{9}$$

It yields that the following quantities are true:

$$u(0)=0\Rightarrow c_{n+1}=0$$
$$D^{\alpha_2}(D^{\alpha_3}...(D^{\alpha_n}(D^{\beta}u(1)+\phi_{k,\lambda}(1)u(1)))...)=0\Rightarrow c_0=-I^{\alpha_1}h(1)$$
$$D^{\beta}u(0))=0\Rightarrow c_{n-1}=0$$
$$D^{\alpha_n}(D^{\beta}u(0))=0\Rightarrow c_{n-2}=0$$
$$D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0)))=0\Rightarrow c_{n-3}=0$$
$$\vdots$$
$$D^{\alpha_4}(D^{\alpha_5}...(D^{\alpha_n}(D^{\beta}u(0)))...)=0\Rightarrow c_2=0$$
$$D^{\alpha_3}(D^{\alpha_4}...(D^{\alpha_n}(D^{\beta}u(0)))...)=0\Rightarrow c_1=0$$
$$u(1)=\theta\Rightarrow$$
$$c_n=-\frac{\int_0^1(1-s)^{\alpha_1-1}h(s)ds}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}$$

$$+\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\left(\frac{1}{\Gamma(\sum_{i=1}^n\alpha_i)}\int_0^s(s-\tau)^{\sum_{i=1}^n\alpha_i-1}h(\tau)d\tau-\frac{k}{s^\mu}u(s)\right)ds-\theta. \tag{10}$$

Replacing $c_0,c_1,c_2,...,c_{n+1}$ we end our proof. □

We introduce to the reader the space:

$$M:=\{x\in C(J,\mathbb{R}),D^\delta x\in C(J,\mathbb{R})\} \tag{11}$$

and also:

$$\|x\|_M=\|x\|_\infty+\|D^\delta x\|_\infty \tag{12}$$

with,

$$\|x\|_\infty=\sup_{t\in J}|x(t)|,\ \|D^\delta x\|_\infty=\sup_{t\in J}|D^\delta x(t)|. \tag{13}$$

We take the application $T:M\to M$:

$$\begin{aligned}Tu(t)&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}[\frac{1}{\Gamma(\sum_{i=1}^n\alpha_i)}\int_0^s(s-\tau)^{\sum_{i=1}^n\alpha_i-1}(f_4(\tau)-f_3(\tau,u(\tau))\\&-A(\tau)f_1(u(\tau),D^\delta u(\tau))-B(\tau)f_2(u(\tau),I^\rho u(\tau)))d\tau-\frac{k}{s^\mu}u(s)]ds\\&-\frac{t^{\sum_{i=2}^n\alpha_i+\beta}+t}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^n\alpha_i+\beta+1)}\int_0^1(1-s)^{\alpha_1-1}(f_4(s)-f_3(s,u(s))\\&-A(s)f_1(u(s),D^\delta u(s))-B(s)f_2(u(s),I^\rho u(s)))ds\\&+\Bigg[\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(\frac{1}{\Gamma(\sum_{i=1}^n\alpha_i)}\times\int_0^s(s-\tau)^{\sum_{i=1}^n\alpha_i-1}(f_4(\tau)-f_3(\tau,u(\tau))\\&-A(\tau)f_1(u(\tau),D^\delta u(\tau))-B(\tau)f_2(\tau,u(\tau),I^\rho u(\tau)))d\tau-\frac{k}{s^\mu}u(s))ds-\theta\Bigg]t.\end{aligned} \tag{14}$$

3. Two Results

We need now to consider:

$(H1)$: The functions f_1 and f_2 defined on $\mathbb{R}^2$ are continuous and f_3 defined on $J \times \mathbb{R}$ is also continuous.

$(H2)$: There are nonnegative constants L_1, L_2, G_1, G_2 with any , $x_i, x_i^* \in \mathbb{R}$,

$$|f_1(x_1, x_2) - f_1(x_1^*, x_2^*)| \leq \sum_{i=1}^{2} L_i |x_i - x_i^*|, \tag{15}$$

$$|f_2(x_1, x_2) - f_2(x_1^*, x_2^*)| \leq \sum_{i=1}^{2} G_i |x_i - x_i^*|, \tag{16}$$

and there is also one positive number r_0; for any $t \in J$, $x, y \in \mathbb{R}$,

$$|f_3(t,x) - f_3(t,y)| \quad \leq \quad r_0 |x - y|. \tag{17}$$

We design by

$$L' := Max(L_1, L_2), G' := Max(G_1, G_2). \tag{18}$$

$(H3)$: We have non negative constants $C_{f_1}, C_{f_2}, C_{f_3}, a, b$; for any $t \in J$, $x \in \mathbb{R}, y \in \mathbb{R}$, we have

$$|f_1(x,y)| \leq C_{f_1}, |f_2(x,y)| \leq C_{f_2}, |f_3(t,y)| \leq C_{f_3}. \tag{19}$$

$$|A(t)| \quad \leq \quad a. |B(t)| \quad \leq \quad b. \tag{20}$$

$(H4)$: We take: $\| f_4 \|_\infty = C_{f_4}$.

Also, we impose

$$Q1 = 2\left[\left(r_0 + aL' + bG' + \frac{bG'}{\Gamma(\rho+1)}\right)\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)}\right) + \frac{k\Gamma(1-\mu)}{\Gamma(\beta - \mu + 1)}\right], \tag{21}$$

$$Q2 = \left(r_0 + aL' + bG' + \frac{bG'}{\Gamma(\rho+1)}\right)\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta - \delta + 1)}\right.$$

$$+\frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta-\delta+1)\Gamma(\alpha_1+1)}+\frac{2}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(2-\delta)\Gamma(\alpha_1+1)}$$

$$\left.+\frac{2}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)\Gamma(2-\delta)}\right)+k\Gamma(1-\mu)\left(\frac{1}{\Gamma(\beta-\mu-\delta+1)}+\frac{2}{\Gamma(\beta-\mu+1)}\right). \tag{22}$$

We have to prove the theorem:

Theorem 6 : *Suppose* $(H\ i)_{i=2,3,4}$ *are satisfied and also we have* $Q<1$, $Q:=Q1+Q2$. *Thus, (1) has a unique solution.*

Proof : We prove that T is a contraction application.

We take $(x,y)\in M^2$. Then,

$$\|Ty-Tx\|_\infty \le 2\left[\left(r_0+aL'+bG'+\frac{bG'}{\Gamma(\rho+1)}\right)\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)}\right.\right.$$

$$\left.\left.+\frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(\alpha_1+1)}\right)+\frac{k\Gamma(1-\mu)}{\Gamma(\beta-\mu+1)}\right]\|y-x\|_M . \tag{23}$$

Also,

$$D^\delta Tu(t) = \frac{1}{\Gamma(\beta-\delta)}\int_0^t(t-s)^{\beta-\delta-1}\left[\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\int_0^s(s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}(f_4(\tau)-f_3(\tau,u(\tau))\right.$$

$$\left.-A(\tau)f_1(u(\tau),D^\delta u(\tau))-B(\tau)f_2(u(\tau),I^\rho u(\tau)))d\tau-\frac{k}{s^\mu}u(s)\right]ds$$

$$-\frac{t^{\sum_{i=2}^{n}\alpha_i+\beta-\delta}}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta-\delta+1)}\times\int_0^1(1-s)^{\alpha_1-1}(f_4(s)-f_3(s,u(s))$$

$$-A(s)f_1(u(s),D^\delta u(s))-B(s)f_2(u(s),I^\rho u(s)))ds$$

$$-\frac{\Gamma(2)t^{1-\delta}}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(2-\delta)}\int_0^1(1-s)^{\alpha_1-1}(f_4(s)-f_3(s,u(s))$$
$$-A(s)f_1(u(s),D^{\delta}u(s))-B(s)f_2(u(s),I^{\rho}u(s)))ds$$
$$+\Bigg[\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\times\int_0^s(s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}(f_4(\tau)-f_3(\tau,u(\tau))$$
$$-A(\tau)f_1(u(\tau),D^{\delta}u(\tau))-B(\tau)f_2(u(\tau),I^{\rho}u(\tau)))d\tau$$
$$-\frac{k}{s^{\mu}}u(s))ds-\theta\Bigg]\frac{\Gamma(2)}{\Gamma(2-\delta)}t^{1-\delta}, \tag{24}$$

thus, we have

$$\begin{aligned}\| D^{\delta}Ty-D^{\delta}Tx\|_{\infty} \le \Bigg[\left(r_0+aL'+bG'+\frac{bG'}{\Gamma(\rho+1)}\right)&\Bigg(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta-\delta+1)}\\ &+\frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta-\delta+1)\Gamma(\alpha_1+1)}\\ &+\frac{2}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(2-\delta)\Gamma(\alpha_1+1)}\\ &+\frac{2}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)\Gamma(2-\delta)}\Bigg)k\Gamma(1-\mu)\Bigg(\frac{1}{\Gamma(\beta-\mu-\delta+1)}\\ &+\frac{2}{\Gamma(\beta-\mu+1)}\Bigg)\Bigg]\| y-x\|_{M} .\end{aligned} \tag{25}$$

By (23) and (25), we see that

$$\| Ty-Tx\|_{M} \quad \le \quad Q\| x-y\|_{M} .$$

□

Next we prove another result:

Theorem 7 : *Suppose that* $(H1)$, $(H3)$ *and* $(H4)$ *are verified. Therefore, (1) has at least one solution.*

Proof : The following steps are needed for the proof:

Continuous of T : The functions f_1, f_2, f_3 and f_4 are continuous, hence T is continuous.

Boundedness of T : Let us take $r > 0$ and then consider the ball $B_r := \{x \in M; \| x \|_M \leq r\}$. For $y \in B_r$, by of $(H3)$ and $(H4)$, we can write

$$\| Ty \|_\infty \leq 2\left[(C_{f_4} + C_{f_3} + aC_{f_1} + bC_{f_2}) \left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)} \right) + \frac{k\Gamma(1-\mu)}{\Gamma(\beta - \mu + 1)} \right] + |\theta| < +\infty \quad (26)$$

and

$$\begin{aligned} \| D^\delta Ty \|_\infty \leq\ & (C_{f_4} + C_{f_3} + aC_{f_1} + bC_{f_2}) \left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta - \delta + 1)} \right. \\ & + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\ & + \frac{2}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\ & \left. + \frac{2}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) \\ & + k\Gamma(1-\mu)\left(\frac{1}{\Gamma(\beta - \mu - \delta + 1)} + \frac{2}{\Gamma(\beta - \mu + 1)} \right) + |\theta| < +\infty. \quad (27) \end{aligned}$$

The tw inequalities show clearly that $\| Ty \|_M < +\infty$.

So, T is uniformly bounded.

Equicontinuity: We prove that for any bounded set B_r for instance, we obtain that $T(B_r)$ is an equicontinuous set of M.

Let us take B_r of M. Let also $y \in B_r$, and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$
\begin{aligned}
|Ty(t_1)-Ty(t_2)| \le\ & \frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)}\left|t_1^{\sum_{i=1}^{n}\alpha_i+\beta}-t_2^{\sum_{i=1}^{n}\alpha_i+\beta}\right| \\
& + kr\frac{\Gamma(1-\mu)}{\Gamma(\beta-\mu+1)}\left|t_1^{\beta-\mu}-t_2^{\beta-\mu}\right| \\
& + \frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(1+\alpha_1)}\left(\left|t_1^{\sum_{i=2}^{n}\alpha_i+\beta}-t_2^{\sum_{i=2}^{n}\alpha_i+\beta}\right|+|t_1-t_2|\right) \\
& + \left(\frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)}+kr\frac{\Gamma(1-\mu)}{\Gamma(\beta-\mu+1)}+\theta\right)|t_1-t_2|
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
& |D^{\delta}Ty(t_1)-D^{\delta}Ty(t_2)| \\
& \le \frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta-\delta+1)}\left|t_1^{\sum_{i=1}^{n}\alpha_i-\delta+\beta}-t_2^{\sum_{i=1}^{n}\alpha_i-\delta+\beta}\right| \\
& + kr\frac{\Gamma(1-\mu)}{\Gamma(1+\beta-\mu-\delta)}\left|t_1^{\beta-\mu-\delta}-t_2^{\beta-\mu-\delta}\right| \\
& + \frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(1+\alpha_1)}\left|t_1^{\sum_{i=2}^{n}\alpha_i+\delta-\beta}-t_2^{\sum_{i=2}^{n}\alpha_i+\delta-\beta}\right| \\
& + \frac{2(C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2})}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)\Gamma(\beta-\delta+1)\Gamma(\alpha_1+1)}\left|t_1^{1-\delta}-t_2^{1-\delta}\right| \\
& + \left(\frac{C_{f_4}+C_{f_3}+aC_{f_1}+bC_{f_2}}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)}+kr\frac{\Gamma(1-\mu)}{\Gamma(1+\beta-\mu)}+\theta\right)\frac{2}{\Gamma(2-\delta)} \\
& \times \left|t_1^{1-\delta}-t_2^{1-\delta}\right|.
\end{aligned}
\tag{29}
$$

By Ascoli-Arzela theorem, we state that T is completely continuous.

Boundedness of $S_{\gamma}=\{x\in X: x=\gamma\, Tx, \gamma\in]0,1[\}$.

Let $y \in S_\gamma$. Then one can see that $y = \gamma T y$ for some $0 < \gamma < 1$. Hence, we can state

$$\| y \|_\infty \leq \gamma \left(2 \left[(C_{f_4} + C_{f_3} + aC_{f_1} + bC_{f_2}) \left(\frac{1}{\Gamma(\sum_{i=1}^{n} \alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n} \alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)} \right) + \frac{k\Gamma(1-\mu)}{\Gamma(\beta - \mu + 1)} \right] + |\theta| \right). \tag{30}$$

One can say also that the inequality holds:

$$\| D^\delta y \|_\infty \leq \gamma \left((C_{f_4} + C_{f_3} + aC_{f_1} + bC_{f_2}) \left(\frac{1}{\Gamma(\sum_{i=1}^{n} \alpha_i + \beta - \delta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n} \alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{2}{\Gamma(\sum_{i=2}^{n} \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{2}{\Gamma(\sum_{i=1}^{n} \alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)} \right) + k\Gamma(1-\mu) \left(\frac{1}{\Gamma(\beta - \mu - \delta + 1)} + \frac{2}{\Gamma(\beta - \mu + 1)} \right) + |\theta| \right). \tag{31}$$

Tanks to Schaefer theorem, we see that T has at least one fixed point. $\square$

Example 8 :

$$\left\{ \begin{array}{l} D^{0.3}(D^{0.2}(D^{0.4}(D^{\frac{1}{2}}(D^{\frac{1}{10}}(D^{0.6}(D^{\frac{3}{2}} + \frac{1}{5t^{0.05}}))))))y(t) + \frac{t}{80}\left(\frac{|y(t) + D^{10^{-2}}y(t)|}{192\pi(1 + |y(t) + D^{10^{-2}}y(t)|)} \right) \\ + \frac{t^2+1}{320}\left(\frac{\sin y(t) + \sin I^{\frac{1}{2}}y(t)}{240e} \right) + \frac{\cos y(t)}{200e^t} + e^{t^2} = 2t,\ t \in]0,1[, \end{array} \right.$$

$$
\begin{cases}
y(0)=0,\\
y(1)=2,\\
D^{\frac{3}{2}}y(0)=0,\\
D^{0.6}(D^{\frac{3}{2}}y(0))=0,\\
D^{0.1}(D^{0.6}(D^{\frac{3}{2}}y(0)))=0,\\
D^{\frac{1}{2}}(D^{0.1}(D^{0.6}(D^{\frac{3}{2}}y(0))))=0,\\
D^{0.4}(D^{\frac{1}{2}}(D^{0.1}(D^{0.6}(D^{\frac{3}{2}}y(0)))))=0,\\
D^{0.2}(D^{0.4}(D^{\frac{1}{2}}(D^{0.1}(D^{0.6}(D^{\frac{3}{2}}y(1)+\phi_{\frac{1}{5},0.05}(1)y(1))))))=0,\\
\phi_{\frac{1}{5},0.05}(t)=\dfrac{1}{5t^{0.05}},
\end{cases}
$$

where

$$
\begin{aligned}
f_1(u_1,u_2)&=\frac{|u_1+u_2|}{192\pi(1+|u_1+u_2|)},\\
f_2(u_1,u_2)&=\frac{\sin u_1+\sin u_2}{240e},\\
f_3(t,u)&=\frac{\cos u}{200e^t}+e^{t^2},\\
f_4(t)&=2t,\\
A(t)&=\frac{t}{80},\\
B(t)&=\frac{t^2+1}{320},
\end{aligned}
$$

and

$$
\begin{aligned}
D1&=0.3231, D2=0.4859,\\
D&=D1+D2=0.809.
\end{aligned}
$$

The conditions of Theorem 6 hold. This example has a unique solution on [0, 1].

The second example is:

Example 9 :

$$
\begin{cases}
D^{0.2}(D^{\frac{4}{5}}(D^{\frac{7}{10}}(D^{0.3}(D^{\frac{5}{3}}+\frac{1}{3t^{0.01}}))))y(t)+\frac{t}{5}\left(\frac{|y(t)|}{80e^2(1+|y(t)|)}+\frac{\cos D^{10^{-1}}y(t)}{40}\right) \\
+\frac{t^2+\pi}{2}\left(\frac{\cos y(t)+\sin I^{\frac{3}{5}}y(t)}{48\ln 2}\right)+\frac{\sin y(t)}{50e^{t^4}+1}+\ln(t+2)=t,\ t\in]0,1[, \\
y(0)=0, \\
y(1)=\frac{1}{2}, \\
D^{\frac{5}{3}}y(0)=0, \\
D^{0.3}(D^{\frac{5}{3}}y(0))=0, \\
D^{\frac{7}{10}}(D^{0.3}(D^{\frac{5}{3}}y(0)))=0, \\
D^{\frac{4}{5}}(D^{\frac{7}{10}}(D^{0.3}(D^{\frac{5}{3}}y(1)+\phi_{\frac{1}{3},0.01}(1)y(1))))=0, \\
\phi_{\frac{1}{3},0.01}(t)=\frac{1}{3t^{0.01}},
\end{cases}
$$

where

$$
\begin{aligned}
f_1(u_1,u_2)&=\frac{|u_1|}{80e^2(1+|u_1|)}+\frac{\cos u_2}{40}, \\
f_2(u_1,u_2)&=\frac{\cos u_1+\sin u_2}{48\ln 2}, \\
f_3(t,u)&=\frac{\sin y(t)}{50e^{t^4}+1}+\ln(t+2), \\
f_4(t)&=t, \\
A(t)&=\frac{t}{5}, \\
B(t)&=\frac{t^2+\pi}{2},
\end{aligned}
$$

and

$$
|f_1(u_1,u_2)|\le\frac{1+2e^2}{80e^2},\ |f_2(u_1,u_2)|\le\frac{1}{24\ln 2},\ |f_3(t,u)|\le\frac{1}{50e}+\ln 3,
$$

$$
\|f_4\|_\infty=1,\ |A(t)|\le\frac{1}{5},\ |A(t)|\le\frac{1+\pi}{2}.
$$

In this case, Theorem 7 allows us to say that this example has at least one solution on [0, 1].

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