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Existence, uniqueness and stability results for coupled systems of fractional integro-differential equations with fixed and nonlocal anti-periodic boundary conditions

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for a new class of coupled system of two Caputo fractional derivatives of different orders and a Riemann-Liouville type integral with fixed and nonlocal anti-periodic boundary conditions, by using the fixed point approach in generalized metric spaces. The Ulam's type stability of the proposed coupled system is also studied. An example is provided to illustrate the obtained theory.

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I. Introduction

Fractional differential equations arise as an excellent tool in the mathematical modeling of systems and processes of many fields of science and engineering such as in physics, chemistry, control, risk theory, circuit theory, viscoelasticity, electromagnetic, etc. For more details about fractional calculus and applications, we refer the reader to the books [1]–[4] and the papers [5]–[8] and the references therein. The subject of fractional differential equations is gaining much importance and attention. For some recent work of such subject, see, for instance, [9]–[12] and the references therein. On the other hand, the study of a coupled differential system of fractional order is also very significant in view of their occurrence in the mathematical modeling of physical phenomena [13]–[15]. In [16] Chalishajar and Kumar investigated the existence and uniqueness of the solutions to a fractional order nonlinear coupled system with integral boundary conditions. Mahmudov et al. [17] studied the existence of the solution for a coupled system of fractional differential equations with integral boundary conditions. Ahmad et al. [18] obtained existence and uniqueness of solution for a coupled impulsive Hilfer-Hadamard type fractional differential system by using Kransnoselskiis fixed point theorem. Houas and Saadi [19] discuss the existence of solutions for a coupled system of nonlinear fractional differential equations involving two Riemann-Liouville fractional orders.

Boundary value problems of fractional-order differential equations and inclusions supplemented with several kinds of conditions such as nonlocal and antiperiodic boundary conditions have extensively been investigated by many researchers. The nonlocal boundary conditions are found to be of great utility in modeling the changes happening within the domain of the given scientific phenomena, where the classical initial/boundary conditions fail to describe some peculiar phenomena occurring inside the domain [20]. On the other hand, the anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention, see a survey paper [21] and the paper [22] and the references cited therein.

Moreover, another important research area in mathematical analysis which has recently gained more attention is the Ulam-Hyers type stability analysis of the differential equations of integer and non-integer order. The Ulam-Hyers type stability was first introduced by Ulam [23] in 1940 and next by Hyers [24] in 1941. Since then, many researchers were interested in Ulam-type stability. For instance, Ali et al. [25] obtained some results for Hyers-Ulam type stability of the coupled system of implicit impulsive boundary value problems of fractional differential equations. Khan et al. [26] studied the Hyers-Ulam stability for a coupled system of fractional differential equations in Caputo's sense with nonlinear p-Laplacian operator. Ali et al. [28] investigated different kinds of Ulam stability such as Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ula-Hyers-Rassias stability for a class of fractional order boundary value problem for implicit fractional differential equations with Riemann-Liouville derivative.

Recently, Ahmad et al. [29] considered the existence and uniqueness of solutions for the following boundary value problem integro-differential equation with two caputo fractional orders supplemented with a combination of fixed and nonlocal anti-periodic boundary conditions:

$$\begin{cases} {}^{c}D^{q}\ [({}^{c}D^{p}+\alpha)y(t)+\beta I^{\sigma}g(t,y(t))] \ = \ f(t,\ y(t)), 1<p,\ q\leq 2,\ t\in 2\ [a,b], \\ y(a)+y(b)=0,\ y'(a)+y'(b)=0, y(\xi)+y(\eta)=0,\ y'(\xi)+y'(\eta)=0, \end{cases} \tag{1}$$

where ${}^{c}D^{\varepsilon}$ denotes the Caputo fractional differential operator of order $\varepsilon(\varepsilon=p,\ q)$, α and β are real numbers, $\sigma>0, I^{\sigma}$ is the Riemann-Liouville fractional integral, $g,f:\ [a,b]\ \mathrm{x}\ \mathbb{R}\to\mathbb{R}$ are given continuous functions, and $-\infty<a<\xi<\eta<b<\infty$.

The present paper is motivated by the above papers and is aimed to study the following system of nonlinear fractional differential equations:

$$\begin{cases} {}^{c}D^{q}\ [({}^{c}D^{p}+\alpha)y(t)+\beta I^{\sigma}g(t,y(t))] \ = \ f(t,\ y(t)), 0<p\leq 1, 1<q\leq 2,\ t\in[a,b], \\ y(a)+y(b)=0,\ y(\xi)+y(\eta)=0,\ y'(\xi)+y'(\eta)=0, \end{cases} \tag{2}$$

where ${}^{c}D^{p}$, ${}^{c}D^{q}$, the quantities $\alpha,\beta,\ \sigma, I^{\sigma}, g, f, \xi, \eta$ are the same as defined in (1). Afterward, we study the following coupled system of nonlinear fractional differential equations:

$$\begin{cases} {}^{c}D^{q_1}[({}^{c}D^{p_1}+\alpha_1)u(t)+\beta_1 I^{\sigma_1}g_1(t,u(t),v(t))] \ = f_1(t,u(t),v(t)), 0<p_1\leq 1, \\ \qquad\qquad\qquad\qquad 1<q_1\leq 2,\ t\in[a,b], \\ {}^{c}D^{q_2}[({}^{c}D^{p_2}+\alpha_2)u(t)+\beta_2 I^{\sigma_2}g_2(t,u(t),v(t))] \ = f_2(t,u(t),v(t)), 0<p_2\leq 1, \\ \qquad\qquad\qquad\qquad 1<q_2\leq 2,\ t\in[a,b], \\ u(a)+u(b)=0,\ u(\xi)+u(\eta)=0,\ u'(\xi)\ +u'(\eta)\ =\ 0, \\ v(a)+v(b)=0,\ v(\xi)+v(\eta)=0,\ v'(\xi)\ +v'(\eta)\ =\ 0, \end{cases} \tag{3}$$

where ${}^{c}D^{\varepsilon}$ denotes the Caputo fractional differential operator of order $\varepsilon\,(\varepsilon = p_1, p_2, q_1, q_2)$, α_1, α_2, β_1 and β_2 are real numbers, $\sigma_1, \sigma_2 > 0$, $I^{\sigma}(\sigma = \sigma_1, \sigma_2)$ is the Riemann-Liouville fractional integral, $g_1, f_1, g_2, f_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $-\infty < a < \xi < \eta < b < \infty$.

The rest of this paper is organized as follows. In Section II, we recall some notations, basic definitions and some fundamental facts of some related definitions of fractional calculus. The main results for the existence of solution of the coupled system of two Caputo fractional differential equations with fixed and nonlocal anti-periodic boundary conditions, using the Banach fixed point theorem and Leray-Schauder alternative fixed point theorem type are given in Section III. The study of Ulams type stability for the coupled system is presented in Section IV. In Section V, an example is provided to demonstrate the application of our main results. Finally, a conclusion is given to simply recall our studies and results obtained in Sections III and IV.

II. Preliminaries

For completeness, in this section, we will give some notations, basic definitions and some fundamental facts of some related definitions of fractional calculus which can been founded in [1].

Definition II.1 : [1] The Riemann-Liouville fractional integral I_a^{α} of order $\alpha > 0$ for a function $v \in L_1[a,b]$, $-\infty < a < b < +\infty$, existing almost everywhere on $[a, b]$, is defined by

$$I_a^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} v(s)\,\mathrm{ds}, \tag{4}$$

where Γ denotes the Euler gamma function.

Definition II.2 : [1] Let $v, v^{(m)} \in L_1[a,b]$. Then the Riemann-Liouville fractional derivative D_a^{α} of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, existing almost everywhere on $[a,b]$, is defined as

$$D_a^{\alpha}\, v(t) = \frac{d^m}{\mathrm{dt}^m} T_a^{m-\alpha}\, v(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{\mathrm{dt}^m} \int_a^t (t-s)^{m-\alpha-1} v(s)\,\mathrm{ds}. \tag{5}$$

The Caputo fractional derivative ${}^{c}D_a^{\alpha} v$ of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$ is defined as

$$ {}^{c}D_a^{\alpha} v(t) = D_a^{\alpha}\left[v(t) - v(a) - v'(a)\frac{(t-a)}{1!} - \cdots - v^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!} \right]. $$

Definition II.3 : If $v \in AC^m[a,b]$, then the Caputo fractional derivative ${}^{c}D_a^{\alpha} v$ of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, existing almost everywhere on [*a*, *b*], is defined as

$$ {}^{c}D_a^{\alpha} v(t) = I_a^{m-\alpha} v^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} v^{(m)}(s)\,\mathrm{ds}. $$

In passing we remark that the fractional integral and derivative operators I_a^{α} and ${}^{c}D_a^{\alpha}$ are respectively written as I^{α} and ${}^{c}D^{\alpha}$ for the sake of convenience. We notice that the Caputo derivative of a constant is zero.

The following properties are useful for the sequel.

Lemma II.4 : *Let* $\alpha, \beta \geq 0$. *If v is continuous, then* $I^{\alpha} I^{\beta} v = I^{\alpha+\beta} v$.

Lemma II.5 : *Let* $n \in \mathbb{N}^{\star}$, $\alpha \in (n-1, n)$. *Then, the general solution of the fractional differential equation* ${}^{c}D^{\alpha} v(t) = 0$ *is given by*

$$ v(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c^{n-1} t^{n-1}, $$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

Now, we state the Leray-Schauder alternative fixed point theorem type in generalized Banach space.

Theorem II.6 : [31] *Let X be a generalized Banach space and let* $G : X \to X$ *be completely continuous operator (i.e., a map restricted to any bounded set in X is compact). Let* $\mathcal{E}(G) = \{x \in X : x = \lambda G(x) \text{ for some } 0 < \lambda < 1\}$. *Then either the set* $\mathcal{E}(G)$ *is unbounded, or G has at least one fixed point.*

Here we prove the following auxiliary lemma that will be used in the next section.

Lemma II. 7 : *Let* $F, G \in C([a,b], \mathbb{R})$. *Then the unique solution for the problem*

$$ \begin{cases} {}^{c}D^{q}[({}^{c}D^{p} + \alpha) y(t) + \beta I^{\sigma} G(t))] = F(t)), 0 < p \leq 1, 1 < q \leq 2, t \in [a,b], \\ y(a) + y(b) = 0, y(\xi) + y(\eta) = 0, y'(\xi) + y'(\eta) = 0, \end{cases} \tag{6} $$

is

$$
\begin{aligned}
y(t) &= I^{p+q}(F(t)) - \alpha I^{p}(y(t)) - \beta I^{\sigma+p}(G(t)) + \theta_{1,p}(t)[\alpha I^{p} y(b) + \beta I^{\sigma+p} G(b) \\
&\quad - I^{p+q} F(b)] + \theta_{2,p}(t)[\alpha (I^{p} y(\xi) + I^{p} y(\eta)) + \beta (I^{\sigma+p} G(\xi) + I^{\sigma+p} G(\eta)) \\
&\quad - (I^{p+q} F(\xi) + I^{p+q} F(\eta))] + \theta_{3,p}(t)[\alpha (I^{p-1} y(\xi) + I^{p-1} y(\eta)) + \beta (I^{\sigma+p-1} G(\xi) \\
&\quad + I^{\sigma+p-1} G(\eta)) - (I^{p+q-1} F(\xi) + I^{p+q-1} F(\eta))], \qquad (7)
\end{aligned}
$$

where

$$
\begin{cases}
\theta_{1,p}(t) = \rho_5 - \dfrac{\rho_3 (t-a)^p}{\Gamma(p+1)} + \dfrac{\rho_1 (t-a)^p}{\Gamma(p+1)} + \dfrac{\rho_1 (t-a)^{p+1}}{\Gamma(p+2)}, \\
\theta_{2,p}(t) = \rho_6 + \dfrac{\rho_3 (t-a)^p}{\Gamma(p+1)} - \dfrac{\rho_1 (t-a)^p}{\Gamma(p+1)} - \dfrac{\rho_1 (t-a)^{p+1}}{\Gamma(p+2)}, \\
\theta_{3,p}(t) = \rho_7 + \dfrac{\rho_4 (t-a)^p}{\Gamma(p+1)} - \dfrac{\rho_2 (t-a)^p}{\Gamma(p+1)} - \dfrac{\rho_2 (t-a)^{p+1}}{\Gamma(p+2)},
\end{cases} \qquad (8)
$$

and it is assumed that $\lambda_0 \neq 0$ *and* $\lambda_0(\gamma_1 - \delta_1) - \lambda_1(\gamma_0 - \delta_0) \neq 0$ *with*

$$
\begin{cases}
\rho_1 = \dfrac{\lambda_0}{\lambda_0(\gamma_1 - \delta_1) - \lambda_1(\gamma_0 - \delta_0)}, \\
\rho_2 = \dfrac{\gamma_0 - \delta_0}{\lambda_0(\gamma_1 - \delta_1) - \lambda_1(\gamma_0 - \delta_0)}, \\
\rho_3 = \dfrac{\rho_1 \lambda_1}{\lambda_0}, \rho_4 = \dfrac{1 + \rho_2 \lambda_1}{\lambda_0}, \\
\rho_5 = \dfrac{1}{2}\left(1 + \gamma_0 \rho_3 - \gamma_1 \rho_1\right), \rho_6 = \dfrac{1}{2}\left(\gamma_1 \rho_1 - \gamma_0 \rho_3\right), \\
\rho_7 = \dfrac{1}{2}\left(\gamma_1 \rho_2 - \gamma_0 \rho_4\right),
\end{cases} \qquad (9)
$$

and the following notation:

$$
\gamma_0 = \frac{(b-a)^p}{\Gamma(p+1)}, \gamma_1 = \frac{a(b-a)^p}{\Gamma(p+1)} + \frac{(b-a)^{p+1}}{\Gamma(p+2)}, \delta_0 = \frac{(\xi-a)^p + (\eta-a)^p}{\Gamma(p+1)},
$$

$$
\lambda_0 = \frac{(\xi-a)^{p-1} + (\eta-a)^{p-1}}{\Gamma(p)}, \delta_1 = \frac{a[(\xi-a)^p + (\eta-a)^p]}{\Gamma(p+1)} + \frac{(\xi-a)^{p+1} + (\eta-a)^{p+1}}{\Gamma(p+2)},
$$

$$
\lambda_1 = \frac{a[(\xi-a)^{p-1} + (\eta-a)^{p-1}]}{\Gamma(p)} + \frac{(\xi-a)^p + (\eta-a)^p}{\Gamma(p+1)}.
$$

Proof : We know that from Lemma II.5 that the general solutions for the problem (6) is defined as

$$\begin{aligned} y(t) &= d_0 + I^p(c_0) + I^p(c_1 t) + I^{p+q}(F(t)) - \alpha I^p(y(t)) - \beta I^{\sigma+p}(G(t)) \\ &= d_0 + \frac{c_0(t-a)^p}{\Gamma(p+1)} + \frac{c_1 a(t-a)^p}{\Gamma(p+1)} + \frac{c_1(t-a)^{p+1}}{\Gamma(p+2)} + I^{p+q}(F(t)) - \alpha I^p(y(t)) \\ &\quad -\beta I^{\sigma+p}(G(t)), \end{aligned} \tag{10}$$

where $c_0, c_1, d_0 \in \mathbb{R}$ are arbitrary constants.

Using the fixed and nonlocal anti-periodic boundary conditions for the problem (6), we find that

$$2d_0 + \gamma_0 c_0 + \gamma_1 c_1 = \alpha I^p y(b) + \beta I^{\sigma+p} G(b) - I^{p+q} F(b), \tag{11}$$

$$\begin{aligned} 2d_0 + \delta_0 c_0 + \delta_1 c_1 &= \alpha\,[I^p y(\xi) + I^p y(\eta)] + \beta\,[I^{\sigma+p} G(\xi) + I^{\sigma+p} G(\eta)] \\ &\quad -[I^{p+q} F(\xi) + I^{p+q} F(\eta)], \end{aligned} \tag{12}$$

$$\begin{aligned} \lambda_0 c_0 + \lambda_1 c_1 &= \alpha\,[I^{p-1} y(\xi) + I^{p-1} y(\eta)] + \beta\,[I^{\sigma+p-1} G(\xi) + I^{\sigma+p-1} G(\eta)] \\ &\quad -[I^{p+q-1} F(\xi) + I^{p+q-1} F(\eta)]. \end{aligned} \tag{13}$$

It follows that

$$\begin{cases} c_0 = -\rho_3(\alpha I^p y(b) + \beta I^{\sigma+p} G(b) - I^{p+q} F(b)) + \\ \rho_3(\alpha\,[I^p y(\xi) + I^p y(\eta)] + \beta\,[I^{\sigma+p} G(\xi) + I^{\sigma+p} G(\eta)] - [I^{p+q} F(\xi) + I^{p+q} F(\eta)]) \\ +\rho_4(\alpha\,[I^{p-1} y(\xi) + I^{p-1} y(\eta)] + \beta\,[I^{\sigma+p-1} G(\xi) + I^{\sigma+p-1} G(\eta)] - [I^{p+q-1} F(\xi) \\ +I^{p+q-1} F(\eta)]), \\ c_1 = \rho_1 + \alpha I^p y(b) + \beta I^{\sigma+p} G(b) - I^{p+q} F(b) - \\ \rho_1(\alpha\,[I^p y(\xi) + I^p y(\eta)] + \beta\,[I^{\sigma+p} G(\xi) + I^{\sigma+p} G(\eta)] - [I^{p+q} F(\xi) + I^{p+q} F(\eta)]) \\ -\rho_2(\alpha\,[I^{p-1} y(\xi) + I^{p-1} y(\eta)] + \beta\,[I^{\sigma+p-1} G(\xi) + I^{\sigma+p-1} G(\eta)] - [I^{p+q-1} F(\xi) \\ +I^{p+q-1} F(\eta)]) \\ d_0 = .\rho_5(\alpha I^p y(b)] + \beta I^{\sigma+p} G(b) - I^{p+q} F(b)) \\ +\rho_6(\alpha\,[I^p y(\xi) + I^p y(\eta)] + \beta\,[I^{\sigma+p} G(\xi) + I^{\sigma+p} G(\eta)] - [I^{p+q} F(\xi) + I^{p+q} F(\eta)]) \\ +\rho_7(\alpha\,[I^{p-1} y(\xi) + I^{p-1} y(\eta)] + \beta\,[I^{\sigma+p-1} G(\xi) + I^{\sigma+p-1} G(\eta)] - [I^{p+q-1} F(\xi) \\ +I^{p+q-1} F(\eta)]), \end{cases}$$

with $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \mathrm{rho}_6$ and ρ_7 are given as in (9).

Substituting the values of c_0, c_1, d_0 in (10), we obtain the solution (7).

On the other hand, it is easy to prove that, if $y(t)$ is a solution of the integral equation (7), then $y(t)$ is also a solution of the problem (6).

III. Main Result

A. Existence results for fractional-Order Integro-Differential equations

In this subsection, we consider a system of nonlinear fractional-Order Integro-Differential equations with fixed and nonlocal anti-periodic boundary conditions (2).

Let $E = C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b] \to \mathbb{R}$ endowed with sup-norm $\| y \| = \sup\{ | y(t) |, t \in [a,b]\}$.

Definition III.1 : A function $y \in AC([a,b],\mathbb{R})$ is said to be a solution of the problem (2) if y fulfills the equation ${}^cD^q [({}^cD^p + \alpha) y(t) + \beta I^\sigma g(t, y(t))] = f(t, y(t))$ in $[a,b]$ and the conditions $y(a) + y(b) = 0, y(\xi) + y(\eta) = 0, y'(\xi) + y'(\eta) = 0.$

Now, we give the following hypotheses.

(H_1) Let $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and let there exist constants $L_1, L_2 > 0$, such that $\forall x, y \in \mathbb{R}$, $\forall \in [a,b]$

$$| f(t,x) - f(t,y | \le L_1 \| x - y \| \quad \text{and} \quad | g(t,x) - g(t,y) | \le L_2 \| x - y \|.$$

In the following theorem, we make use of Banach's fixed point theorem.

Theorem III.2 : *Let $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and suppose that the condition (H_1) is satisfied.*

Then, the boundary value problem (2) has a unique solution on $[a,b]$ if

$$\Theta_1 + | \alpha | \Theta_2 + | \beta | \Theta_3 < 1, \tag{14}$$

where

$$\Theta_1 = L_1 \left\{ (1 + \overline{\theta}_1 + 2\overline{\theta}_2) \frac{(b-a)^{p+q+1}}{\Gamma(p+q+1)} + \frac{2\overline{\theta}_3 (b-a)^{p+q}}{\Gamma(p+q)} \right\},$$

$$\Theta_2 = (1 + \overline{\theta}_1 + 2\overline{\theta}_2) \frac{(b-a)^{p+1}}{\Gamma(p+1)} + \frac{2\overline{\theta}_3 (b-a)^{p}}{\Gamma(p)},$$

$$\Theta_3 = L_2 \left\{ (1 + \overline{\theta}_1 + 2\overline{\theta}_2) \frac{(b-a)^{\sigma+p+1}}{\Gamma(\sigma+p+1)} + \frac{2\overline{\theta}_3 (b-a)^{\sigma+p}}{\Gamma(\sigma+p)} \right\},$$

and

$$\overline{\theta}_{1,p} = \sup_{t\in[a,b]} |\theta_{1,p}(t)|, \quad \overline{\theta}_{2,p} = \sup_{t\in[a,b]} |\theta_{2,p}(t)|, \quad \overline{\theta}_{3,p} = \sup_{t\in[a,b]} |\theta_{3,p}(t)|.$$

Proof : We transform the problem (2) into a fixed point problem by introducing an operator $\Theta : E \to E$ as

$$\begin{aligned}\Theta y(t) = {} & I^{p+q}(f(t,y(t))) - \alpha I^{p}(y(t)) - \beta I^{\sigma+p}(g(t,y(t))) + \theta_{1,p}(t)[\alpha I^{p} y(b) \\ & + \beta I^{\sigma+p} g(b,y(b)) - I^{p+q} f(b,y(b))] + \theta_{2,p}(t)[(\alpha\,[I^{p} y(\xi) + I^{p} y(\eta)] \\ & + \beta\,[I^{\sigma+p} g(\xi,y(\xi)) + I^{\sigma+p} g(\eta,y(\eta))] - [I^{p+q} f(\xi,y(\xi)) + I^{p+q} f(\eta,y(\eta)))] \\ & + \theta_{3,p}(t)[(\alpha\,[I^{p-1} y(\xi) + I^{p-1} y(\eta)] + \beta\,[I^{\sigma+p-1} g(\xi,y(\xi)) \\ & + I^{\sigma+p-1} g(\eta,y(\eta))] - [I^{p+q-1} f(\xi,y(\xi)) + I^{p+q-1} f(\eta,y(\eta)))].\end{aligned} \tag{15}$$

Notice that Θ is well defined. Indeed for $t \in [a,b]$, the map $t \to \Theta y(t)$ is continuous. Now we need to show that Θ is a contraction map. Let y, $\overline{y} \in E$,

$$\begin{aligned}\|\Theta y - \Theta\overline{y}\| \le {} & \frac{(b-a)^{p+q}}{\Gamma(p+q+1)} L_1 \|y-\overline{y}\| + |\alpha| \frac{(b-a)^{p}}{\Gamma(p+1)} \|y-\overline{y}\| \\ & + |\beta| \frac{(b-a)^{\sigma+p}}{\Gamma(\sigma+p+1)} L_2 \|y-\overline{y}\| + \overline{\theta}_{1,p}\Bigg[|\alpha| \frac{(b-a)^{p}}{\Gamma(p+1)} \|y-\overline{y}\| \\ & + |\beta| \frac{(b-a)^{\sigma+p}}{\Gamma(\sigma+p+1)} L_2 \|y-\overline{y}\| + \frac{(b-a)^{p+q}}{\Gamma(p+q+1)} L_1 \|y-\overline{y}\| \Bigg] \\ & \overline{\theta}_{2,p}\Bigg[2|\alpha| \frac{(b-a)^{p}}{\Gamma(p+1)} \|y-\overline{y}\| + 2|\beta| \frac{(b-a)^{\sigma+p}}{\Gamma(\sigma+p+1)} L_2 \|y-\overline{y}\| \\ & + 2\frac{(b-a)^{p+q}}{\Gamma(p+q+1)} L_1 \|y-\overline{y}\| \Bigg] \overline{\theta}_{3,p}\Bigg[2|\alpha| \frac{(b-a)^{p-1}}{\Gamma(p)} \|y-\overline{y}\| \\ & + 2|\beta| \frac{(b-a)^{\sigma+p-1}}{\Gamma(\sigma+p)} L_2 \|y-\overline{y}\| + 2\frac{(b-a)^{p+q-1}}{\Gamma(p+q)} L_1 \|y-\overline{y}\| \Bigg] \\ = {} & [\Theta_1 + |\alpha|\Theta_2 + |\beta|\Theta_3] \|y-\overline{y}\|.\end{aligned}$$

As $\Theta_1 + |\alpha|\Theta_2 + |\beta|\Theta_3 < 1$, Θ is a contraction, and hence, by Banach fixed point theorem, Θ has a unique fixed point which is a solution of (2). This completes the proof.

B. Existence results for coupled system of fractional-Order Integro-Differential equations

In this subsection, we establish existence and uniqueness criteria of solutions for the coupled system of nonlinear fractional-Order Integro-Differential equations with fixed and nonlocal anti-periodic boundary conditions (3).

Let $E=\{u/u\in C([a,b],\mathbb{R})\}$ denote the space of all continuous functions from $[a,b]$, $(a\leq b)$, into $\mathbb{R}$ equipped with the norm $\|u\|=\sup\{|u(t)|,t\in[a,b]\}$. Obviously, $(E,\|.\|)$ is a Banach space. Then the product space $(\mathcal{U}=E\times E,\|(u,v)\|)$ is also a Banach space equipped with the norm $\|(u,v)\|=\|u\|+\|v\|$.

Definition III.3 : A function $(u_1,u_2)\in E\times E$ is a solution of the coupled system (3) if it satisfies the coupled system of integral equations

$$\begin{aligned}
u_i(t)=&\,I^{p_i+q_i}(f_i(t,u_i(t),u_i(t)))-\alpha_i I^{p_i}(u_i(t))-\beta_i I^{\sigma_i+p_i}(g_i(t,u_i(t),u_i(t)))\\
&+\theta_{1,p_1}(t)[\alpha_i I^{p_i}u_i(b)+\beta_i I^{\sigma_i+p}g_i(b,u_i(b),u_i(b))-I^{p_i+q_i}f_i(b,u_i(b),u_i(b))+]\\
&\theta_{2,p_i}(t)[\alpha_i[I^{p_i}u_i(\xi)+I^{p_i}u_i(\eta)]+\beta_i[I^{\sigma_i+p_i}g_i(\xi,u_i(\xi),u_i(\xi))\\
&+I^{\sigma_i+p_i}g_i(\eta,u_i(\eta),u_i(\eta))]-[I^{p_i+q_i}f_i(\xi,u_i(\xi),u_i(\xi))\\
&+I^{p_i+q_i}f_i(\eta,u_i(\eta),u_i(\eta))]+\theta_{3,p_i}(t)[\alpha_1[I^{p_i-1}u_i(\xi)+I^{p_i-1}u_i(\eta)]\\
&+\beta_i[I^{\sigma_1+p_i-1}g_i(\xi,u_i(\xi),u_i(\xi))+I^{\sigma_i+p_i-1}g_i(\eta,u_i(\eta),u_i(\eta))]\\
&-[I^{p_i+q_i-1}f_i(\xi,u_i(\xi),u_i(\xi))+I^{p_i+q_i-1}f_i(\eta,u_i(\eta),u_i(\eta))],\ i=1,2.
\end{aligned}\tag{16}$$

Here we establish the existence and uniqueness of the solutions for the boundary value problem (3) by using Banach's contraction mapping principle. We considered the following assumptions:

H_1 The functions $f_i,g_i:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous, for i = 1, 2,

H_2 There exist positive constants k_i, l_i, $i=1,2,3,4$ such that $\forall u,\overline{u},\overline{v},v\in\mathbb{R}$ $\forall t\in[a,b]$:

$$|f_i(t,u,v)-f_i(t,\overline{u},\overline{v}|\leq k_i|u-\overline{u}|+l_i|v-\overline{v}|,$$

and

$$|g_i(t,u,v)-g_i(t,\overline{u},\overline{v}|\leq k_{2+i}|u-\overline{u}|+l_{2+i}|v-\overline{v}|,$$

Define $\sup_{t\in[a,b]} f_1(t,0,0) = N_1 < \infty$, $\sup_{t\in[a,b]} g_1(t,0,0) = N_2 < \infty$, $\sup_{t\in[a,b]} f_2(t,0,0) = M_1 < \infty$ and $\sup_{t\in[a,b]} g_2(t,0,0) = M_2 < \infty$ and for the sake of convenience, we set

$$A_1 = \frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}, A_2 = \frac{(b-a)^{p_1}}{\Gamma(p_1+1)}, A_3 = \frac{(b-a)^{\sigma_1+p_1}}{\Gamma(\sigma_1+p_1+1)}, B_1 = \frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)},$$

$$B_2 = \frac{(b-a)^{p_2}}{\Gamma(p_2+1)}, B_3 = \frac{(b-a)^{\sigma_2+p_2}}{\Gamma(\sigma_1+p_1+1)},$$

$$\Phi_1 = (1+\bar\theta_{1,p_1} + 2\bar\theta_{2,p_1} + 2\bar\theta_{3,p_1})(A_1k_1 + |\alpha_1| A_2 + |\beta_1| A_3k_3), \tag{17}$$
$$\Phi_2 = (1+\bar\theta_{1,p_1} + 2\bar\theta_{2,p_1} + 2\bar\theta_{3,p_1})(A_1l_1 + |\beta_1| A_3l_3), \tag{18}$$
$$\Phi_3 = (1+\bar\theta_{1,p_1} + 2\bar\theta_{2,p_1} + 2\bar\theta_{3,p_1})(A_1N_1 + |\beta_1| A_3N_2), \tag{19}$$
$$\Psi_1 = (1+\bar\theta_{1,p_2} + 2\bar\theta_{2,p_2} + 2\bar\theta_{3,p_2})(B_1k_2 + |\alpha_2| B_2 + |\beta_2| B_3k_4), \tag{20}$$
$$\Psi_2 = (1+\bar\theta_{1,p_2} + 2\bar\theta_{2,p_2} + 2\bar\theta_{3,p_2})(B_1l_2 + |\beta_2| B_3l_4), \tag{21}$$
$$\Psi_3 = (1+\bar\theta_{1,p_2} + 2\bar\theta_{2,p_2} + 2\bar\theta_{3,p_2})(B_1M_1 + |\beta_2| B_3M_2). \tag{22}$$

Theorem III.4 : (*Uniqueness result via Banachs contraction mapping principle*)
Assume that assumptions $(H1)$-$(H2)$ *holds. If* $\Phi_1 + \Phi_2 + \Psi_1 + \Psi_2 < 1$, *where* Φ_1, Φ_2, Ψ_1 *and* Ψ_1 *are given in* (17), (18), (20) *and* (21) *respectively, then there exists a unique solution for* (3) *on the interval* $[a,b]$.

Proof : In view of Lemma (II.7), we define the operator $\Theta : \mathcal{U} \to \mathcal{U}$ defined by

$$\Theta(u(t),v(t)) = (\Theta_1(u,v)(t), \Theta_2(u,v)(t)),$$

where $\Theta_1, \Theta_1 : \mathcal{U} = E \times E \to E$ are as under

$$\begin{aligned}\Theta_1(u,v)(t) = {} & I^{p_1+q_1}(f_1(t,u(t),v(t))) - \alpha_1 I^{p_1}(u(t)) - \beta_1 I^{\sigma_1+p_1}(g_1(t,u(t),v(t))) \\ & + \theta_{1,p_1}(t)[\alpha_1 I^{p_1}u(b) + \beta_1 I^{\sigma_1+p} g_1(b,u(b),v(b)) - I^{p_1+q_1} f_1(b,u(b),v(b))] \\ & + \theta_{2,p_1}(t)[\alpha_1[I^{p_1}u(\xi) + I^{p_1}u(\eta)] + \beta_1[I^{\sigma_1+p_1} g_1(\xi,u(\xi),v(\xi)) \\ & + I^{\sigma_1+p_1} g_1(\eta,u(\eta),v(\eta))] - [I^{p_1+q_1} f_1(\xi,u(\xi),v(\xi)) \\ & + I^{p_1+q_1} f_1(\eta,u(\eta),v(\eta))] + \theta_{3,p_1}(t)[\alpha_1[I^{p_1-1}u(\xi) + I^{p_1-1}u(\eta)] \\ & + \beta_1[I^{\sigma_1+p_1-1} g_1(\xi,u(\xi),v(\xi)) + I^{\sigma_1+p_1-1} g_1(\eta,u(\eta),v(\eta))] \\ & - [I^{p_1+q_1-1} f_1(\xi,u(\xi),v(\xi)) + I^{p_1+q_1-1} f_1(\eta,u(\eta),v(\eta)],\end{aligned} \tag{23}$$

$$
\begin{aligned}
\Theta_2(u,v)(t) = {} & I^{p_2+q_2}(f_2(t,u(t),v(t))) - \alpha_2 I^{p_2}(v(t)) - \beta_2 I^{\sigma_2+p_2}(g_2(t,u(t),v(t))) \\
& + \theta_{1,p_2}(t)[\alpha_2 I^{p_2} u(b) + \beta_2 I^{\sigma_2+p_2} g_2(b,u(b),v(b)) \\
& - I^{p_2+q_2} f_2(b,u(b),v(b))] + \theta_{2,p_2}(t)[\alpha_2 [I^{p_2} u(\xi) + I^{p_2} u(\eta)] \\
& + \beta_2 [I^{\sigma_2+p_2} g_2(\xi,u(\xi),v(\xi)) + I^{\sigma_2+p_2} g_2(\eta,u(\eta),v(\eta))] \\
& - [I^{p_2+q_2} f_2(\xi,u(\xi),v(\xi)) + I^{p_2+q_2} f_2(\eta,u(\eta),v(\eta))] \\
& + \theta_{3,p_2}(t)[(\alpha_2[I^{p_2-1}v(\xi) + I^{p_2-1}v(\eta)] + \beta_2 [I^{\sigma_2+p_2-1} g_2(\xi,u(\xi),v(\xi)) \\
& + I^{\sigma_2+p_2-1} g_2(\eta,u(\eta),v(\eta))] - [I^{p_2+q_2-1} f_2(\xi,u(\xi),v(\xi)) \\
& + I^{p_2+q_2-1} f_2(\eta,u(\eta),v(\eta))]. \qquad (24)
\end{aligned}
$$

Let $(u_2,v_2),(u_1,v_1) \in E \times E$, the for any $t \in [a,b]$, we get

$$
\begin{aligned}
\| \Theta_1(u_2,v_2) - \Theta_1(u_1,v_1) \| \le {} & (1+\overline{\theta}_{1,p_1} + 2\overline{\theta}_{2,p_1} + 2\overline{\theta}_{3,p_1})(A_1k_1 + |\alpha_1| A_2 \\
& + |\beta_1| A_3k_3) \| u_2 - u_1 \| + (1+\overline{\theta}_{1,p_1} + 2\overline{\theta}_{2,p_1} + 2\overline{\theta}_{3,p_1}) \\
& (A_1l_1 + |\beta_1| A_3l_3) \| v_2 - v_1 \|
\end{aligned}
$$

consequently

$$\| \Theta_1(u_2,v_2) - \Theta_1(u_1,v_1) \| \le (\Phi_1 + \Phi_2)(\| u_2 - u_1 \| + \| v_2 - v_1 \|).$$

Similarly, we have

$$\| \Theta_2(u_2,v_2) - \Theta_2(u_1,v_1) \| \le (\Psi_1 + \Psi_2)(\| u_2 - u_1 \| + \| v_2 - v_1 \|).$$

It follows that

$$\| \Theta(u_2,v_2) - \Theta(u_1,v_1) \| \le (\Phi_1 + \Phi_2 + \Psi_1 + \Psi_2)(\| u_2 - u_1 \| + \| v_2 - v_1 \|).$$

Since $\Phi_1 + \Phi_2 + \Psi_1 + \Psi_2 < 1$, therefore, Θ is a contraction operator. So, by Banach's fixed point theorem, the operator Θ has a unique fixed point, which is the unique solution of system (3). This completes the proof.

Next we will prove the existence of solutions by applying the Leray-Schauder alternative fixed point theorem.

Let us introduce the following hypotheses which are used hereafter.

(H_3)Assume that there exist real constants $\widetilde{k_i}$, $\widetilde{li}, \widetilde{m_i}, \widetilde{n_i} \ge 0$ $(i = 0, 1, 2)$ and such that $\forall x_i \in \mathbb{R}, i = 1, 2,$ we have

$$| f_1(t,u,v) | \le \widetilde{k_0} + \widetilde{k_1} |u| + \widetilde{k_2} |v|, | f_2(t,u,v) | \le \widetilde{l_0} + \widetilde{l_1} |u| + \widetilde{l_2} |v|,$$

and

$$| g_1(t,u,v_1) | \le \widetilde{m_0} + \widetilde{m_1} |u| + \widetilde{m_2} |v|, | g_2(t,u,v) | \le \widetilde{n_0} + \widetilde{n_0} |u| + \widetilde{n_2} |v|.$$

For the sake of convenience, we set

$$\Delta_1 = A_1\left(1+\overline{\theta}_{1,p_1} + 2\overline{\theta}_{2,p_1} + 2\frac{(p_1+q_1)}{(b-a)}\overline{\theta}_{3,p_1}\right), \tag{25}$$

$$\Delta_2 = A_2\left(1+\overline{\theta_{1\,p_1}} + 2\overline{\theta_{2\,p_1}} + 2\frac{p_1}{(b-a)}\overline{\theta_{3\,p_1}}\right), \tag{26}$$

$$\Delta_3 = A_3\left(1+\overline{\theta}_{1,p_1} + 2\overline{\theta}_{2,p_1} + 2\frac{(\sigma_1+p_1)}{(b-a)}\overline{\theta}_{3,p_1}\right), \tag{27}$$

$$\Lambda_1 = B_1\left(1+\overline{\theta}_{1,p_2} + 2\overline{\theta}_{2,p_2} + 2\frac{(p_2+q_2)}{(b-a)}\overline{\theta}_{3,p_2}\right), \tag{28}$$

$$\Lambda_2 = B_2\left(1+\overline{\theta_{1\,p_2}} + 2\overline{\theta_{2\,p_2}} + 2\frac{p_2}{(b-a)}\overline{\theta_{3\,p_2}}\right), \tag{29}$$

$$\Lambda_3 = B_3\left(1+\overline{\theta}_{1,p_2} + 2\overline{\theta}_{2,p_2} + 2\frac{(\sigma_2+p_2)}{(b-a)}\overline{\theta}_{3,p_2}\right), \tag{30}$$

$$\begin{aligned}\Delta_0 = \min(1-(\Delta_1\widetilde{k_1}+|\alpha_1|\Delta_2+|\beta_1|\widetilde{m_1}+\Lambda_1\widetilde{l_1}+|\beta_2|\widetilde{n_1}),1-(\Lambda_1\widetilde{l_2}+|\beta_2|\widetilde{n_2}\\ +|\beta_1|\widetilde{m_2}+\Delta_1\widetilde{k_2}+|\alpha_2|\Lambda_2)).\end{aligned} \tag{31}$$

Theorem III.5 : (*Existence result via the Leray-Schauder alternative*) *Assume that* (H_3) *holds. In addition it is assumed that*

$$\begin{aligned}\Delta_1\widetilde{k_1}+|\alpha_1|\Delta_2+|\beta_1|\widetilde{m_1}+\Lambda_1\widetilde{l_1}+|\beta_2|\widetilde{n_1}<1, \Lambda_1\widetilde{l_2}+|\beta_2|\widetilde{n_2}\\ +|\beta_1|\widetilde{m_2}+\Delta_1\widetilde{k_2}+|\alpha_2|\Lambda_2<1,\end{aligned}$$

and

$$1-2|\alpha_1|\Delta_2>0,\quad 1-2|\alpha_2|\Lambda_2>0,$$

where Δ_1, Δ_2, Δ_3, Λ_1, Λ_2 *and* Λ_3 *are given by* (25), (26), (27), (28), (29) *and* (30) *respectively. Then the boundary value problem* (3) *has at least one solution on* $[a,b]$.

Proof : The proof will be divided into several steps.

Step 1 : We show that the operator $\Theta : E\times E \to E\times E$ is completely continuous. Since f_1, f_2, g_1 and g_2 are continuous functions, then by the same method used by Majouba *et al.* [27, Theorem 3.1], we can prove that operator Θ is continuous and compact.

Step 2 : Boundedness of $\mathcal{E}=\{(u,v)\in E\times E\,|\,(u,v)=\lambda\Theta(u,v), 0\le\lambda\le 1\}$.

Let $(u,v)\in\mathcal{E}$, then $(u,v)=\lambda\Theta(u,v)$. For any $t\in[a,b]$, we have

$$u(t)=\lambda\Theta_1(u,v)(t), v(t)=\lambda\Theta_2(u,v)(t).$$

By (H_3) we have

$$\begin{aligned}
|u(t)| &= |\lambda\Theta_1(u,v)(t)| \\
&\le \frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}(\widetilde{k_0}+\widetilde{k_1}\|u\|+\widetilde{k_2}\|v\|)+|\alpha_1|\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}\|u\| \\
&+|\beta_1|\frac{(b-a)^{\sigma_1+p_1}}{\Gamma(\sigma_1+p_1+1)}(\widetilde{m_0}+\widetilde{m_1}\|u\|+\widetilde{m_2}\|v\|) \\
&+\overline{\theta}_{1,p_1}\left\{\frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}(\widetilde{k_0}+\widetilde{k_1}\|u\|+\widetilde{k_2}\|v\|)+|\alpha_1|\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}\|u\|\right. \\
&\left.+|\beta_1|\frac{(b-a)^{\sigma_1+p_1}}{\Gamma(\sigma_1+p_1+1)}(\widetilde{m_0}+\widetilde{m_1}\|u\|+\widetilde{m_2}\|v\|)\right\} \\
&+\overline{\theta}_{2,p_1}\left\{\frac{2(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}(\widetilde{k_0}+\widetilde{k_1}\|u\|+\widetilde{k_2}\|v\|)+2|\alpha_1|\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}\|u\|\right. \\
&\left.+2|\beta_1|\frac{(b-a)^{\sigma_1+p_1}}{\Gamma(\sigma_1+p_1+1)}(\widetilde{m_0}+\widetilde{m_1}\|u\|+\widetilde{m_2}\|v\|)\right\} \\
&+\overline{\theta}_{3,p_1}\left\{\frac{2(b-a)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}(\widetilde{k_0}+\widetilde{k_1}\|u\|+\widetilde{k_2}\|v\|)+2|\alpha_1|\frac{(b-a)^{p_1-1}}{\Gamma(p_1)}\|u\|\right. \\
&\left.+2|\beta_1|\frac{(b-a)^{\sigma_1+p_1-1}}{\Gamma(\sigma_1+p_1)}(\widetilde{m_0}+\widetilde{m_1}\|u\|+\widetilde{m_2}\|v\|)\right\}.
\end{aligned}$$

Hence, we have

$$\|u\|\le(\Delta_1\widetilde{k_0}+|\beta_1|\Delta_3\widetilde{m_0})+(\Delta_1\widetilde{k_1}+|\alpha_1|\Delta_2+|\beta_1|\widetilde{m_1})\|u\|+,(\Delta_1\widetilde{k_2}+|\beta_1|\widetilde{m_2})\|v\|,$$

and

$$\|v\|\le(\Lambda_1\widetilde{l_0}+|\beta_2|\Lambda_3\widetilde{n_0})+(\Lambda_1\widetilde{l_1}+|\beta_2|\widetilde{n_1})\|u\|+(\Lambda_1\widetilde{l_2}+|\alpha_2|\Lambda_2+|\beta_2|\widetilde{n_2})\|v\|,$$

which imply that

$$\begin{aligned}
\|u\|+\|v\| \le\ & (\Delta_1\widetilde{k_0}+|\beta_1|\Delta_3\widetilde{m_0}+\Lambda_1\widetilde{l_0}+|\beta_2|\Lambda_3\widetilde{n_0}) \\
&+(\Delta_1\widetilde{k_1}+|\alpha_1|\Delta_2+|\beta_1|\widetilde{m_1}+\Lambda_1\widetilde{l_1}+|\beta_2|\widetilde{n_1})\|u\| \\
&+(\Delta_1\widetilde{k_2}+|\alpha_2|\Lambda_2+|\beta_1|\widetilde{m_2}+\Lambda_1\widetilde{l_2}+|\beta_2|\widetilde{n_2})\|v\|.
\end{aligned}$$

Consequently, we get

$$\|(u,v)\| \leq \frac{\Delta_1 \widetilde{k_0} + |\beta_1| \Delta_3 \widetilde{m_0} + \Lambda_1 \widetilde{l_0} + |\beta_2| \Lambda_3 \widetilde{n_0}}{\Delta_0},$$

for any $t \in [a,b]$, where Δ_0 is defined by (31), which proves that $\mathcal{E}$ is bounded. Thus, by Theorem (II.6), the operator Θ has at least one fixed point. Hence the proposed system (3) has at least one solution.

IV. HYERS-ULAM stability of the system (3)

In this subsection, we study Ulams type stability for the coupled system (3). For some $\epsilon = (\epsilon_{p_1,q_1}, \epsilon_{p_2,q_2}) > 0$, we consider the following inequality

$$\begin{cases} |{}^cD^{q_1}\left[({}^cD^{p_1} + \alpha_1)u(t) + \beta_1 I^{\sigma_1} g_1(t,u(t),v(t))\right] \\ \qquad\qquad - f_1(t,u(t),v(t))| \leq \varepsilon_{p_1,q_1}, t \in [a,b], \\ |{}^cD^{q_2}\left[({}^cD^{p_2} + \alpha_2)v(t) + \beta_2 I^{\sigma_2} g_2(t,u(t),v(t))\right] \\ \qquad\qquad - f_2(t,u(t),v(t))| \leq \varepsilon_{p_2,q_2}, t \in [a,b]. \end{cases} \tag{32}$$

Definition IV.1 : A function $(u_1,u_2) \in E \times E$ is said to be a solution of the system of inequality (32) if there exist functions $\phi_1, \phi_2 \in C(J,\mathbb{R})$ which depend upon u and v respectively, such that

(A_1) $|\phi_1(t)| \leq \epsilon_{p_1q_1}$, $|\phi_2(t)| \leq \epsilon_{p_2q_2}$, $t \in J$,

(A_2) and

$$\begin{cases} {}^cD^{q_1}[({}^cD^{p_1} + \alpha_1)u(t) + \beta_1 I^{\sigma_1} g_1(t,u(t),v(t))] = f_1(t,u(t),v(t)) + \phi_1(t), t \in [a,b], \\ {}^cD^{q_2}[({}^cD^{p_2} + \alpha_2)v(t) + \beta_2 I^{\sigma_2} g_2(t,u(t),v(t))] = f_2(t,u(t),v(t)) + \phi_2(t), t \in [a,b]. \\ u(a) + u(b) = 0, u(\xi) + u(\eta) = 0, u'(\xi) + u'(\eta) = 0, \\ v(a) + v(b) = 0, v(\xi) + v(\eta) = 0, v'(\xi) + v'(\eta) = 0. \end{cases} \tag{33}$$

Definition IV.2 : [16] The coupled system (3) is said to be Ulam-Hyers stable, if there exist $(C_{p_1q_1p_2q_2} = (C_{p_1q_1}, C_{p_2q_2}) > 0$ such that for every solution $(u,v) \in E \times E$ of the inequality (32), there exists a unique solution $(x,y) \in E \times E$ of the system (3) with

$$|(u,v)(t)-(x,y)(t)| \le C_{p_1q_1p_2q_2} \in, t \in [a,b]. \tag{34}$$

Definition IV.3 : [16] The coupled system (3) is said to be generalized Ulam-Hyers stable, if there exist $\Psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Psi(0)=0,$ such that for every solution $(u,v) \in E \times E$ of the inequality (32), there exist a unique solution $(x,y) \in E \times E$ of the system (3) which satisfies

$$|(u,v)(t)-(x,y)(t)| \le \Psi(\in), t \in [a,b]. \tag{35}$$

Remark IV.4 : Definition IV.2 $\Rightarrow$ Definition IV.3.

Lemma IV.5 : *Let* $(u_1,u_2) \in E \times E$ *be the solution of the inequality* (32), *then the following inequality will be satisfied*:

$$\begin{aligned}
&|u_i(t)-I^{p_i+q_i}(f_i(t,u_i(t),u_i(t)))+\alpha_i I^{p_i}(u_i(t))+\beta_i I^{\sigma_i+p_i}(g_i(t,u_i(t),u_i(t)))\\
&\quad -\theta_{1,p_i}(t)\,[\alpha_i I^{p_i}u_i(b)+\beta_i I^{\sigma_i+p}g_i(b,u_i(b),u_i(b))-I^{p_i+q_i}f_1(b,u_i(b),u_i(b))]\\
&\quad -\theta_{2,p_i}(t)[\alpha_1[I^{p_i}u_i(\xi)+I^{p_i}u_i(\eta)]+\beta_i[I^{\sigma_1+p_i}g_i(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{\sigma_i+p_i}g_i(\eta,u_i(\eta),u_i(\eta))]-[I^{p_i+q_i}f_i(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{p_i+q_i}f_i(\eta,u_i(\eta),u_i(\eta))]-\theta_{3,p_i}(t)\,[\alpha_i[I^{p_i-1}u_i(\xi)+I^{p_i-1}u_i(\eta)]\\
&\quad +\beta_i[I^{\sigma_i+p_i-1}g_i(\xi,u_i(\xi),u_i(\xi))+I^{\sigma_i+p_i-1}g_i(\eta,u_i(\eta),u_i(\eta))]\\
&\quad -[I^{p_i+q_i-1}f_i(\xi,u_i(\xi),u_i(\xi))+I^{p_i+q_i-1}f_i(\eta,u_i(\eta),u_i(\eta))]|\\
&\quad \le C_{p_iq_i}\in_{p_iq_i}, t \in [a,b],\ i=1,2.
\end{aligned}$$

Proof : By Applying Lemma II.7 and Definition (IV.1), the solution of (33) will be as follows:

$$\begin{aligned}
u_i(t) &= I^{p_i+q_i}f_i(t,u_i(t),u_i(t))+I^{p_i+q_i}\phi(t)-\alpha_i I^{p_i}u_i(t)-\beta_1 I^{\sigma_i+p_1}g_i(t,u_i(t),u_i(t))\\
&\quad +\theta_{1,p_i}(t)[\alpha_i I^{p_i}u_i(b)+\beta_i I^{\sigma_i+p_i}g_i(b,u_i(b),u_i(b))-I^{p_i+q_i}f_i(b,u_i(b),u_i(b))\\
&\quad -I^{p_i+q_i}\phi_i(b)]+\theta_{2,p_i}(t)[\alpha_i[I^{p_i}u_i(\xi)+I^{p_i}u_i(\eta)]+\beta_i[I^{\sigma_i+p_i}g_i(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{\sigma_i+p_i}g_i(\eta,u_i(\eta),u_i(\eta))]-[I^{p_i+q_i}f_i(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{p_i+q_i}f_i(\eta,u_i(\eta),u_i(\eta))]-[I^{p_i+q_i}\phi_i(\xi)+I^{p_i+q_i}\phi_i(\eta)]]\\
&\quad +\theta_{3,p_i}(t)[\alpha_1[I^{p_i-1}u_i(\xi)+I^{p_i-1}u(\eta)]+\beta_i[I^{\sigma_i+p_i-1}g_1(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{\sigma_i+p_i-1}g_1(\eta,u(\eta),v(\eta))]-[I^{p_i+q_i-1}f_1(\xi,u_i(\xi),u_i(\xi))\\
&\quad +I^{p_i+q_i-1}f_i(\eta,u_i(\eta),u_i(\eta))]-[I^{p_i+q_i-1}\phi(\xi)+I^{p_i+q_i-1}\phi_i(\eta)]].
\end{aligned} \tag{36}$$

Considering the first equation of the system (36), we have

$$\begin{aligned}
&|u_i(t) - I^{p_i+q_i}(f_1(t,u_i(t),u_i(t))) + \alpha_1 I^{p_i}(u_i(t)) + \beta_i I^{\sigma_i+p_i}(g_i(t,u_i(t),u_i(t))) \\
&\quad -\theta_{1,p_i}(t)[\alpha_i I^{p_i}u_i(b) + \beta_i I^{\sigma_i+p} g_i(b,u_i(b),u_i(b)) - I^{p_i+q_i} f_i(b,u_i(b),u_i(b))] \\
&\quad -\theta_{2,p_i}(t)[\alpha_1[I^{p_i}u_i(\xi) + I^{p_i}u_i(\eta)] + \beta_i[I^{\sigma_i+p_i} g_i(\xi,u_i(\xi),u_i(\xi)) \\
&\quad +I^{\sigma_i+p_i} g_i(\eta,u_i(\eta),u_i(\eta))] - [I^{p_i+q_i} f_i(\xi,u_i(\xi),u_i(\xi)) \\
&\quad +I^{p_i+q_i} f_1(\eta,u_i(\eta),u_i(\eta))] - \theta_{3,p_i}(t)[\alpha_i[I^{p_i-1}u_i(\xi) + I^{p_1-1}u_i(\eta)] \\
&\quad +\beta_i[I^{\sigma_i+p_i-1} g_i(\xi,u_i(\xi),u_i(\xi)) + I^{\sigma_i+p_i-1} g_i(\eta,u_i(\eta),u_i(\eta))] \\
&\quad -[I^{p_i+q_i-1} f_1(\xi,u_i(\xi),u_i(\xi)) + I^{p_i+q_i-1} f_i(\eta,u_i(\eta),u_i(\eta))]| \\
&\quad \le I^{p_i+q_i}|\phi_i(t)| + |\theta_{1,p_i}(t)| I^{p_i+q_i}|\phi_i(b)| + |\theta_{2,p_i}(t)| I^{p_i+q_i}|\phi_i(\xi)| \\
&\quad +I^{p_i+q_i}|\phi_i(\eta)| + |\theta_{3,p_i}(t)| I^{p_i+q_i-1}|\phi_i(\xi)| + I^{p_i+q_i-1}|\phi_i(\eta)| \\
&\quad \le \Delta_i \in_{p_i q_i} = C_{p_i q_i} \in_{p_i q_i},
\end{aligned}$$

where $\Delta_i, i = 1, 2$ are given as in (25) and (26).

Theorem IV.6 : *If the assumption* (H_2), (A_1) *and* (A_2) *are fulfilled, then the fractional order coupled system* (3 *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable provided that* $(1-\Phi_1)(1-\Psi_1) - \Phi_2\Psi_2 \neq 0$, *where* Φ_1, Φ_2, Ψ_1 *and* Ψ_2 *are given in* (17), (18), (20) *and* (21) *respectively*.

Proof: Let $(u,v) \in E \times E$ be the solution of the system (33) and $(x_1, x_2) \in E \times E$ be the unique solution to the following considered system:

$$\begin{cases}
{}^cD^{q_1}[({}^cD^{p_1} + \alpha_1)x(t) + \beta_1 I^{\sigma_1} g_1(t,x(t),y(t))] = f_1(t,x(t),y(t)), \\
0 < p_1, p_2 \le 1, 1 < q_1, q_2 \le 2, t \in [a,b], \\
{}^cD^{q_2}[({}^cD^{p_2} + \alpha_2)y(t) + \beta_2 I^{\sigma_2} g_2(t,x(t),y(t))] = f_2(t,x(t),y(t)), \\
0 < p_1, p_2 \le 1, 1 < q_1, q_2 \le 2, t \in [a,b], \\
x(a) + x(b) = 0, x(\xi) + x(\eta) = 0, x'(\xi) + x'(\eta) = 0, \\
y(a) + y(b) = 0, y(\xi) + y(\eta) = 0, y'(\xi) + y'(\eta) = 0.
\end{cases} \tag{37}$$

The solution of (37) is given by

$$\begin{aligned}
x_i(t) &= I^{p_i+q_i}(f_i(t,x_1(t),x_2(t))) - \alpha_1 I^{p_i}x_i(t) - \beta_i I^{\sigma_i+p_i}(g_i(t,x_1(t),x_2(t))) \\
&\quad +\theta_{1,p_i}(t)[\alpha_i I^{p_i}x_i(b) + \beta_i I^{\sigma_i+p} g_i(b,x_1(b),x_2(b)) - I^{p_i+q_i} f_i(b,x_1(b),x_2(b))+] \\
&\quad \theta_{2,p_i}(t)[\alpha_i[I^{p_i}x_i(\xi) + I^{p_i}x_i(\eta)] + \beta_1[I^{\sigma_i+p_i} g_i(\xi,x_1(\xi),x_2(\xi))
\end{aligned}$$

$$
\begin{aligned}
&+I^{\sigma_i+p_i} g_i(\eta,x_1(\eta),x_2(\eta))]-[I^{p_i+q_i} f_i(\xi,x_1(\xi),x_2(\xi))\\
&+I^{p_i+q_i} f_i(\eta,x_1(\eta),x_2(\eta))]+\theta_{3,p_i}(t)[\alpha_i[I^{p_i-1}x_1(\xi)+I^{p_i-1}x_i(\eta)]\\
&+\beta_i[I^{\sigma_i+p_i-1} g_i(\xi,x_1(\xi),x_2(\xi))+I^{\sigma_i+p_i-1} g_i(\eta,x_1(\eta),x_2(\eta))]\\
&-[I^{p_i+q_i-1} f_i(\xi,x_1(\xi),x_2(\xi))+I^{p_i+q_i-1} f_i(\eta,x_1(\eta),x_2(\eta))],\ i=1,2. \qquad (38)
\end{aligned}
$$

We have

$$
\begin{aligned}
|u(t)-x(t)| \le & |u(t)-I^{p_1+q_1} f_1(t,u(t),v(t))+\alpha_1 I^{p_1}(u(t))+\beta_1 I^{\sigma_1+p_1} g_1(t,u(t),v(t))\\
&-\theta_{1,p_1}(t)[\alpha_1 I^{p_1} u(b)+\beta_1 I^{\sigma_1+p} g_1(b,u(b),v(b))-I^{p_1+q_1} f_1(b,u(b),v(b))]\\
&-\theta_{2,p_1}(t)[\alpha_1[I^{p_1}u(\xi)+I^{p_1}u(\eta)]+\beta_1[I^{\sigma_1+p_1} g_1(\xi,u(\xi),v(\xi))\\
&+I^{\sigma_1+p_1} g_1(\eta,u(\eta),v(\eta))]-[I^{p_1+q_1} f_1(\xi,u(\xi),v(\xi))\\
&+I^{p_1+q_1} f_1(\eta,u(\eta),v(\eta))]-\theta_{3,p_1}(t)[\alpha_1[I^{p_1-1}u(\xi)+I^{p_1-1}u(\eta)]\\
&+\beta_1[I^{\sigma_1+p_1-1} g_1(\xi,u(\xi),v(\xi))+I^{\sigma_1+p_1-1} g_1(\eta,u(\eta),v(\eta))]\\
&-[I^{p_1+q_1-1} f_1(\xi,u(\xi),v(\xi))+I^{p_1+q_1-1} f_1(\eta,u(\eta),v(\eta))]|\\
&+I^{p_1+q_1}|f_1(t,u(t),v(t))-f_1(t,x(t),y(t))|+|\alpha_1| I^{p_1}|u(t)-x(t)|\\
&+|\beta_1| I^{\sigma_1+p_1}|g_1(t,u(t),v(t))-g_1(t,x(t),y(t))|\\
&+|\theta_{1,p_1}(t)|[|\alpha_1| I^{p_1}|u(b)-x(b)|+|\beta_1| I^{\sigma_1+p}|g_1(b,u(b),v(b))\\
&-g_1(b,x(b),y(b))|+I^{p_1+q_1}|f_1(b,u(b),v(b))-f_1(b,x(b),y(b))|]\\
&+|\theta_{2,p_1}(t)|[|\alpha_1|[I^{p_1}|u(\xi)-x(\xi)|+I^{p_1}|u(\eta)-x(\eta)|]\\
&+|\beta_1|[I^{\sigma_1+p_1}|g_1(\xi,u(\xi),v(\xi))-g_1(\xi,x(\xi),y(\xi))|\\
&+I^{\sigma_1+p_1}|g_1(\eta,u(\eta),v(\eta))-g_1(\eta,x(\eta),y(\eta))|]\\
&+[I^{p_1+q_1}|f_1(\xi,u(\xi),v(\xi))-f_1(\xi,x(\xi),y(\xi))|\\
&+I^{p_1+q_1}|f_1(\eta,u(\eta),v(\eta))-f_1(\eta,x(\eta),y(\eta))|]\\
&+|\theta_{3,p_1}(t)|[\alpha_1[I^{p_1-1}|u(\xi)-x(\xi)|+I^{p_1-1}|u(\eta)-x(\eta)|]\\
&+|\beta_1|[I^{\sigma_1+p_1-1}|g_1(\xi,u(\xi),v(\xi))-g_1(\xi,x(\xi),y(\xi))|\\
&+I^{\sigma_1+p_1-1}|g_1(\eta,u(\eta),v(\eta))-g_1(\eta,x(\eta),y(\eta))|]\\
&+[I^{p_1+q_1-1}|f_1(\xi,u(\xi),v(\xi))-f_1(\xi,x(\xi),y(\xi))|\\
&+I^{p_1+q_1-1}|f_1(\eta,u(\eta),v(\eta))-f_1(\eta,x(\eta),y(\eta))|],\\
&\le C_{p_1q_1}\ \in_{p_1q_1}+\Phi_1\|u-x\|+\Phi_2\|v-y\|).
\end{aligned}
$$

Hence, we get

$$(1-\Phi_1)\|u-x\|-\Phi_2\|v-y\|)\leq C_{p_1q_1}\in_{p_1q_1}. \tag{39}$$

In similar way we get

$$(1-\Psi_1)\|v-y\|-\Psi_2\|u-x\|)\leq C_{p_2q_2}\in_{p_2q_2}. \tag{40}$$

The matrix representation of (39) and (40) is as follows

$$\begin{pmatrix}1-\Phi_1 & -\Phi_2\\ -\Psi_2 & 1-\Psi_1\end{pmatrix}\begin{pmatrix}\|u-x\|\\ \|v-y\|\end{pmatrix}\leq\begin{pmatrix}C_{p_1q_1}\in_{p_1q_1}\\ C_{p_2q_2}\in_{p_2q_2}\end{pmatrix},$$

which implies that

$$\begin{pmatrix}\|u-x\|\\ \|v-y\|\end{pmatrix}\leq\begin{pmatrix}\frac{1-\Psi_1}{\Xi} & \frac{\Phi_2}{\Xi}\\ \frac{\Psi_2}{\Xi} & \frac{1-\Phi_1}{\Xi}\end{pmatrix}\begin{pmatrix}C_{p_1q_1}\in_{p_1q_1}\\ C_{p_2q_2}\in_{p_2q_2}\end{pmatrix},$$

where $\Xi=(1-\Phi_1)(1-\Psi_1)-\Phi_2\Psi_2\neq 0.$

It follows that

$$\|u-x\|\leq\frac{1-\Psi_1}{\Xi}C_{p_1q_1}\in_{p_1q_1}+\frac{\Phi_2}{\Xi}C_{p_2q_2}\in_{p_2q_2}, \tag{41}$$

and

$$\|v-y\|\leq\frac{\Psi_2}{\Xi}C_{p_1q_1}\in_{p_1q_1}+\frac{1-\Phi_1}{\Xi}C_{p_2q_2}\in_{p_2q_2}. \tag{42}$$

Therefore, from inequalities (41) and (42), we have that

$$\|u-x\|+\|v-y\|\leq\frac{1-\Psi_1}{\Xi}C_{p_1q_1}\in_{p_1q_1}+\frac{\Phi_2}{\Xi}C_{p_2q_2}\in_{p_2q_2}+\frac{\Psi_2}{\Xi}C_{p_1q_1}\in_{p_1q_1}+\frac{1-\Phi_1}{\Xi}C_{p_2q_2}\in_{p_2q_2},$$

Hence, we can conclude that

$$\|(u,v)-(x,y)\|_{E\times E}\leq C_{p_1q_1p_2q_2}\in, \tag{43}$$

where $\in=\max(\in_{p_1q_1},\in_{p_2q_2})$ and $C_{p_1q_1p_2q_2}=\frac{1-\Psi_1}{\Xi}C_{p_1q_1}+\frac{\Phi_2}{\Xi}C_{p_2q_2}+\frac{\Psi_2}{\Xi}C_{p_1q_1}+\frac{1-\Phi_1}{\Xi}C_{p_2q_2}.$

Hence, by inequality (43), we can conclude that the coupled system (3) is Ulam-Hyers stable. Further, inequality (43) can be written as

$$\|(u,v)-(x,y)\|_{E\times E}\leq \Psi(\varepsilon),\quad \text{where}\quad \Psi(0)=0. \tag{44}$$

Thus, it follows from the inequalities (44) that the coupled system (3) is generalized Ulam-Hyers stable.

V. Example

In this section we present example to illustrate the usefulness and applicability of our results. Consider the following coupled system of two Caputo fractional derivatives of different orders:

$$\begin{cases} {}^cD^{\frac{3}{2}}\left[({}^cD^{\frac{4}{5}}+\alpha_1)u(t)+\beta_1 I^{\frac{8}{3}}g_1(t,u(t),v(t))\right]=f_1(t,u(t),v(t)), t\in[0,1],\\ {}^cD^{\frac{7}{4}}\left[({}^cD^{\frac{2}{3}}+\alpha_2)v(t)+\beta_2 I^{\frac{5}{3}}g_2(t,u(t),v(t))\right]=f_2(t,u(t),v(t)), t\in[0,1],\\ u(0)+u(1)=0, u(\xi)+u(\eta)=0, u'(\xi)+u'(\eta)=0,\\ v(0)+v(1)=0, v(\xi)+v(\eta)=0, v'(\xi)+v'(\eta)=0, \end{cases} \tag{45}$$

where, $f_1(t,u(t),v(t))=\frac{1}{75}t^2\sin u(t)\ +\frac{1}{3(t+5)^2}\left(\frac{|v(t)|}{1+|v(t)|}+\frac{1}{2}\right)$, $f_2(t,u(t),v(t))=\frac{e^{-2t}}{7(t^2+5)}(\cos u(t)+v(t))$, $g_1(t,u(t),v(t))=\frac{1}{7}u(t)+\frac{e^{-t}}{(t^2+7)}(\cos v(t)+2)$, and $g_2(t,u(t),v(t))=\frac{1}{9}\sin u(t)+\frac{1}{9}(\arctan v(t)+1)$, The suggested parameter values are $\alpha_1=\frac{1}{80}$, $\alpha_2=\frac{2}{95}$, $\beta_1=\frac{1}{11}$, $\beta_2=\frac{3}{17}$, $\xi=\frac{1}{3}$ and $\eta=\frac{2}{3}$. It is clear that $f_1,f_2,g_1,g_2\in\mathcal{C}([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and for any $t\in[0,1]$ and $u_1,u_2,v_1,v_2\in\mathbb{R}$ we have

$$|f_1(t,u_2,v_2)-f_1(t,u_1,v_1)|\leq\frac{1}{75}|u_2-u_1|+\frac{1}{75}|v_2-v_1|,$$

$$|f_2(t,u_2,v_2)-f_2(t,u_1,v_1)|\leq\frac{1}{35}|u_2-u_1|+\frac{1}{35}|v_2-v_1|,$$

$$|g_1(t,u_2,v_2)-g_1(t,u_1,v_1)|\leq\frac{1}{7}|u_2-u_1|+\frac{1}{7}|v_2-v_1|,$$

$$|g_2(t,u_2,v_2)-g_2(t,u_1,v_1)|\leq\frac{1}{9}|u_2-u_1|+\frac{1}{9}|v_2-v_1|.$$

Using the given parameters, we find that, $\Phi_1 = 0.2426$, $\Phi_2 = 0.0761$, $\Psi_1 = 0.4104$ and $\Psi_2 = 0.1698$. Hence, $\Phi_1 + \Phi_2 + \Psi_1 + \Psi_2 = 0.8991 < 1$. Thus, all the conditions of Theorem III.4 are satisfied and so the problem (3) has a unique solution on [0, 1]. Further, we have, $(1-\Phi_1)(1-\Psi_1) - \Phi_2\Psi_2 = 0.4335 \neq 0$, then, the solution is Hyers-Ulam stable as well as generalized Ulam-Hyers stable by using Theorem IV.6.

VI. Conclusion

In this paper, we have investigated the existence and uniqueness of solutions for a new coupled system of fractional differential equations involving two Caputo fractional derivatives of different orders with fixed and nonlocal anti-periodic boundary conditions. By using the the Leray-Schauder alternative fixed point theorem and Banach contraction principle, results about existence and uniqueness of solutions of the system (3) are obtained. We have also discussed the Ulam-Hyers stability and the generalized Ulam-Hyers stability of the system. Finally, an example is presented to demonstrate the main result.

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