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Differential sandwich subordination of convolution operator acting on complex meromorphic functions

Layth T. Khudhuir *
College of Computer Science and Mathematics
University of Mosul
Mosul
Iraq

Hiba F. Al-Janaby †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Ahmed M. Ali §
College of Computer Science and Mathematics
University of Mosul
Mosul
Iraq

Abstract

In the present study, a new complex convolution operator is introduced in terms of the meromorphic formula of the generalized Mittag-Leffler functions (MLfs), namely R–function. By employing the subordination methodology, appropriate classes of admissible functions are considered as well as the dual features of the third-order(3rd-order) differential subordination and superordination for this new operator are discussed. As a sequel, the sandwich-type outcome is also examined.

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* *E-mail:* **Layth.t.k@gmail.com (Corresponding Author)**
† *E-mail:* **fawzihiba@yahoo.com**
§ *E-mail:* **ahmedgraph@uomosul.edu.iq**

1. Introduction

Special functions have a vast importance in developing numerous aspects of mathematics, such as complex analysis, number theory and fractional calculus. Therefore, with the resolution of fractional (differential and integral) equations, interest in the theme of special functions has developed and grown remarkably. Further studies on this topic have led several scientists to investigate a lot of implementations in engineering and other allied disciplines. Outstanding instances include the Wright function, Fox H-function, gamma function, sigmoid function, hypergeometric function, Mittag-Leffler function and others. In complex analysis, such functions have wide implementations in Geometric Function Theory (GFT), which investigates the geometric features of complex functions, especially after the Bieberbach problem has been evidenced by de Branges, [1]. Since then, numerous papers have been devoted to discussing and analyzing geometric aspects of regular univalent functions in terms of special functions, [2-3]. In this context, the most significant class of special functions in the study of fractional differential equations and Abel sort integral equations is the Mittag-Leffler function (MLf), as a generalized exponential function. And then, the usage of such equations in modeling natural and technical processes is becoming more popular [4-5]. Initially, in 1903, the scholar Mittag-Leffler [6] put forward the first MLf of one parameter. After two years, Wiman [7] proposed a general formula of MLf from two complex parameters and later Prabhakar [8] in 1971 successfully investigated a generalized MLf of three complex parameters. Then, since 2009, Srivastava has contributed greatly to the investigation of assorted properties, generalizations, extensions and applications of MLf. For example, Srivastava and Tomovski [9] introduced a more general formula of MLf and in [10]. Srivastava provided a review of several recent growths involving diverse classes of MLf types which are related to different families of generalized Riemann-Liouville and other connected fractional derivative operators. Afterwards, in [11], Kumar, Choi and Srivastava investigated the solution of the generalized fractional kinetic equation that includes a generalized MLf through the usage of the Sumudu transform and the Laplace transform manners. Later, Srivastava et al. [12] examined new relations between the MLf of different parameters employing fractional calculus. Whilst, in 2010, Saxena and Nishimoto [13] considered and studied an interesting generalized multi-index MLf type. Then recently, Singh and Dubey [14] introduced and investigated a fractional calculus with an integral operator which contains the family of generalized k-Mittag-Leffler function. Also, Ghanim and Al-Janaby [15] defined a new MLf named Mittag-Leffler-confluent hypergeometric function and discussed some analytical applications.

Let A symbolize the class of all regular functions g in U, normalized by the conditions $g(0)=g'(0)-1=0$, of the formula: $g(z)=z+\sum_{n=2}^{\infty} a_n z^n, (z \in U)$.

Further, let Σ symbolize the class of functions $g(z)$ of the form $g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} a_n z^n$, which are regular in the punctured open unit disc $U^*=U\setminus\{0\}=\{z \in C: 0<|z|<1\}$. The function $g(z)$ has a simple pole at $z=0$.

Analogous to the convolution tool, we consider the following new convolution operator $L_{\lambda,\mu}^{k,v}: \Sigma \to \Sigma$ as:

$$\begin{aligned} L_{\lambda,\mu}^{k,v} g(z) &= {}_sV_r^{\lambda,k} * g(z) \\ &= \frac{1}{z}+\sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{s}(\chi_j)_{n+1}}{\Pi_{j=1}^{r}(\omega_j)_{n+1}} \frac{\Gamma(\mu)(v)_{(n+1)k}}{(n+1)!\,\Gamma(\lambda(n+1)+\mu)} a_n z^n, \end{aligned} \tag{1.1}$$

such that the function ${}_sV_r^{\lambda,k}$ defined by:

$${}_sV_r^{\lambda,k} = \frac{\Gamma(\mu)\,{}_sR_r^{k,\mu}}{z}, \tag{1.2}$$

where ${}_sR_r^{k,\mu}$ is the generalized MLf (R–function) [16] having Taylor series representation as:

$${}_sR_r^{k,\mu}(\chi_1,\chi_2,\ldots,\chi_s;\omega_1,\omega_2,\ldots,\omega_r;z)=\sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{s}(\chi_j)_n}{\Pi_{j=1}^{r}(\omega_j)_n} \frac{(v)_{nk}}{n!\,\Gamma(\lambda n+\mu)} z^n, \tag{1.3}$$

where $z,\lambda,\mu,v \in C$, $\mathrm{Re}(\lambda)>\max\{0,\mathrm{Re}(k)-1\}$, $\mathrm{Re}(k)>0$, $k \in C$, $\chi_j \in C (j=1,2,\ldots,s)$, such that $\omega_j \in C$ $(\omega_j \neq 0,-1,-2,\ldots; j=1,2,\ldots,r)$, $s \le r+1$; $s,r \in \{0,1,\ldots\}$.

Note that from (1.1), this operator fulfills the following recurrence relations:

$$kz(L_{\lambda,\mu}^{k,v} g(z))' = v(L_{\lambda,\mu}^{k,v+1} g(z)) - (k+v)(L_{\lambda,\mu}^{k,v} g(z)) \tag{1.4}$$

and

$$\lambda z(L_{\lambda,\mu+1}^{k,v} g(z))' = \mu(L_{\lambda,\mu}^{k,v} g(z)) - (\lambda+\mu)(L_{\lambda,\mu+1}^{k,v} g(z)) \tag{1.5}$$

Remark 1.1 : The following special states of the operator $L_{\lambda,\mu}^{k,v} g(z)$ presented by (1.1) are yielded by taking appropriate choices of the parameters as:

1. For $s=2,\ r=1,\ \chi_1=\chi_2=1,$ and $\lambda=\mu=k=\omega_1=v,$ the operator $L^{k,v}_{\lambda,\mu}g(z)$ coincides to $g(z)$.
2. For $s=3, r=2, \chi_1=\chi_2=1, \chi_3=2, \omega_1=v, \omega_2=1, \mu=\lambda=1$ and $k=1$ the operator $L^{k,v}_{\lambda,\mu}g(z)$ corresponds to $zg'(z)+2g(z)$.

Definition 1.1 : [19] Let $g(z)$ and $G(z)$ be regular functions. The function $g(z)$ is said to be subordinate to $G(z)$, written $g(z)\prec G(z)$, if there exists a function $\omega(z)$ regular in U with $|\omega(z)|<1$, such that $g(z)=G(\omega(z))$, If $G(z)$ is univalent, then $g(z)\prec G(z)$ if and only if $g(0)=G(0)$ and $g(U)\subset G(U)$.

Definition 1.2 : [20] Let D be the set of regular functions $\lambda(z)$ and univalent on $\overline{U}\setminus E(\lambda)$, where $E(\lambda)=\{\zeta\in\partial U:\lim\limits_{z\to\zeta}\lambda(z)=\infty\}$, is such that $\min|\lambda'(\zeta)|=\rho>0$ for $\zeta\in\partial U\setminus E(\lambda)$. Further, let $D(a)=\{\lambda(z)\in D:\lambda(0)=a\}$ and $D_1=D(1)$.

Definition 1.3 : [17] Let $\psi:C^4\times U\to C$ and $h(z)$ be univalent in U. If $\wp(z)$ is regular in U and achieves the third-order differential subordination:

$$\psi(\wp(z),z\wp'(z),z^2\wp''(z),z^3\wp'''(z);z)\prec h(z), \tag{1.6}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $\lambda(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $\wp(z)\prec\lambda(z)$ for all $\wp(z)$ achieving (1.6). A dominant $\tilde{\lambda}(z)$ that achieves $\tilde{\lambda}(z)\prec\lambda(z)$ for all dominants of (1.6) is called the best dominant of (1.6).

Definition 1.4 : [17] If Ω is a set in C, $\lambda\in D$ and $n\in\mathbb{N}\setminus\{1\}$. The class of admissible functions $\Psi_n[\Omega,\lambda]$ consists of those functions $\psi:C^4\times U\to C$ that achieve the admissibility condition :

$$\psi(\delta_1,\delta_2,\delta_3,\delta_4;z)\notin\Omega$$

whenever $\delta_1=\lambda(\zeta), \delta_2=\alpha\zeta\lambda'(\zeta)$, $\mathrm{Re}\left(\frac{\delta_3}{\delta_2}+1\right)\geq\alpha\ \mathrm{Re}\left(\frac{\zeta\lambda''(\zeta)}{\lambda'(\zeta)}+1\right)$

and $\mathrm{Re}\left(\frac{\delta_4}{\delta_2}\right)\geq\alpha^2\ \mathrm{Re}\left(\frac{\zeta^2\lambda'''(\zeta)}{\lambda'(\zeta)}\right)$, where $\alpha\geq n, z\in U$ and $\zeta\in\partial U\setminus E(q)$.

Definition 1.5 : [18] Let $\psi:C^4\times U\to C$ and the function $h(z)$ be regular in U. If the functions $\lambda(z)$ and $\psi(\wp(z),z\wp'(z),z^2\wp''(z),z^3\wp'''(z);z)$, are univalent in and achieve the following third-order differential superordination:

$$h(z) \prec \psi(\wp(z), z\wp'(z), z^2\wp''(z), z^3\wp'''(z); z), \tag{1.7}$$

then $\wp(z)$ is called a solution of the differential superordination. A regular function $\lambda(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $\lambda(z) \prec \wp(z)$ for $\wp(z)$ achieving (1.7). A univalent subordinant $\tilde{\lambda}(z)$ that achieves $\tilde{\lambda}(z) \prec \wp(z)$ for all subordinants $\lambda(z)$ of (1.7) is said to be the best subordinant.

Definition 1.6 : [18] Let Ω be a set in C, $\lambda \in H[a,n]$ and $\lambda'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, \lambda]$ consists of those functions $\psi : C^4 \times \bar{U} \to C$ that achieve the following admissibility condition : $\psi(\delta_1, \delta_2, \delta_3, \delta_4; z) \in \Omega$, whenever $\delta_1 = \lambda(z)$, $\delta_2 = \frac{z\lambda'(z)}{\beta}$, $\operatorname{Re}\left(\frac{\delta_3}{\delta_2}+1\right) \geq \frac{1}{\beta} \operatorname{Re}\left(\frac{z\lambda''(z)}{\lambda'(z)}+1\right)$

and $\operatorname{Re}\left(\frac{\delta_4}{\delta_2}\right) \geq \frac{1}{\beta^2} \operatorname{Re}\left(\frac{z^2\lambda'''(z)}{\lambda'(z)}\right)$, where $\beta \geq n \geq 2, z \in U$ and $\zeta \in \partial U \setminus E(\lambda)$.

Also, we need the following theorems in our investigations:

Theorem 1.1 : [17] *Let* $\wp(z) \in H[a,n]$ *with* $n \in \mathbb{N} \setminus \{1\}$ *Also, let* $\lambda(z) \in D(a)$ *and achieve the following conditions:* $\operatorname{Re}\left(\frac{\zeta\lambda''(\zeta)}{\lambda'(\zeta)}\right) \geq 0$, $\left|\frac{z\wp'(z)}{\lambda'(z)}\right| \leq \alpha$, *where* $z \in U; \alpha \geq n$ *and* $\zeta \in \partial U \setminus E(\lambda)$. *If* Ω *is a set in* C, $\Psi_n[\Omega, \lambda]$ *and* $\psi(\wp(z), z\wp'(z), z^2\wp''(z), z^3\wp'''(z); z) \in \Omega$, *then* $\wp(z) \prec \lambda(z)$.

Theorem 1.2 : [16] *Let* $\lambda(z) \in H[a,n]$ *and* $\psi \in \Psi'_n[\Omega, \lambda]$. *If* $\psi(\wp(z), z\wp'(z), z^2\wp''(z), z^3\wp'''(z); z)$ *is univalent in Uand* $\wp(z) \in D(a)$ *achieve conditions:*

$$\operatorname{Re}\left(\frac{z\lambda''(z)}{\lambda'(z)}\right) \geq 0, \left|\frac{\zeta\wp'(\zeta)}{\lambda'(\zeta)}\right| \leq \beta, \ (z \in U; \beta \geq n \geq 2; \zeta \in \partial U),$$

and $\Omega \subset \{\psi(\wp(z), z\wp'(z), z^2\wp''(z), z^3\wp'''(z); z) : z \in U\}$, then $\lambda(z) \prec \wp(z)$.

2. Third Order Differential Subordination with $L_{\lambda,\mu}^{k,v}$

Definition 2.1 : Let Ω be a set in C and $\lambda(z) \in D$. The class of admissible functions $\Phi_\Gamma[\Omega, \lambda]$ consists of those functions $\phi : C^4 \times U^* \to C$ that achieve the admissibility condition:

$$\phi(b_1, b_2, b_3, b_4; z) \notin \Omega,$$

whenever $b_1 = \lambda(\zeta),\ b_2 = \frac{k}{v}\alpha\zeta\lambda'(\zeta) + \lambda(\zeta),\ b_3 = \frac{k^2\delta_3 + k(k+2v+1)\delta_2}{v(v+1)} + \delta_1$, and

$$b_4 = \frac{k^3}{v(v+1)(v+2)}\delta_4 + \frac{3k^2(k+v+1)}{v(v+1)(v+2)}\delta_3 + \frac{(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))}{v(v+1)(v+2)}\delta_2 + \delta_1$$

$$\operatorname{Re}\left(\frac{(v+1)(b_3-b_1)}{k(b_2-b_1)} - \frac{(2v+1)}{k}\right) \geq \alpha \operatorname{Re}\left(\frac{\zeta\lambda''(\zeta)}{\lambda'(\zeta)} + 1\right),$$

$$\operatorname{Re}\left(\frac{(v+1)\big((v+2)(b_4-b_1) - 3(k+v+1)(b_3-b_1)\big)}{k^2(b_2-b_1)} + \frac{(3v^2+3v+1)}{k^2} + 3\frac{(2v+1)}{k} + 2\right)$$
$$\geq \alpha^2 \operatorname{Re}\left(\frac{\zeta^2\lambda'''(\zeta)}{\lambda'(\zeta)}\right),$$

where $v, k \in C \setminus Z_0^-, \alpha \in \mathbb{N} \setminus \{1\}, z \in U^*$ and $\zeta \in \partial U^* \setminus E(\lambda)$.

Theorem 2.1 : *Let* $\phi \in \Phi_\Gamma[\Omega, \lambda]$, , *If* $g(z) \in \Sigma$ *and* $\lambda(z) \in D_1$ *achieve the following conditions:*

$$\operatorname{Re}\left(\frac{\zeta\lambda''(\zeta)}{\lambda'(\zeta)}\right) \geq 0, \left|\frac{z\left(L_{\lambda,\mu}^{k,v+1}g(z) - L_{\lambda,\mu}^{k,v}g(z)\right)}{\wp'(\zeta)}\right| \leq \alpha\left|\frac{k}{v}\right|, \tag{2.1}$$

and $\{\phi(zL_{\lambda,\mu}^{k,v}f(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) : z \in U^*\} \subset \Omega,$ (2.2)

then
$$zL_{\lambda,\mu}^{k,v}g(z) \prec \lambda(z). \tag{2.3}$$

Proof: Let us define the regular function $\wp(z)$ as:

$$\wp(z) = zL_{\lambda,\mu}^{k,v}g(z),\ (z \in U^*) \tag{2.4}$$

then we get
$$zL_{\lambda,\mu}^{k,v+1}g(z) = \frac{k}{v}z\wp'(z) + \wp(z), \tag{2.5}$$

which implies
$$zL_{\lambda,\mu}^{k,v+2}g(z) = \frac{k^2z^2\wp''(z) + k(k+2v+1)z\wp'(z)}{v(v+1)} + \wp(z), \tag{2.6}$$

also, we can see that

$$zL_{\lambda,\mu}^{k,v+3}g(z)=\frac{k^3}{v(v+1)(v+2)}z^3\wp'''(z)+\frac{3k^2(k+v+1)}{v(v+1)(v+2)}z^2\wp''(z)$$
$$+\frac{k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2))}{v(v+1)(v+2)}z\wp'(z)+\wp(z), \tag{2.7}$$

Let us define the parameters b_1, b_2, b_3 and b_4 as:

$$b_1=\delta_1,\quad b_2=\frac{k}{v}\delta_2+\delta_1,\quad b_3=\frac{k^2\delta_3+k(k+2v+1)\delta_2}{v(v+1)}+\delta_1,\ \text{and}$$

$$b_4=\frac{k^3}{v(v+1)(v+2)}\delta_4+\frac{3k^2(k+v+1)}{v(v+1)(v+2)}\delta_3+\frac{k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2))}{v(v+1)(v+2)}\delta_2+\delta_1.$$

Now, we define the following transformation $\psi: C^4\times U^*\to C$

$$\psi(\delta_1,\delta_2,\delta_3,\delta_4;z)=\phi(b_1,b_2,b_3,b_4;z). \tag{2.8}$$

From (2.4.) to (2.7), we have

$$\psi(\wp(z),z\wp'(z),z^2\wp''(z),z^3\wp'''(z);z)=$$
$$\phi(zL_{\lambda,\mu}^{k,v}g(z),zL_{\lambda,\mu}^{k,v+1}g(z),zL_{\lambda,\mu}^{k,v+2}g(z),zL_{\lambda,\mu}^{k,v+3}g(z);z). \tag{2.9}$$

Then, we can rewrite (2.2) as $\psi(\wp(z),z\wp'(z),z^2\wp''(z),z^3\wp'''(z);z)\in\Omega.$

Therefore the proof is completed by showing that the admissibility condition is equivalent to the admissibility condition as given in Definition (1.3), since

$$\frac{\delta_3}{\delta_2}+1=\frac{(v+1)(b_3-b_1)}{k(b_2-b_1)}-\frac{(2v+1)}{k}, \tag{2.10}$$

and

$$\frac{\delta_4}{\delta_2}=\frac{(v+1)((v+2)(b_4-b_1)-3(k+v+1)(b_3-b_1))}{k^2(b_2-b_1)}+\frac{(3v^2+3v+1)}{k^2}+3\frac{(2v+1)}{k}+2$$

we also note that $\left|\frac{z\wp'(z)}{\lambda'(\zeta)}\right|=\left|\frac{\frac{v}{k}z(L_{\lambda,\mu}^{k,v+1}g(z)-L_{\lambda,\mu}^{k,v}g(z))}{\lambda'(\zeta)}\right|\le\alpha.$ Therefore, $\psi\in\Psi_2[\Omega,\lambda]$ and hence by Theorem 1.1, $\wp(z)\prec\lambda(z).$

If $\Omega\neq C$ is a simply connected domain, then $\Omega=h(U^*)$ for some conformal mapping $h(z)$ of U^* onto Ω. In this case the class $\Phi_\Gamma[h(U^*),\lambda]$ is written as $\Phi_\Gamma[h,\lambda]$.

The next theorem is a direct consequence of Theorem 2.1.

Theorem 2.2 : *Let $\phi \in \Phi_\Gamma[h, \lambda]$. If $g(z) \in \Sigma$ and $\lambda(z) \in D_1$ achieve the following conditions:*

$$\operatorname{Re}\left(\frac{\zeta \lambda''(\zeta)}{\lambda'(\zeta)}\right) \geq 0, \left|\frac{z(L_{\lambda,\mu}^{k,v+1} g(z) - L_{\lambda,\mu}^{k,v} g(z))}{\lambda'(\zeta)}\right| \leq \alpha \left|\frac{k}{v}\right| \tag{2.11}$$

and $$\phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z) \prec h(z), \tag{2.12}$$

then $zL_{\lambda,\mu}^{k,v} g(z) \prec \lambda(z)$.

The following corollary is an extension of Theorem 2.1 to the case where the behavior of $\lambda(z)$ on ∂U^* is not known.

Corollary 2.1 : *Let $\Omega \subset C$ and let $\lambda(z)$ be univalent in $U^*, \lambda(0) = 1$ Let $\phi \in \Phi_\Gamma[\Omega, \lambda_\rho]$ for some $\rho \in (0,1)$ where $\lambda_\rho(z) = \lambda(\rho z)$. If $g(z) \in \Sigma$ achieves*

$$\operatorname{Re}\left(\frac{\zeta \lambda_\rho''(\zeta)}{\lambda_\rho'(\zeta)}\right) \geq 0, \left|\frac{z\left(L_{\lambda,\mu}^{k,v+1} g(z) - L_{\lambda,\mu}^{k,v} g(z)\right)}{\lambda_\rho'(\zeta)}\right| \leq \alpha \left|\frac{k}{v}\right| \tag{2.13}$$

and $$\phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z) \in \Omega, \tag{2.14}$$

then $zL_{\lambda,\mu}^{k,v} g(z) \prec \lambda(z)$, where $z \in U^*$, and $\zeta \in \partial U^* \setminus E(\lambda_\rho)$.

Proof: By utillzing Theorem 2.1, we have $zL_{\lambda,\mu}^{k,v} g(z) \prec \lambda_\rho(z)$, Then we get the result from $\lambda_\rho(z) \prec \lambda(z)$.

Corollary 2.2 : *Let $\Omega \subset C$ and let $\lambda(z)$ be univalent in $U^*, \lambda(0) = 1$ Let $\phi \in \Phi_\Gamma[h, \lambda_\rho]$ for some $\rho \in (0,1)$ where $\lambda_\rho(z) = \lambda(\rho z)$. If $g(z) \in \Sigma$ achieves*

$$\operatorname{Re}\left(\frac{\zeta \lambda_\rho''(\zeta)}{\lambda_\rho'(\zeta)}\right) \geq 0, \left|\frac{z\left(L_{\lambda,\mu}^{k,v+1} g(z) - L_{\lambda,\mu}^{k,v} g(z)\right)}{\lambda_\rho'(\zeta)}\right| \leq \alpha \left|\frac{k}{v}\right| \tag{2.15}$$

and $$\phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z) \prec h(z), \tag{2.16}$$

then $zL_{\lambda,\mu}^{k,v} g(z) \prec \lambda(z)$, where $z \in U^*$, and $\zeta \in \partial U^* \setminus E(\lambda_\rho)$.

Theorem 2.3 : *Let $h(z)$ be univalent in U^*. Let $\phi : C^4 \times U^* \to C$. Suppose that the differential equation:*

$$\phi\left(\begin{array}{l}\lambda(z), \frac{k}{v} z\lambda'(z)+\lambda(z), \frac{k^2 z^2 \lambda''(z)+k(k+2v+1)z\lambda'(z)}{v(v+1)}+\lambda(z), \\ \frac{k^3}{v(v+1)(v+2)} z^3 \lambda'''(z)+\frac{3k^2(k+v+1)}{v(v+1)(v+2)} z^2 \lambda''(z) \\ +\frac{(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))}{v(v+1)(v+2)} z\lambda'(z)+\lambda(z); z\end{array}\right)=h(z) \quad (2.17)$$

has a solution $\lambda(z)$ *with* $\lambda(0)=1$ *which achieves (2.1). If* $g(z)\in\Sigma$ *achieves (2.16) and*

$$\phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z)$$

is regular in U^*, *then*

$$zL_{\lambda,\mu}^{k,v} g(z) \prec \lambda(z) \quad (2.18)$$

and $\lambda(z)$ *is the best dominant of (2.18).*

Proof: By using Theorem 2.1 that $\lambda(z)$ is a dominant of (2.16). Since $\lambda(z)$ achieves (2.17), it is also a solution of (2.16) and therefore $\lambda(z)$ will be dominated by all dominants. Hence $\lambda(z)$ is the best dominant.

In the case $\lambda(z)=1+Mz, (M>0)$ and in view of the Definition 2.1, the class of admissible functions $\Phi_\Gamma[\Omega,\lambda]$ symbolized by $\Phi_\Gamma[\Omega,M]$ is defined below.

Definition 2.2 : Let Ω be a set in C and $M>0$ The class of admissible functions $\Phi_\Gamma[\Omega,M]$ consists of those functions $\phi: C^4\times U^*\to C$ that achieve the admissibility condition

$$\phi\left(\begin{array}{l}1+Me^{i\theta}, 1+\frac{(\alpha k+v)}{v} Me^{i\theta}, 1+\frac{k^2 L+(\alpha k(k+2v+1)+v(v+1))Me^{i\theta}}{v(v+1)}, \\ 1+\frac{k^3 N+3k^2(k+v+1)L}{v(v+1)(v+2)} \\ +\frac{(\alpha(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))+v(v+1)(v+2))Me^{i\theta}}{v(v+1)(v+2)}; z\end{array}\right)\notin\Omega$$

(2.19)

Where $\alpha\in\mathbb{N}\setminus\{1\}$ and $z\in U^*, \mathrm{Re}(Le^{i\theta})\geq(\alpha-1)\alpha M$ and $\mathrm{Re}(Ne^{i\theta})\geq 0$ for all real θ.

Corollary 2.3 : Let $\phi \in \Phi_{\Gamma}[\Omega, M]$. If $g(z) \in \Sigma$ achieves the following conditions:

$$\left| z(L_{\lambda,\mu}^{k,v+1} g(z) - L_{\lambda,\mu}^{k,v} g(z)) \right| \leq \alpha \left| \frac{k}{v} \right| M \tag{2.20}$$

and $$\phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z) \in \Omega, \tag{2.21}$$

then $\left| zL_{\lambda,\mu}^{k,v} g(z) - 1 \right| < M.$

In the case $\Omega = \lambda(U^*) = \{\omega : |\omega - 1| < M, (M > 0)\}$, for facilitation we symbolize by $\Phi_{\Gamma}[M]$ to the class $\Phi_{\Gamma}[\Omega, M]$.

Corollary 2.4 : *Let* $\phi \in \Phi_{\Gamma}[\Omega, M]$. *If* $g(z) \in \Sigma$ *achieves the condition (2.20) and*

$$\left| \phi(zL_{\lambda,\mu}^{k,v} g(z), zL_{\lambda,\mu}^{k,v+1} g(z), zL_{\lambda,\mu}^{k,v+2} g(z), zL_{\lambda,\mu}^{k,v+3} g(z); z) - 1 \right| < M, \tag{2.22}$$

then $|zL_{\lambda,\mu}^{k,v} g(z) - 1| < M.$

***Proof* :** Putting $\phi(b_1, b_2, b_3, b_4; z) = b_2 = 1 + \frac{(k\alpha + v)Me^{i\theta}}{v}$ in Corollary 2.4, we have the following corollary:

Corollary 2.5 : *Let* $M > 0$ *and* $v, k \in C \setminus Z_0^-$. *with* $\mathrm{Re}\left(\frac{v}{k}\right) < \frac{-\alpha}{2} (\alpha \in \mathbb{N} \setminus \{1\})$. *If* $g(z) \in \Sigma$ *achieves the condition (2.20) and* $|zL_{\lambda,\mu}^{k,v+1} g(z) - 1| < M$, *then* $|zL_{\lambda,\mu}^{k,v} g(z) - 1| < M$.

Corollary 2.6 : *Let* $\alpha \in \mathbb{N} \setminus \{1\}, v, k \in C \setminus Z_0^-$ *and* $M > 0$. *If* $g(z) \in \Sigma$ *achieves the condition (2.20) and achieves one of the following:*

i. $$\left| z(L_{\lambda,\mu}^{k,v+1} g(z) - L_{\lambda,\mu}^{k,v} g(z)) \right| < \alpha \left| \frac{k}{v} \right| M, \tag{2.23}$$

ii. $$\left| z(L_{\lambda,\mu}^{k,v+2} g(z) - L_{\lambda,\mu}^{k,v+1} g(z)) \right| < \left(\left| \frac{k^2}{v(v+1)} \right| + \left| \frac{\alpha k^2 + \alpha kv}{v(v+1)} \right| \right) 2M, \tag{2.24}$$

iii. $$\left| z(L_{\lambda,\mu}^{k,v+3} g(z) - L_{\lambda,\mu}^{k,v+2} g(z)) \right| < \left(\left| \frac{3k^2(k+v+1) - k^2(v+2)}{v(v+1)(v+2)} \right| + \left| \frac{k(k^2 + v^2 + 2kv + k + v)}{v(v+1)(v+2)} \right| \right) 2M, \tag{2.25}$$

then $\left|zL_{\lambda,\mu}^{k,v}g(z)-1\right|<M.$

Proof :

(i) Let $\phi(b_1,b_2,b_3,b_4;z)=b_2-b_1$. Using Corollary 2.3 with $\Omega=h(U^*)$ and $h(z)=\alpha\left|\frac{k}{v}\right|Mz\quad(z\in U^*)$. Now we show that $\phi\in\Phi_\Gamma[\Omega,M]$. Since

$$\left|\phi\left(\begin{array}{l}1+Me^{i\theta},1+\dfrac{(\alpha k+v)}{v}Me^{i\theta},1+\dfrac{k^2L+(\alpha k(k+2v+1)+v(v+1))Me^{i\theta}}{v(v+1)},\\ 1+\dfrac{k^3N+3k^2(k+v+1)L}{v(v+1)(v+2)}\\ +\dfrac{(\alpha(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))+v(v+1)(v+2))Me^{i\theta}}{v(v+1)(v+2)};z\end{array}\right)\right|$$
$$=\left|\frac{k}{v}\alpha Me^{i\theta}\right|=\left|\frac{k}{v}\right|\alpha M.$$

(ii) We define $\phi(b_1,b_2,b_3,b_4;z)=b_3-b_2$. Using Corollary 2.3 with $\Omega=h(U^*)$ and

$$h(z)=\left(\left|\frac{k^2}{v(v+1)}\right|+\left|\frac{\alpha k^2+\alpha kv}{v(v+1)}\right|\right)2Mz\quad(z\in U^*).\ g(z).\ \text{Since}$$

$$\left|\phi\left(\begin{array}{l}1+Me^{i\theta},1+\dfrac{(\alpha k+v)}{v}Me^{i\theta},1+\dfrac{k^2L+(\alpha k(k+2v+1)+v(v+1))Me^{i\theta}}{v(v+1)},\\ 1+\dfrac{k^3N+3k^2(k+v+1)L}{v(v+1)(v+2)}\\ +\dfrac{(\alpha(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))+v(v+1)(v+2))Me^{i\theta}}{v(v+1)(v+2)};z\end{array}\right)\right|$$

$$\geq\left(\left|\frac{k^2}{v(v+1)}\right|+\left|\frac{\alpha k^2+\alpha kv}{v(v+1)}\right|\right)2M.$$

(iii) We define $\phi(b_1,b_2,b_3,b_4;z)=b_4-b_3$. Using Corollary 2.3 with $\Omega=h(U^*)$ and

$$h(z)=\left(\left|\frac{3k^2(k+v+1)-k^2(v+2)}{v(v+1)(v+2)}\right|+\left|\frac{k(k^2+v^2+2kv+k+v)}{v(v+1)(v+2)}\right|\right)2Mz \quad (z\in U^*).g(z).$$

Since

$$\left|\phi\left(\begin{array}{l}1+Me^{i\theta},1+\dfrac{(\alpha k+v)}{v}Me^{i\theta},1+\dfrac{k^2L+(\alpha k(k+2v+1)+v(v+1))Me^{i\theta}}{v(v+1)},\\ 1+\dfrac{k^3N+3k^2(k+v+1)L}{v(v+1)(v+2)}\\ +\dfrac{(\alpha(k^2(k+2v+1)+k(2(v+1)^2+(k+v)(v+2)))+v(v+1)(v+2))Me^{i\theta}}{v(v+1)(v+2)};z\end{array}\right)\right|$$

$$\geq\left(\left|\frac{3k^2(k+v+1)-k^2(v+2)}{v(v+1)(v+2)}\right|+\left|\frac{k(k^2+v^2+2kv+k+v)}{v(v+1)(v+2)}\right|\right)2M.$$

3. Third Order Differential Superordination with $L^{k,v}_{\lambda,\mu}$

Definition 3.1 : Let Ω be a set in C and $\lambda(z)\in H$ with $\lambda'(z)\neq 0$. The class of admissible functions $\Phi'_{\Gamma}[\Omega,\lambda]$ consists of those functions $\phi:C^4\times\overline{U}^*\to C$ that achieve the admissibility condition: $\phi(b_1,b_2,b_3,b_4;\zeta)\in\Omega$, whenever

$$b_1=\lambda(z), b_2=\frac{k}{\beta v}z\lambda'(z)+\lambda(z),\ \operatorname{Re}\left(\frac{(v+1)(b_3-b_1)}{k(b_2-b_1)}-\frac{(2v+1)}{k}\right)\geq\frac{1}{\beta}\operatorname{Re}\left(\frac{\zeta\lambda''(\zeta)}{\lambda'(\zeta)}+1\right),$$

$$\operatorname{Re}\left(\frac{(v+1)((v+2)(b_4-b_1)-3(k+v+1)(b_3-b_1))}{k^2(b_2-b_1)}+\frac{(3v^2+3v+1)}{k^2}+3\frac{(2v+1)}{k}+2\right)$$
$$\geq\frac{1}{\beta^2}\operatorname{Re}\left(\frac{\zeta^2\lambda'''(\zeta)}{\lambda'(\zeta)}\right),$$

where $v,k\in C\setminus Z_0^-, z\in U^*,\zeta\in\partial U^*$ and $\beta\in\mathbb{N}\setminus\{1\}$.

Theorem 3.1 : *Let* $\phi\in\Phi'_{\Gamma}[\Omega,\lambda]$, *If* $g(z)\in\Sigma$ *and* $\lambda(z)\in D_1$ *achieve the following conditions:*

$$\operatorname{Re}\left(\frac{z\lambda''(z)}{\lambda'(z)}\right)\geq 0,\left|\frac{z(L^{k,v+1}_{\lambda,\mu}g(z)-L^{k,v}_{\lambda,\mu}g(z))}{\lambda'(\zeta)}\right|\leq\beta\left|\frac{k}{v}\right| \tag{3.1}$$

$$\{\phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) : z \in U^*\}$$

is univalent and

$$\Omega \subset \{\phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) : z \in U^*\}, \tag{3.2}$$

then $zL_{\lambda,\mu}^{k,v}g(z) \prec \lambda(z)$.

Proof: Let two functions $\lambda(z)$ and ψ be defined by (2.4) and (2.8). Since $\phi \in \Phi'_\Gamma[\Omega, \lambda]$ Therefore (2.9) and (3.2) imply $\Omega \subset \psi(\wp(z), z\wp'(z), z^2\wp''(z), z^3\wp'''(z); z)$.

The admissible condition for $\phi \in \Phi'_\Gamma[\Omega, \lambda]$ is equivalent to the admissible condition for ψ in Definition 1.6 with $\Gamma = 2$ Therefore, $\psi \in \Psi'_2[\Omega, \lambda]$, and by using (3.1) and Theorem 1.2, we have $\lambda(z) \prec \wp(z)$, which yields $\lambda(z) \prec zL_{\lambda,\mu}^{k,v}g(z)$.

Theorem 3.2 : *Let* $\phi \in \Phi'_\Gamma[h, \lambda]$. *Also, let* $h(z)$ *be regular in* U^*. *If* $g(z) \in \Sigma$ *and* $\lambda(z) \in D_1$ *achieve the conditions (3.1),*

$$\{\phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) : z \in U^*\},$$

is univalent in U^*, *and*

$$h(z) \prec \phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) \tag{3.3}$$

then $\lambda(z) \prec zL_{\lambda,\mu}^{k,v}g(z)$.

Theorem 3.3 : *Let* $h(z)$ *be regular in* U. *Let* $\phi : C^4 \times \overline{U}^* \to C$. *and* ψ *be given by (2.8). assume that the differential equation (2.17) has a solution* $\lambda(z) \in D_1$. *If* $g(z) \in \Sigma$ *achieves the condition (3.1),*

$$\{\phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z) : z \in U^*\}$$

is univalent in U^*, *and*

$$h(z) \prec \phi(zL_{\lambda,\mu}^{k,v}g(z), zL_{\lambda,\mu}^{k,v+1}g(z), zL_{\lambda,\mu}^{k,v+2}g(z), zL_{\lambda,\mu}^{k,v+3}g(z); z)$$

then

$$\lambda(z) \prec zL_{\lambda,\mu}^{k,v}g(z), \tag{3.4}$$

and $\lambda(z)$ is the best subordinant of (3.3).

We get the following sandwich type result by combining Theorem 2.2 and Theorem 3.2.

Corollary 3.1 : *Let* $h_1(z)$ *and* $\lambda_1(z)$ *be regular in* U^*. *Let* $h_2(z)$ *be univalent in* U^*, $\lambda_2(z) \in D_1$ *with* $\lambda_1(0) = \lambda_2(0) = 1$ *and* $\phi \in \Phi_\Gamma[h, \lambda] \cap \Phi'_\Gamma[h, \lambda]$. *If* $f(z) \in \Sigma, zL^{k,v}_{\lambda,\mu}g(z) \in D_1 \cap H$,

$$\{\phi(zL^{k,v}_{\lambda,\mu}g(z), zL^{k,v+1}_{\lambda,\mu}g(z), zL^{k,v+2}_{\lambda,\mu}g(z), zL^{k,v+3}_{\lambda,\mu}g(z); z) : z \in U^*\}$$

is univalent in U^*, *and the conditions (2.11) and (3.1) are achieved, Also, let*

$$h_1(z) \prec \phi(zL^{k,v}_{\lambda,\mu}g(z), zL^{k,v+1}_{\lambda,\mu}g(z), zL^{k,v+2}_{\lambda,\mu}g(z), zL^{k,v+3}_{\lambda,\mu}g(z); z) \prec h_2(z), \tag{3.5}$$

then $\lambda_1(z) \prec zL^{k,v}_{\lambda,\mu}g(z) \prec \lambda_2(z)$.

4. Conclusion

Numerous implementations of Mittag-Leffler functions types have a great influence on the study of engineering, mathematics and other associated areas of science. In the present research, we have applied the subordination and superordination methodologies to introduce appropriate classes of admissible functions. Moreover, we have yielded new outcomes on differential subordination and superordination for meromorphic functions, which are formulated by a new complex convolution operator based on the generalized Mittag-Leffler functions (or named R-function) in the punctured unit disk. Furthermore, the sandwich-type outcome (upper bonded and lower bonded) is also gained for this operator. In addition, it's possible to utilize the relation (1.5) to obtain a corresponding sandwich-outcome. Finally, in terms of this operator, a lot geometric classes may be investigated in the Geometric Function Theory.

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