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A note on the Lyapunov functions for SIR and SIRS epidemic model

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Abstract

O'Regan et al [Journal of Applied Mathematics Letters. 23 (2010) , pp. 446-448] constructed a function as a Lyapunov function for getting stability of endemic equilibrium of a mathematical model for an epidemic. In this paper, we prove the introduced function does not satisfy some of the conditions of Lyapunov function. At last, we correct the introduced function which can help us to reach the aims in stability.

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1. Introduction

Although Euler and Bernoulli wanted to explain the process of epidemics by mathematics and its rules, but we can say the first modern developed mathematical model was introduced in 1927 by Kermack and Mckendrick (see [2]). This model was an analytic one for black plauge. They approximated solution for $I(t)$, in the SIR model [2]. In 1933, another new model was introduced as SIS modeling. It was a beginning for threshold theory. After that, many applied and valuable mathematical modeling was written by mathematicians, for example see [6]. In 2019, a nonlinear epidemic model SEIRS was introduced with respect to media impact on the transmission dynamics of infectious diseases [5]. Lyapunov function as an applied and helpful tool in dynamical systems has good

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role for developing mathematical modeling for epidemic. We directly use Lyapunov method, introduced by Russian great mathematician, Lyapunov, for consideration, working and analysing of stability of fixed point in epidemic model [3]. Lyapunov gave us a helpful tool by introducing a function with respect to dynamical system of epidemic. O'Regan et al [4] considered a mathematical modeling and introduced a function and claimed that their introduced function is a Lyapunov function. Then they showed the equilibrium point is globally asymptotically stable. In this paper we see the introduced function is not a Lyapunov function. Then we correct the introduced function as a suitable Lyapunov function.

Now we see one definition and one theorem in the following manner:

Definition 1.1 : [1] Suppose x_0 be a fixed point of $\dot{x} = g(x),$, where $x = (x_1, x_2, ...x_n)$. A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be a Lyapunov function for x_0 if there exists a neighborhood T of x_0 such that the following conditions hold:

(1) $f(x_0) = 0$,

(2) $f(x) > 0$ for all $x \neq x_0$,

(3) $\dot{f}(x) \leq 0$ for all x in T.

if the following inequality

$$\dot{f}(x) < 0 \text{ for all } x \neq x_0 \text{ in } T$$

can be held instead of (3), then f is said to be a strict Lyapunov function.

Theorem 1.2 : [1] *Let $\overline{x}$ be a fixed point of $\dot{x} = f(x)$. If we have a strict Lyapunov function for $\overline{x}$, then $\overline{x}$ will be asymptotically stable. Also, if a radially unbounded strict Lyapunov function can be found for $\overline{x}$, then $\overline{x}$ is globally asymptotically stable.*

2. On the Lyapunov function for SIR and SIRS epidemic model

We rewrite an SIRS mathematical model of [4]:

$$\dot{I} = (q - \beta I - \beta R)\mathrm{I},$$
$$\dot{R} = \gamma I - \omega R.$$

$I(t)$ and $R(t)$ are infected and recovered (removed) people, respectively. b and w are the birth and death rate, respectively. w_1 is the disease related death rate. γ is the rate of recovery and β is applied to show the incident rate. All of the coefficients are assumed to be strictly positive. Also the rate of the loss of immunity is denoted by α. p is the percent of offspring of the infectives are born infected. Obviously, these new born people enter into the infective group. In the above system by considering $q = \beta N - \sigma$ and $\sigma = \gamma + w_1 + w - pb$, if $R_0 = \frac{\beta N}{\sigma} > 1$ then $(I_*, R_*) = \left(\frac{\omega q}{\beta(\gamma+\omega)}, \frac{\gamma q}{\beta(\gamma+\omega)}\right)$ is the positive equilibrium point.

O'Regan et al [4] introduced the following function as the Lyaponov function $U(I,R) = I - I_* ln(I) + a(R - R_*)^2$, where $a = \frac{\beta}{2\gamma} > 0$.

It is easy to see that $U(I,R)$ is not a Lyaponov function. Because $U(I_*, R_*) = 0$ if and only if $I_* = e$, where e is the Napier number. On the other hand, I_* is not limited just to e. I_* is related to ω, q, β and γ.

Then, we can not say Q_* is globally asymptotically stable with respect to the introduced function.

Now by introducing the Lyaponov function as the following aims of O'Regan et al [4] about stability can be correct.

$$U(I,R) = I - I^* - I_*(ln(I) - ln(I^*)) + a(R - R_*)^2$$

Also, the system 3 in [4], is given from the system 2 in [4] by omitting the tildes for simplicity of notaions.

But, here, omitting the tildes, in the case, is not completely correct; Because by omitting the tildes, we get β and ω, where, are not β and ω in system 2.

Using the other alphabets, is more useful than omitting tildes.

3. Conclusion

In this paper, we worked on a mathematical modeling and in the following on an introduced function as a Lyapunov function in [4]. We proved that the mentioned function is not a Lyapunov function. Hence this function is not useful for considering the stability of positive equilibrium point. At last, we corrected the introduced function where the stability results remain valid as all other properties in Definition 1.1 are satisfied.

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