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Minimum mean covering Gutman energy of certain graphs

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Abstract

The mean-energy orm-energy of a graph was introduced by Laura Buggy, Amaliailiuc, Katelyn Mc.Call, Duyguyen in 2010. The minimum mean covering energy (MMCE) is a special type of graph energy, symbolically represented by $E_c(G)$, introduced by C. Adiga et. al. The sum of the magnitudes of the characteristic values obtained from the Gutman matrix is named as Gutman energy. we present the results of minimum mean covering

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Gutman energies (MMCGE) of certain standard graphs $(E_{CG}^{M}(G))$ and the lower and upper bounds of $E_{CG}^{M}(G)$ were also established.

Subject Classification: *05C50, 05C99.*

Keywords: *Gutman energy, Minimum mean covering energy, Minimum mean covering Gutman energy.*

1. Introduction

"Energy of graphs" is one of the key parameter in the study of graphs especially minimum mean covering energy in its minimum covering. A special type of graph energy named MMCE was introduced by Adiga et. al. [1]. The sum of the magnitudes of characteristic values from the Gutman matrix is named as Gutman energy[2]. Some of the preliminary definitions and examples are mentioned here:

Definition 1.1: The smallest set $\check{C} \subseteq V$ of a graph $G = (V, E)$ such that for every $e \in E$ at least one $v \in \check{C}$ as an end vertex of e is minimum covering set (*MCS*) of G.

Definition 1.2: Let Č be the *MCS* of $G = (V, E)$. Then the minimum covering Gutmanmatrix (*MCGM*)[5] of G is a $n \times n$ matrix defined as:

$$\bar{\bar{A}}_{CGM}(G) = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in CD \\ 0 & \text{if } i = j \text{ and } v_i \notin CD \\ d_i d_j d(v_i, v_j) & \text{otherwise} \end{cases}$$

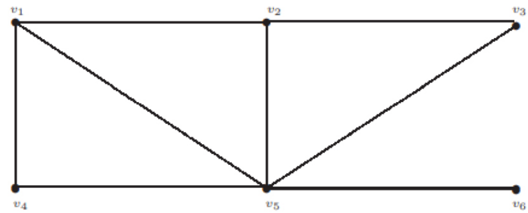
Where d_i, d_j are the degrees of vertices v_i, v_j and $d(v_i, v_j)$ is the shortest distance between v_i, v_j in G.

Definition 1.3: For a graph G of order "n" with $\bar{\bar{A}}_{CGM}(G)$ as *MCGM*, the minimum covering Gutman energy(MCGE) of graph [1, 2, 5] is $\mathbb{E}_{CG}(G) = \sum_{i=1}^{n} |g_i|$, where $g_1, g_2, \ldots, g_n$ are the characteristic values of $\bar{\bar{A}}_{CGM}(G)$.

Definition 1.4: The *MMCGE* [3,4] of a graph "G" is then defined as $\mathbb{E}_{CG}^{M}(G) = \sum_{i=1}^{n} |g_i - \bar{g}|$, where $g_1, g_2, \ldots, g_n$ are the characteristic values of $\bar{\bar{A}}_{CGM}(G)$ and $\bar{g}$ their average.

Example 1.5: The *MMCGE* depends upon the choice of *MCS* of **G**.

Proof: For the following graph G, with minimum covering sets $\check{C}_1 = \{v_1, v_2, v_5\}$, $\check{C}_2 = \{v_2, v_4, v_5\}$



(i) For $\check{C}_1 = \{v_1, v_2, v_5\}$,

the *MCGM* is $\overline{\overline{A}}_{\check{C}_1}(G) = \begin{pmatrix} 1 & 9 & 12 & 6 & 15 & 6 \\ 9 & 1 & 6 & 12 & 15 & 6 \\ 12 & 6 & 0 & 8 & 10 & 4 \\ 6 & 12 & 8 & 0 & 10 & 4 \\ 15 & 15 & 10 & 10 & 1 & 5 \\ 6 & 6 & 4 & 4 & 5 & 0 \end{pmatrix}$

The characteristic values are –16.3417, –14.00, –7.2977, -2.5721, -2 and 45.2115.

Mean = $\frac{3}{6}$ = 0.5.

The Minimum Mean Covering Gutman Energy (MMCGE) is computed as:

$$\mathbb{E}^{M}_{C_1}(G) = |-16.3417-3| + |-14-3| + |-7.2977-3| + |-2.5721-3| + |-2-3| + |45.2115-3|$$
$$= 19.3417 + 17 + 10.2977 + 5.5721 + 5 + 42.2115 = 99.423.$$

Similarly**,** we can find the Minimum Mean Covering Gutman Energy of $C_2 = \{v_2, v_4, v_5\}$ and we can observe that the minimum mean covering Gutman energy depends upon minimum covering set.

In the following sections, MMCGE for certain graphs and lower and upper bounds of few standard graphs are discussed.

2. The Minimum Mean Covering Gutman Energy of graphs

Theorem 2.1: *The MMCGE of star graph* $\mathcal{K}_{1,g-1}$ *is* $\frac{(g-2)(2g+1)}{g} + \sqrt{(2g-3)^2 + 4(g^3 - 3g^2 + g + 3)}$ when $g \geq 3$.

Proof: The minimum covering set is $CD = \{v_1\}$ (where v_1 is the centre vertex) for $\mathcal{K}_{1,g-1}$ with $V = \{v_1, v_2, \ldots, v_g\}$.

The MCGM of $\mathcal{K}_{1,g-1}$ is

$$\bar{\bar{A}}_{CGM}(\mathcal{K}_{1,g-1}) = \begin{pmatrix} 1 & g-1 & g-1 & & g-1 & g-1 \\ g-1 & 0 & 2 & \cdots & 2 & 2 \\ g-1 & 2 & 0 & & 2 & 2 \\ & \vdots & & \ddots & & \vdots \\ g-1 & 2 & 2 & \cdots & & 0 \end{pmatrix}_{g\times g}$$

Then the characteristic Equation (CE) $|\bar{\bar{A}}_{CGM}(\mathcal{K}_{1,g-1}) - \vartheta I| = 0$ is

$$(\vartheta + 2)^{g-2}(\vartheta^2 - (2g-3)\vartheta - (g^3 - 3g^2 + g + 3)) = 0.$$

The Characteristic Equation (CE) of $MCGM$ of $k_{1,g-1}$ are -2 with (g-2) times and $\frac{(2g-3)+\sqrt{(2g-3)^2+4(g^3-3g^2+g+3)}}{2}$ each one time.

Mean $\bar{\vartheta} = \frac{1}{g}$.

The $MMCGE$ is computed as:

$$\begin{aligned}\mathbb{E}^M_{CG}(\mathcal{K}_{1,g-1}) &= \sum_{i=1}^{g-2} |-2 - \frac{1}{g}| + |\frac{(2g-3)+\sqrt{(2g-3)^2+4(g^3-3g^2+g+3)}}{2} - \frac{1}{n}| \\ &= \frac{(g-2)(2g+1)}{g} | + | \frac{(2g^2-3g-2) \pm g\sqrt{4g^2-16g+21}}{2g} | \\ &= \frac{(g-2)(2g+1)}{g} + \sqrt{(2g-3)^2 + 4(g^3 - 3g^2 + g + 3)} \text{ when } g \geq 3.\end{aligned}$$

Theorem 2.2: The $MMCGE$ of *Crown graph* S_g^0 *is*

$$4(g-1)^3 + 1 + (g-1)\sqrt{(2g-3)^2(2g-1)^2 + 8(g-1)^2} \ \textit{when}\ g \geq 3.$$

Proof: Let $V = \{u_1, u_2, \ldots, u_g, v_1, v_2, \ldots, v_g\}$ be vertex set of S_g^0.

Then MCS is $C = \{u_1, u_2, \ldots, u_n\}$.

Then $MCGM$, is an $2g \times 2g$ matrix,

$$A_{CGM}(S_g^0) =$$

$$\begin{bmatrix}
1 & 2(g-1)^2 & 2(g-1)^2 & \cdot\ \cdot & 2(g-1)^2 & 3(g-1)^2 & (g-1)^2 & (g-1)^2 & \cdot\ \cdot & (g-1)^2 \\
2(g-1)^2 & 1 & 2(g-1)^2 & \cdot\ \cdot & 2(g-1)^2 & (g-1)^2 & 3(g-1)^2 & (g-1)^2 & \cdot\ \cdot & (g-1)^2 \\
2(g-1)^2 & 2(g-1)^2 & (g-1)^2 & \cdot\ \cdot & 2(g-1)^2 & (g-1)^2 & (g-1)^2 & 3(g-1)^2 & \cdot\ \cdot & (g-1)^2 \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\
2(g-1)^2 & 2(g-1)^2 & 2(g-1)^2 & \cdot\ \cdot & (g-1)^2 & (g-1)^2 & (g-1)^2 & (g-1)^2 & \cdot\ \cdot & 3(g-1)^2 \\
3(g-1)^2 & (g-1)^2 & (g-1)^2 & \cdot\ \cdot & (g-1)^2 & 0 & 2(g-1)^2 & 2(g-1)^2 & \cdot\ \cdot & 2(g-1)^2 \\
(g-1)^2 & 3(g-1)^2 & (g-1)^2 & \cdot\ \cdot & (g-1)^2 & 2(g-1)^2 & 0 & 2(g-1)^2 & \cdot\ \cdot & 2(g-1)^2 \\
(g-1)^2 & (g-1)^2 & 3(g-1)^2 & \cdot\ \cdot & (g-1)^2 & 2(g-1)^2 & 2(g-1)^2 & 0 & \cdot\ \cdot & 2(g-1)^2 \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\
(g-1)^2 & (g-1)^2 & (g-1)^2 & \cdot\ \cdot & 3(g-1)^2 & 2(g-1)^2 & 2(g-1)^2 & 2(g-1)^2 & \cdot\ \cdot & 0
\end{bmatrix}$$

Then the *CE*, $|A_{CGM}(S_g^0) - \vartheta I| = 0$ is

$$(\vartheta^2 + (2g-3)(2g-1)\vartheta - 2(g-1)^2)^{g-1}(\vartheta^2 - (4(g-1)^3+1)\vartheta + (g-1)^3(3g(g-1)(g-4)+2)) = 0.$$

The *CV*of *MCGM*of S_g^0 are

$$\frac{-(2g-3)(2g-1) \pm \sqrt{(2g-3)^2(2g-1)^2 + 8(g-1)^2}}{2} \ (g-1) \text{ times}$$

$$\frac{(4(g-1)^3+1) \pm \sqrt{(4(g-1)^3+1)^2 - 4(g-1)^3(3g(g-1)(g-4)+2)}}{2} \text{ each one time.}$$

Mean $\vartheta = \frac{1}{2}$.

Then *MMCGE* of Crown graph

$$E_{CG}^M(S_g^0) = \sum_{i=1}^{g-1} \left| \left(\frac{-(2g-3)(2g-1) \pm \sqrt{(2g-3)^2(2g-1)^2 + 8(g-1)^2}}{2} - \frac{1}{2} \right) \right| + \left| \frac{(4(g-1)^3+1) \pm \sqrt{(4(g-1)^3+1)^2 - 4(g-1)^3(3g(g-1)(g-4)+2)}}{2} - \frac{1}{2} \right|$$

$$= 4(g-1)^3 + 1 + (g-1)\sqrt{(2g-3)^2(2g-1)^2 + 8(g-1)^2}$$

Theorem 2.3: *The MMCGE of the cocktail party graph $\mathcal{K}_{g\times 2}$ is equal to*

$$\frac{8g^4 - 16g^3 + 8g^2 + g - 2}{g} + \sqrt{(8g(g-1)^2 + 1)^2 - 32(g-1)^2} \text{ when } g \geq 3.$$

Proof: For $\mathcal{K}_{g\times 2}$ with vertex set $V = \bigcup_{i=1}^{g}\{u_i, v_i\}$, the *MCS* is $C = \bigcup_{i=1}^{g-1}\{u_i, v_i\}$.

Hence, the *MCGM* is

$$\bar{\bar{A}}_{CGM}(\mathcal{K}_{g\times 2}) =$$

$$\begin{bmatrix}
1 & 8(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
8(g-1)^2 & 1 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 8(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
4(g-1)^2 & 4(g-1)^2 & 8(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 8(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 8(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 \\
4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 0 & 8(g-1)^2 \\
4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & 4(g-1)^2 & \cdot & \cdot & \cdot & \cdot & 4(g-1)^2 & 4(g-1)^2 & 8(g-1)^2 & 0
\end{bmatrix}$$

The *CE*, $|\bar{\bar{A}}_{CGM}(\mathcal{K}_{g\times 2}) - \vartheta I| = 0$ is

$$(\vartheta + 8(g-1)^2)(\vartheta - 1)^{g-2}(\vartheta + (8(g-1)^2 - 1))^{g-1}(\vartheta^2 - (8g(g-1)^2 + 1)\vartheta + 8(g-1)^2) = 0.$$

The *CVs* of *MCGM* of $\mathcal{K}_{g\times 2}$ are $-8(g-1)^2$ of one time, $-(8(g-1)^2 - 1)$ of (g-1) times, 1 of (g-2) times and $\frac{(8g(g-1)^2+1)\pm\sqrt{(8g(g-1)^2+1)^2-32(g-1)^2}}{2}$ each one time.

Mean $\bar{e} = \frac{g-1}{g}$.

The *MMCGE* of $\mathcal{K}_{g\times 2} = \mathbb{E}^{M}_{CG}(\mathcal{K}_{g\times 2})$

$$= \left|-8(g-1)^2 - \tfrac{g-1}{g}\right| + \sum_{i=1}^{g-1}\left|-(8(g-1)^2 - 1) - \tfrac{g-1}{g})\right| + \sum_{i=1}^{g-2}\left|\left(1 - \tfrac{g-1}{g}\right)\right| +$$

$$+\left|\frac{(8g(g-1)^2+1)\pm\sqrt{(8g(g-1)^2+1)^2-32(g-1)^2}}{2} - \frac{g-1}{g}\right|$$

$$= \frac{g-1}{g}(8g(g-1)+1) + \frac{g-1}{g}(8g(g-1)^2 - g + g - 1) + \frac{g-2}{g}$$

$$+\frac{g(8g(g-1)^2+1) - g + 1 + g\sqrt{(8g(g-1)^2+1)^2 - 32(g-1)^2}}{2g}$$

$$= \frac{8g^4 - 16g^3 + 8g^2 + g - 2}{g} + \sqrt{(8g(g-1)^2+1)^2 - 32(g-1)^2}.$$

3. Bounds of *MMCGE*

Result 3.1 [2,5]: Let $G=(V,E)$ be a simple (g, f) graph with $|C|=\mathcal{K}$, where CisMCS. If $c_1, c_2,\ldots, c_g$ are the *CV*s of *MCGM* of *G i.e* $A_{CGM}(G)$ then

(i) $\sum_{i=2}^{g} c_i = \mathcal{K}$ (ii) $\sum_{i=1}^{g} c_i^2 = \mathcal{K}+2\zeta$ where $\zeta = \sum_{1\le i<j\le g}(d_i d_i d_{ij})^2$.

Theorem 3.2 [5]: *Let $G=(V,E)$ be a simple (g,f) graph with $|C|=\mathcal{K}$, where C is MCS. Then*

$$\sqrt{(\mathcal{K}+2\zeta\ )} \le \mathbb{E}_{CG}(G) \le \sqrt{g(\mathcal{K}+2)\zeta}.$$

Theorem 3.3: *Let G be a simple graph (g, f) with $|C|=\mathcal{K}$, where C is MCS. And $c_1, c_2,\ldots, c_g$ are the CVs of $A_{CGM}(G)$ with mean $\overline{c}$ then,*

$$\sqrt{(\mathcal{K}+2\zeta)-2|\overline{c}|\sqrt{g(\mathcal{K}+2\zeta)}\,|} \le \mathbb{E}_{CG}^{M}(G) \le \sqrt{g((\mathcal{K}+2\zeta)+g\,\overline{c}^{2})}.$$

Proof: By Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{g} |l_i m_i|\right)^2 = \left(\sum_{i=1}^{g} l_i^{2}\right)\left(\sum_{i=1}^{g} m_i^{2}\right)$$

Take $l_i = 1$, $m_i = |c_i - \overline{c}|$ then,

$$\begin{aligned}[\mathbb{E}_{CG}^{M}(G)]^2 &= (\textstyle\sum_{i=1}^{g} 1)(\sum_{i=1}^{g}|c_i-\overline{c}|^2)\\ &= (g)(\textstyle\sum_{i=1}^{g}|c_i-\overline{c}|^2)\\ &\le g(\textstyle\sum_{i=1}^{g} c_i^{2} + \sum_{i=1}^{g}\overline{c}^{2})\\ &= g(\textstyle\sum_{i=1}^{g} c_i^{2} + g\overline{c}^{2})\\ &= g(\mathcal{K}+2\zeta)+g\,\overline{c}^{2})\end{aligned}$$

Then

$$\mathbb{E}_{CG}^{M}(G) \le \sqrt{g((\mathcal{K}+2\zeta)+g\,\overline{c}^{2})}$$

Also

$$\begin{aligned}[\mathbb{E}_{CG}^{M}(G)]^2 &= (\textstyle\sum_{i=1}^{g}|c_i-\overline{c}|)^2\\ &\ge \textstyle\sum_{i=1}^{g}|c_i-\overline{c}|^2\\ &\ge \textstyle\sum_{i=1}^{g}|c_i|^2 - 2|\overline{c}|\sum_{i=1}^{g}|c_i|\\ &\ge (\mathcal{K}+2\zeta)-2|\overline{c}|\sqrt{g(\mathcal{K}+2\zeta)}\end{aligned}$$

Then

$$\mathbb{E}_{CG}^{M}(G) \geq \sqrt{(\mathcal{K}+2\zeta)-2\,\overline{c}\,|\sqrt{g(\mathcal{K}+2\zeta)}}.$$

Theorem 3.4 [5]: *Let G = (V, E) be a simple graph with* $|C|=\mathcal{K}$, *where C is MCS and P = det* $A_{CGM}(G)$. *Then*

$$\sqrt{(\mathcal{K}+2\zeta)+g(g-1)P^{2/g}}\ | \leq \mathbb{E}_{CG}(G) \leq \sqrt{(\mathcal{K}+2g\zeta)}.$$

Theorem 3.5: *Let G = (V, E) be a simple (g, f) graph with* $|C|=\mathcal{K}$, *where C is MCS and P = det* $A_{CGM}(G)$*and* $c_1, c_2,\ldots, c_g$ *are the CVs of MCGM,* $A_{CGM}(G)$ *with mean* $\overline{c}$ *then,*

$$\sqrt{(\mathcal{K}+2\zeta)-2|\overline{c}|\sqrt{g(\mathcal{K}+2g\zeta)}}\ | \leq \mathbb{E}_{CG}^{M}(G) \leq \sqrt{g((\mathcal{K}+2g\zeta)+g\,\overline{c}^2)}.$$

Proof: By Cauchy-Schwarz inequality

$$(\textstyle\sum_{i=1}^{g}|l_i m_i|)^2 = (\sum_{i=1}^{g} l_i^2)(\sum_{i=1}^{g} m_i^2)$$

Take $l_i = 1$, $m_i = |c_i - \overline{c}|$ then,

$$\begin{aligned}
[\mathbb{E}_{CG}^{M}(G)]^2 &= (\textstyle\sum_{i=1}^{g} 1)(\sum_{i=1}^{g}|c_i-\overline{c}|^2) \\
&= (g)\,(\textstyle\sum_{i=1}^{g}|c_i-\overline{c}|^2) \\
&\leq g\,(\textstyle\sum_{i=1}^{g} c_i^2+\sum_{i=1}^{g}\overline{c}^2) \\
&= g(\textstyle\sum_{i=1}^{g} c_i^2+g\overline{c}^2) \\
&\leq g((\textstyle\sum_{i=1}^{g} c_i)^2+g\overline{c}^2) \\
&= g((\mathcal{K}+2g\zeta)+g\,\overline{c}^2)
\end{aligned}$$

Then

$$\mathbb{E}_{CG}^{M}(G) \leq \sqrt{g((\mathcal{K}+2g\zeta)+g\,\overline{c}^2\ )}$$

Also

$$\begin{aligned}
[\mathbb{E}_{CG}^{M}(G)]^2 &= (\textstyle\sum_{i=1}^{g}|c_i-\overline{c}|)^2 \\
&\geq \textstyle\sum_{i=1}^{g}|c_i-\overline{c}|^2 \\
&\geq \textstyle\sum_{i=1}^{g}|c_i|^2-2|\overline{c}|\sum_{i=1}^{g}|c_i| \\
&\geq (\mathcal{K}+2\zeta)-2|\overline{c}|\sqrt{g(\mathcal{K}+2g\zeta)}
\end{aligned}$$

Then

$$\mathbb{E}_{CG}^{M}(G) = \sqrt{(\mathcal{K}+2\zeta)-2|\bar{c}|\sqrt{g(\mathcal{K}+2g\zeta)}}$$

Theorem 3.6 [5]: *Let G = (V, E) be a simple (g, f) graph with* $|C| = \mathcal{K}$, *where C is MCS, then*

$$\mathbb{E}_{CG}(G) \le \frac{\mathcal{K}+2\zeta}{g} + \sqrt{(g-1)\left[(\mathcal{K}+2\zeta)-\left(\frac{\mathcal{K}+2\zeta}{g}\right)^2\right]}.$$

Theorem 3.7: *Let G = (V, E) be a simple (g, f) graph with* $|C| = \mathcal{K}$, *where C is MCS and then*

$$\mathbb{E}_{CG}^{M}(G) \le \sqrt{\left(g\left[\left(\frac{\mathcal{K}+2\zeta}{g} + \sqrt{(g-1)\left[(\mathcal{K}+2\zeta)-\left(\frac{\mathcal{K}+2\zeta}{g}\right)^2\right]}\right)^2 + g\bar{c}^2\right]\right)}.$$

Proof: By Cauchy-Schwarz inequality

$$(\textstyle\sum_{i=1}^{g}|l_i m_i|)^2 = (\sum_{i=1}^{g} l_i^2)(\sum_{i=1}^{g} m_i^2)$$

Take $l_i = 1$, $m_i = |c_i - \bar{c}|$ then,

$$\begin{aligned}
[\mathbb{E}_{CG}^{M}(G)]^2 &= (\textstyle\sum_{i=1}^{g} 1)(\sum_{i=1}^{g}|c_i - \bar{c}|^2) \\
&= (g)(\textstyle\sum_{i=1}^{g}|c_i - \bar{c}|^2) \\
&\le g(\textstyle\sum_{i=1}^{g} c_i^2 + \sum_{i=1}^{g}\bar{c}^2) \\
&= g(\textstyle\sum_{i=1}^{g} c_i^2 + g\bar{c}^2) \\
&\le g[\,(\textstyle\sum_{i=1}^{g}|c_i|)^2 + g\bar{c}^2] \\
&\le g\left[\left(\frac{\mathcal{K}+2\zeta}{g} + \sqrt{(g-1)\left[(\mathcal{K}+2\zeta)-\left(\frac{\mathcal{K}+2\zeta}{g}\right)^2\right]}\right)^2 + g\bar{c}^2\right].
\end{aligned}$$

Hence

$$\mathbb{E}_{CG}^{M}(G) \le \sqrt{\left(g\left[\left(\frac{\mathcal{K}+2\zeta}{g} + \sqrt{(g-1)\left[(\mathcal{K}+2\zeta)-\left(\frac{\mathcal{K}+2\zeta}{g}\right)^2\right]}\right)^2 + g\bar{c}^2\right]\right)}.$$

Conclusion

The MMCGE for standard graphs is discussed and some of the good results for it were also discussed. Also discussed are the upper bounds of $\mathbb{E}_{CG}^{M}(G)$ in this article. These results are very useful in finding the energies between the atoms in chemical bonding and also in molecular orbital theory. The MMCGE can also be obtained for the graphs discussed by Mario Gionfriddo [6], Huiming Duan, et. al. [7] and Pintu Bhunia, et. al. [8].

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