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Group analysis of a Hamilton–Jacobi type equation

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Abstract

In this paper, we apply Lie group theory to a Hamilton-Jacobi type equation, which is a nonlinear partial differential equation of the first order. We then employ the local inverse theorem to build the Lie invariance condition for this equation. The determining equations are then obtained by using this condition. We split and solve these obtained equations to obtain the symmetries of the equation under study. We obtain the largest solvable Lie algebra. We also obtain the symmetry algebra and make a group classification of this equation. Further, some exact solutions are deduced and represented graphically.

Subject Classification: *(2020) 35R10, 35F20, 35B06, 35C99.*

Keywords: *Group analysis, Hamilton-Jacobi equation, Lie algebra, Partial differential equations, Symmetry.*

1. Introduction

Differential equations have several applications in modeling physical phenomenon. However, it is seldom understood why differential equations can be solved. Symmetry analysis provides the only unified understanding on the solvability of differential equations. Sophus Lie in the nineteenth century introduced the concept of one-parameter Lie groups to solve differential equations - both ordinary and partial. We study geometric objects in terms of linear objects through the Lie group-Lie algebra correspondence. A very important application of using the group analysis is the reduction in order of an Ordinary Differential Equation (ODE) (if it

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is of first-order, it becomes separable) and in the reduction of the number of independent variables for Partial Differential Equations (PDEs). More details on the application of symmetry analysis can be found in [4, 6, 7].

A transformation is called a *symmetry* if it leaves an object unchanged. Symmetries are an important tool in studying various laws that determine natural phenomenon, as seen in [11]. This concept that finds a wide range of applications in physical and applied sciences, has been used by scientists in the past to study the laws of motion as a symmetry principle. A noteworthy application of symmetries was the relationship between symmetries and conservation laws. This was observed in 1918 by Emma Nöether.

To answer the question of solvability of polynomials, Galois groups were associated with polynomials. The solvability of the Galois group (which was easier to establish) was used to imply the solvability of polynomials. This stood as a motivation for Lie as he thought on similar lines and used Lie groups for modeling the continuous symmetries of differential equations just as Galois used Galois groups for modeling the discrete symmetries of algebraic equations.

The Hamilton-Jacobi equation is a nonlinear PDE of the first-order and has the form

$$H(x, u_x(x, \beta, t)) + u_t(x, \beta, t) = J(\beta, t),$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The parameter $\beta \in \mathbb{R}^n$.

A simpler form of the Hamilton-Jacobi equation is

$$u_t + H(u_x, x) = 0, \tag{1}$$

where H mentioned above is a given nonlinear function.

Equation (1) finds applications in quantum theory, optics and mechanics. It's solutions, which are equations of motion of a mechanical system or ray approximations in optical systems, determine infinite families of solutions of Hamilton's ordinary differential equations.

The existence of continuous point symmetries for nonlinear PDEs is found in [14]. Sahadevan and Prakash [2] demonstrate the application of group analysis to time fractional nonlinear PDEs. Further, in [8], PDEs with fractional order derivatives are studied and exact solutions are obtained. Using similarity transformation method, [16] analyzes dispersive wave equations. Recently, in [2], similarity reduction and solitary wave solutions are carried out using Lie group theoretic method. Studies on PDEs include the explicit linear and implicit Euler finite difference schemes to estimate the numerical blow-up solutions of semi-linear heat equations

[12] and reaction-diffusion equations [13]. Research interests have also been extended to functional differential equations. Extending the Lie symmetry theory to Poisson differential equations, symmetries of their delay fractional counterpart are researched in [10]. Lie point symmetries for differential equations with neutral delay have been found in [9]. Such neutral differential equations with variable time lags are shown to have globally stable zero solutions by [18]. An emerging area of interest namely dynamic equations on time scales are already provided with existence of solutions based on fixed point theory by [17]. Group analysis can also be applied to study the properties of such equations.

In this paper, we choose $H = -u_x^2$ in (1) and do the analysis and computation to illustrate the application of group analysis in studying differential equations.

That is, we will study the equation

$$u_t = u_x^2, \tag{2}$$

where u, t and x are functions of a real variable on $\mathbb{R}$.

We organize the rest of the paper as follows:

In the following section, Lie groups, Lie algebras and their related results are given. In the next section, the local inverse theorem is used to build the invariance condition for PDEs. The solvability of the equation under study is then tackled using the derived condition. We then construct exact solutions of the Hamilton-Jacobi equation and present them graphically with an analysis of the solutions. The concluding section summarizes the results obtained.

2. Lie Groups and Lie Algebras

This section is concerned with introducing the definitions, concepts and standard results. We begin with the following definition.

Definition 1: Transformations, $\overline{t}_i = \xi_i(t_j, \delta), \forall\, 1 \le i, j \le n,$ continuously depending on the parameter δ with analytic $\xi_i\ \forall\ i$ are said to form a one-parameter group of transformations, if:

(1) (*Existence of Identity*) There exists a value of the parameter δ, say $\delta = 0$ for which the source and image points coincide. Precisely, $t_i = \xi_i(t_j, 0), \forall\, 1 \le i, j \le n.$

(2) (*Existence of Inverse*) The inverse transformation is characterized by the parameter $-\delta$. Precisely,

$$t_i = \xi_i(\overline{t}_j, -\delta), \forall\, 1 \le i, j \le n.$$

(3) (*Closure Property*) Carrying out of two transformations one after another results in a transformation which is again a member of the set. Precisely, if
$\overline{t}_i = \xi_i(t_j, \delta), \forall\, 1 \le i, j \le n.$, and $\hat{t}_i = \xi_i(\overline{t}_j, \varepsilon), \forall\, 1 \le i, j \le n.$, then
$\hat{t}_i = \xi_i(t_j, \delta + \varepsilon), \forall\, 1 \le i, j \le n.$

Definition2 : If the multiplication operation is smooth, then the group is a Lie group.

Example 1: Any finite group is a Lie group. Other examples include the orthogonal group, the unitary group, and the symplectic group to name a few. Further, the set of all matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $a, b, c, d \in \mathbb{C}$ with $\det(ad - bc) = 1$ form Lie groups.

Remark 1: The group is **abelian** if the transformations can be carried out in any order.

In the study of symmetry analysis, it becomes very important to note a characteristic of the determining equations. These equations, which are a system of PDEs can often be solved explicitly. The solutions obtained form a special vector space called as *Lie Algebras*.

These special vector spaces were first brought out by H. Weyl. In fact, Lie himself never called them *Lie algebras* but used the word *infinitesimal groups*. At this point, it becomes necessary to introduce Lie algebras.

Definition 3: Given a field $\mathbb{F}$, a Lie algebra L is a vector space over $\mathbb{F}$, together with a bilinear map known as the Lie bracket

$$L \times L \to L, \quad (S_1, S_2) \mapsto [S_1, S_2],$$

satisfying the following properties:

(1) $[S_1, S_1] = 0, \forall S_1 \in L.$

(2) (Jacobi identity)
$[S_1, [S_2, S_3]] + [S_2, [S_3, S_1]] + [S_3, [S_1, S_2]] = 0, \forall S_1, S_2, S_3 \in L.$

Our main results include the demonstration of the largest solvable ideal associated with the Hamilton-Jacobi type equation. As such, we need the concept of an ideal first, which is now defined below.

Definition 4: An ideal $\mathcal{I}$ is a Lie subalgebra of L such that $[S_1, S_2]$ is in $\mathcal{I}$, whenever S_1 is in $\mathcal{I}$ and S_2 is in L.

Definition 5: A $k-$ dimensional algebra is solvable if there exists nested chain of subalgebras L_i of dimension i such that L_i is an ideal in L_{i+1} for all $1 \le i \le k$. It should be noted that $L = \{0\}$ is the smallest algebra of the chain.

Definition 6: (Radical) An ideal of the largest dimension which is also solvable is said to be the radical of the Lie algebra L and is denoted by $radL$.

Remark 2: While any Lie algebra of dimension two is certainly solvable, the solvability of higher dimensional algebras have to be determined. In case a three or higher dimensional algebra isn't solvable, it makes sense to speak about its radical.

Interested readers may refer [5] for a comprehensive literature on Lie algebras.

3. Symmetry Analysis of PDEs of First-Order

We shall now use the (local) inverse theorem to develop the infinitesimal generators of the associated algebra for first-order PDEs.

Let u be a function dependent on t and x. With $\Theta = O(\delta^2)$, and for analytic functions h_1, h_2, h_3 in the dependent and independent variables we can express $h_{i's}$ as a Taylor series in δ as,

$$\begin{cases} \overline{t} = \overline{h_1} = t + \delta\lambda + \Theta, \\ \overline{x} = \overline{h_2} = x + \delta\mu + \Theta, \\ \overline{u} = \overline{h_3} = u + \delta\nu + \Theta, \end{cases} \tag{3}$$

where $\overline{h_i}, i = 1,2,3,$ denote h_i as functions of the dependent, independent variables and the parameter δ. The notations λ, μ, ν denote functions of the dependent and independent variable, namely t, x and u only.

$$\lambda = \left.\frac{\partial h_1}{\partial \delta}\right|_{\delta=0}, \quad \mu = \left.\frac{\partial h_2}{\partial \delta}\right|_{\delta=0}, \quad \nu = \left.\frac{\partial h_3}{\partial \delta}\right|_{\delta=0}.$$

The *total derivatives* introduced below can be written as:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{4}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots. \tag{5}$$

We can locally invert the first two equations of (3) to obtain t and x in terms of $\bar{t}$ and $\bar{x}$, under the assumption that the Jacobian does not vanish, i.e.,

$$\mathcal{J} = \begin{vmatrix} D_t \bar{t} & D_t \bar{x} \\ D_x \bar{t} & D_x \bar{x} \end{vmatrix} \neq 0, \quad \text{when} \quad u = u(x,t). \tag{6}$$

For equation (6) to hold, an application of the local inverse theorem allows us to write the last equation of (3) as:

$$\bar{u} = \bar{u}(\bar{t}, \bar{x}). \tag{7}$$

Applying chain rule to equation (7), results in

$$\begin{bmatrix} D_t \bar{u} \\ D_x \bar{u} \end{bmatrix} = \begin{bmatrix} D_t \bar{t} & D_t \bar{x} \\ D_x \bar{t} & D_x \bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_{\bar{t}} \\ \bar{u}_{\bar{x}} \end{bmatrix},$$

which by using Cramer's rule yields

$$\bar{u}_{\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t \bar{u} & D_t \bar{x} \\ D_x \bar{u} & D_x \bar{x} \end{vmatrix}, \quad \bar{u}_{\bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t \bar{t} & D_t \bar{u} \\ D_x \bar{t} & D_x \bar{u} \end{vmatrix} \tag{8}$$

We can simplify equation (8) to obtain the extended infinitesimal representation,

$$\bar{u}_{\bar{t}} = u_t + \delta \nu_{[t]} + \Theta, \quad \bar{u}_{\bar{x}} = u_x + \delta \nu_{[x]} + \Theta, \tag{9}$$

in which,

$$\nu_{[t]} = D_t(\nu) - u_x D_t(\mu) - u_t D_t(\lambda), \tag{10}$$

$$\nu_{[x]} = D_x(\nu) - u_x D_x(\mu) - u_t D_x(\lambda). \tag{11}$$

On expanding and simplifying (11), we get

$$\nu_{[t]} = \nu_t - \mu_t u_x + (\nu_u - \lambda_t) u_t - \mu_u u_x u_t - \lambda_u u_t^2, \tag{12}$$

$$\nu_{[x]} = \nu_x + (\nu_u - \mu_x) u_x - \lambda_x u_t - \mu_u u_x^2 - \lambda_u u_x u_t. \tag{13}$$

For any given Lie group, we can write the infinitesimal generator as:

$$\zeta^* = \lambda \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial u}. \tag{14}$$

Putting all the facts above together, the celebrated Lie invariance condition for PDEs $\Delta(t, x, u, u_x, u_t) = 0$ is given by $\zeta^{(1)^*} \Delta|_{\Delta=0} = 0$, where

$$\zeta^{(1)^*} = \zeta^* + \nu_{[t]} \frac{\partial}{\partial u_t} + \nu_{[x]} \frac{\partial}{\partial u_x}. \tag{15}$$

4. Symmetries of the Equation Under Study

An application of the operator given by equation (15) to equation (2), yields

$$\nu_{[t]} - 2u_x \nu_{[x]} = 0.$$

On substituting equation (12) and eliminating u_t from equation (2), we get

$$\nu_t - (\mu_t + 2\nu_x)u_x + (2\mu_x - \lambda_t - \nu_u)u_x^2 + (\mu_u + 2\lambda_x)u_x^3 + \lambda_u u_x^4 = 0.$$

Equating various coefficients of u_x to 0, we get

$$\nu_t = 0, \quad \lambda_u = 0, \quad \mu_t + 2\nu_x = 0, \quad \mu_u + 2\lambda_x = 0, \quad 2\mu_x - \lambda_t - \nu_u = 0.$$

These equations on solving yield

$$\lambda = \alpha(t, x), \quad \mu = -2\alpha_x u + \beta(t, x), \quad \nu = -2\alpha_{xx} u^2 + (2\beta_x - \alpha_t)u + \gamma(t, x),$$

where α, β, γ are arbitrary functions in *t* and *x*.

These equations when put into the determining equations give,

$$\alpha_{xxx} = 0, \alpha_{tx} - \beta_{xx} = 0, \beta_t + 2\gamma_x = 0, \alpha_{txx} = 0, \alpha_{tt} - 2\beta_{tx} = 0, \gamma_t = 0.$$

Solving these equations we get,

$$\alpha = c_1 x^2 + (2c_2 t + c_3)x + 4c_4 t^2 + c_5 t + c_6,$$

$$\beta = c_2 x^2 + (4c_4 t + c_7)x + 2c_8 t + c_9,$$

$$\gamma = -c_4 x^2 - c_8 x + c_{10}.$$

This in turn gives the infinitesimals,

$$\lambda(t, x, u) = c_1 x^2 + (2c_2 t + c_3)x + 4c_4 t^2 + c_5 t + c_6,$$

$$\mu(t,x,u) = -2(2c_1x + 2c_2t + c_3)u + c_2x^2 + (4c_4t + c_7)x + 2c_8t + c_9,$$

$$\nu(t,x,u) = -4c_1u^2 + (2c_2x - c_5 + 2c_7)u - c_4x^2 - c_8x + c_{10},$$

where $c_1, \cdots, c_{10}$ are constants of integration.

We immediately see that there are ten elements in the basis set. We can enumerate the generators of the Lie algebra as,

$$S_1 = x^2\frac{\partial}{\partial t} - 4xu\frac{\partial}{\partial x} - 4u^2\frac{\partial}{\partial u}, \quad S_2 = 2tx\frac{\partial}{\partial t} + (x^2 - 4tu)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u},$$
$$S_3 = x\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial x}, \quad S_4 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial u}, \quad S_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad S_6 = \frac{\partial}{\partial t},$$
$$S_7 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \quad S_8 = 2t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \quad S_9 = \frac{\partial}{\partial x}, \quad S_{10} = \frac{\partial}{\partial u}.$$

Table 1

The Commutator Table

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_1	0	0	0	0	S_1	0	$-2S_1$	$-2S_2$	$-2S_3$	$4S_7$
S_2	0	0	$-S_1$	0	0	$-2S_3$	$-S_2$	$-S_4$	$-2(S_5+S_7)$	$2S_8$
S_3	0	S_1	0	$2S_2$	S_3	0	$-S_3$	$-2S_5$	$-S_6$	$2S_9$
S_4	0	0	$-2S_2$	0	$-S_4$	$4S_7-8S_5$	0	0	$-2S_8$	0
S_5	$-S_1$	0	$-S_3$	S_4	0	$-S_6$	0	S_8	0	S_{10}
S_6	0	$2S_3$	0	$8S_5-4S_7$	S_6	0	0	$2S_9$	0	0
S_7	$2S_1$	S_2	S_3	0	0	0	0	$-S_8$	$-S_9$	$-2S_{10}$
S_8	$2S_2$	S_4	$2S_5$	0	$-S_8$	$-2S_9$	S_8	0	S_{10}	0
S_9	$2S_3$	$2(S_5+S_7)$	S_6	$2S_8$	0	0	S_9	$-S_{10}$	0	0
S_{10}	$-4S_7$	$-2S_8$	$-2S_9$	0	$-S_{10}$	0	$2S_{10}$	0	0	0

From Table 1, we see that $L' = L$, and therefore L is not solvable. Consider the following subalgebras,

$$L_1 = span\{S_1\},$$

$$L_2 = span\{S_1, S_2\},$$

$$L_3 = span\{S_1, S_2, S_3\},$$

$$L_4 = span\{S_1, S_2, S_3, S_4\},$$

$$L_5 = span\{S_1, S_2, S_3, S_4, S_5\},$$

$$L_6 = span\{S_1, S_2, S_3, S_4, S_5, S_8\},$$

$$L_7 = span\{S_1, S_2, S_3, S_4, S_5, S_7, S_8\}.$$

We see that,

$$\{0\} \subset L_1 \subset L_2 \subset L_3 \subset L_4 \subset L_5 \subset L_6 \subset L_7.$$

Consequently L_7 is the radical of L.

The largest possible dimension of the solvable Lie algebra cannot exceed 7. This is because, if I is a solvable algebra of dimension 8, then L/I becomes two dimensional and thus solvable. Now having got L/I as solvable, this would imply the existence of some $m \geq 1$ such that $(L/\mathrm{I})^{(m)} = 0$. This would mean that $L^{(m)} + \mathrm{I} = \mathrm{I}$ and thus $L^{(m)} \subset \mathrm{I}$. If I were also to be solvable, then this would imply that for some $k \geq 1$ we must have $\mathrm{I}^{(k)} = 0$, and hence $(L^{(m)})^{(k)} = L^{(m+k)} \subset \mathrm{I}^{(k)} = 0$. Consequently L becomes solvable, which is not so.

The above arguments lead to the following theorem:

Theorem 1: *The Hamilton-Jacobi type equation (2) admits a 10–dimensional Lie algebra with a 7–dimensional radical.*

5. Solutions of the Hamilton-Jacobi Type Equation

5.1 *Solutions Using Canonical Coordinates*

As we are interested in solutions of equation (2), we shall now introduce three new variables in t, x and u, namely ϕ, ψ, θ, such that ϕ and ψ are independent and θ is dependent. An application of the change of variables $(t, x, u) \to (\phi, \psi, \theta)$ yields

$$\begin{aligned}\zeta^* &= \lambda \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial u} \\ &= \lambda \left(\phi_t \frac{\partial}{\partial \phi} + \psi_t \frac{\partial}{\partial \psi} + \theta_t \frac{\partial}{\partial \theta} \right) + \mu \left(\phi_x \frac{\partial}{\partial \phi} + \psi_x \frac{\partial}{\partial \psi} + \theta_x \frac{\partial}{\partial \theta} \right) \\ &\quad + \nu \left(\phi_u \frac{\partial}{\partial \phi} + \psi_u \frac{\partial}{\partial \psi} + \theta_u \frac{\partial}{\partial \theta} \right)\end{aligned}$$

$$= (\lambda\phi_t + \mu\phi_x + \nu\phi_u)\frac{\partial}{\partial\phi} + (\lambda\psi_t + \mu\psi_x + \nu\psi_u)\frac{\partial}{\partial\psi}$$
$$+ (\lambda\theta_t + \mu\theta_x + \nu\theta_u)\frac{\partial}{\partial\theta},$$

wherein if we choose,

$$\begin{aligned} \lambda\phi_t + \mu\phi_x + \nu\phi_u &= 0, \\ \lambda\psi_t + \mu\psi_x + \nu\psi_u &= 1, \\ \lambda\theta_t + \mu\theta_x + \nu\theta_u &= 0, \end{aligned} \tag{16}$$

gives $\zeta^* = \frac{\partial}{\partial\psi}$, which represents a translation in the variable *s*.

We therefore see that an application of the change of variables reduces the equation into one in which an independent variable is removed. More precisely, an equation of the form $\Phi(t, x, u, u_t, u_x) = 0$ is transformed to $\Psi(\phi, \theta, \theta_\phi, \theta_\psi) = 0$. Although this is still a PDE, we can insist on having solutions of the form $\theta = \theta(\phi)$ only. This transformed equation will then get reduced to $\eta(\phi, \theta, \theta_\phi) = 0$ which is a first-order ODE!

These arguments lead to the following result:

Theorem 2: *On introducing canonical variables* ϕ, ψ, θ, *equation (3) can be converted to the translation group*

$$\bar{\phi} = \phi, \bar{\psi} = \psi + \delta, \bar{\theta} = \theta.$$

We shall now analyze equation (2) by taking some special cases. To begin with, consider $c_5 = 1 = c_7$, and $c_i = 0$ for other *i*. This gives

$$\lambda = t, \quad \mu = x, \quad \nu = u.$$

By applying the change of variables procedure discussed above,

$$t\phi_t + x\phi_x + u\phi_u = 0,$$

$$t\psi_t + x\psi_x + u\psi_u = 1,$$

$$t\theta_t + x\theta_x + u\theta_u = 0.$$

This solution of this system is found to be,

$$\phi = A\left(\frac{x}{t}, \frac{u}{t}\right), \quad \psi = \ln t + B\left(\frac{x}{t}, \frac{u}{t}\right), \quad \theta = C\left(\frac{x}{t}, \frac{u}{t}\right).$$

Here A, B, C are arbitrary functions of $\frac{x}{t}$ and $\frac{u}{t}$.

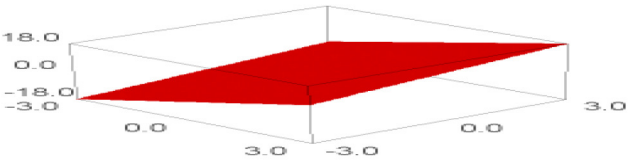


Figure 1

Representation of $u = c_{11}x + c_{11}^2 t, t \in [-3,3], x \in [-3,3], c_{11} = 2$

Suitably choosing $A, B, C,$ we can get

$$\phi = \frac{x}{t}, \quad \psi = \ln t, \quad \theta = \frac{u}{t}.$$

This will then transform equation (2) to yield,

$$\theta + \theta_\psi - \phi\theta_\phi = \theta_\phi^2.$$

If we put $\theta_\psi = 0,$ and simplify the equation further, we get the Clairaut equation,

$$\theta_\phi^2 + \phi\theta_\phi - \theta = 0,$$

whose general solution is,

$$\theta = -\frac{1}{4}\phi^2, \quad \theta = c_{11}\phi + c_{11}^2,$$

where c_{11} is a constant of integration. The exact solution obtained on passing through the transformation is,

$$u = -\frac{x^2}{4t}, t \in [-5,5], x \in [-5,5],$$

The graphical representation of this solution is given in Figure 1.

We observe that the solution $u(t,x) = c_{11}x + c_{11}^2 t$ becomes unbounded with the passage of time t.

5.2 *Solutions Using Direct Reduction*

In Section 5.1, we considered invariance of equation (2) and obtained infinitesimals λ, μ and ν leaving the original equation invariant. We introduced new variables ϕ, ψ and θ transforming the equation into a new equation independent of ψ. Setting $\theta_\psi = 0$ gave rise to an ordinary differential equation which we solved giving rise to an exact solution of

the original partial differential equation. In this section, we shall show that we can go directly to the ordinary differential equation without the intermediate step of introducing new variables ϕ, ψ and θ. For this, it suffices to choose

$$\frac{dt}{\lambda} = \frac{dx}{\mu} = \frac{du}{\nu}. \tag{17}$$

Once this is solved, it will give us the solution form or ansatz and upon substitution into the original equation, will yield an ordinary differential equation which we need to solve.

To understand this, let us consider the choices of the infinitesimals made in Section 5.1, namely

$$\lambda = t, \quad \mu = x, \quad \nu = u.$$

To go directly to the solution ansatz, we solve equation (17), namely

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{u}.$$

The solution of this system is found to be

$$u = tF\left(\frac{x}{t}\right). \tag{18}$$

The substitution of equation (18) in equation (2), gives

$$F'^2 + rF' - F = 0,$$

where $F = F(r)$ and $r = \frac{x}{t}$, which is exactly the same ordinary differential equation obtained earlier (with θ instead of F).

We know illustrate another choice of infinitesimals which will yield another solution of equation (2).

Choosing $c_1 = 1$ and all other constants as 0, we get the infinitesimals

$$\lambda = x^2, \quad \mu = -4xu, \quad \nu = -4u^2.$$

Equation (17), then yields

$$\frac{dt}{x^2} = \frac{dx}{-4xu} = \frac{du}{-4u^2}.$$

The solution of this system is

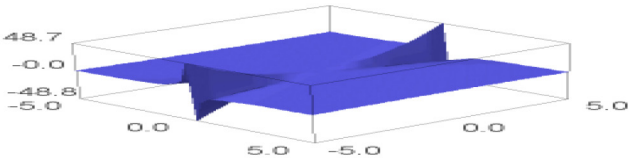


Figure 2

Representation of $u = -\frac{x^2}{4t}, t \in [-5,5], x \in [-5,5], c_{12} = 0$

$$x + \frac{4tu}{x} = F\left(\frac{u}{x}\right). \tag{19}$$

We note that the solution in this case is implicit. Substituting equation (19) in equation (2), we get the linear ordinary differential equation

$$rF' = F,$$

whose solution is $F(r) = c_{12}r$, where c_{12} is an arbitrary constant.

Substituting this in equation (19), we get

$$x + \frac{4tu}{x} = c_{12}\frac{u}{x},$$

which on solving for u yields the solution

$$u = \frac{x^2}{c_{12} - 4t}.$$

This solution can be graphically represented as in Figure 2.

We observe that the solution tends to 0 with the passage of time t, under the assumption that x lies in a set which is bounded.

As a final illustration, we consider another choice of infinitesimals which will yield another solution of equation (2).

Choose $c_6 = 1, c_8 = -3$ and all other constants equal to 0, to get the infinitesimals

$$\lambda = 1, \quad \mu = -6t, \quad \nu = 3x.$$

Equation (17), then yields

$$\frac{dt}{1} = \frac{dx}{-6t} = \frac{du}{3x}.$$

The solution of this system is

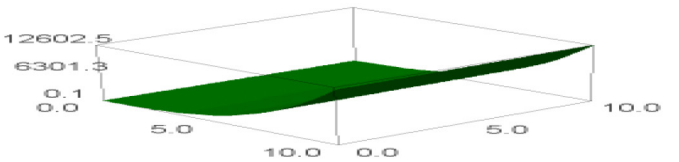


Figure 3

Representation of $u = 3xt + 6t^3 + \frac{2\sqrt{3}}{3}(x + 3t^2)^{3/2} + c_{13}$, $t \in [0, 10]$, $x \in [0, 10]$, $c_{13} = 0$

$$u = 3xt + 6t^3 + F(x + 3t^2). \tag{20}$$

Substituting equation (20) in equation (2), we get

$$F'^2 = 3r,$$

where $r = x + 3t^2$, whose solution is $F(r) = \frac{2\sqrt{3}}{3} r^{3/2} + c_{13}$, where c_{13} is an arbitrary constant.

This choice leads to the exact solution given by

$$u = 3xt + 6t^3 + \frac{2\sqrt{3}}{3}(x + 3t^2)^{3/2} + c_{13}.$$

The graphical representation of this solution is given in Figure 3.

In this case the solution becomes unbounded with the passage of time t.

6. Invariant Surface Condition of the Hamilton-Jacobi Type Equation

To the characteristic equations (17), we can associate the PDE of first-order

$$\lambda u_t + \mu u_x = \nu. \tag{21}$$

We call equation (21) as the "invariant surface condition."

Considering the Hamilton-Jacobi type equation under study, equation (21) now reads

$$\begin{aligned} &(c_1 x^2 + (2c_2 t + c_3)x + 4c_4 t^2 + c_5 t + c_6)u_t \\ &+ (-2(2c_1 x + 2c_2 t + c_3)u + c_2 x^2 + (4c_4 t + c_7)x + 2c_8 t + c_9)u_x \\ &= -4c_1 u^2 + (2c_2 x - c_5 + 2c_7)u - c_4 x^2 - c_8 x + c_{10}. \end{aligned} \tag{22}$$

Equation (22) is the invariant surface condition for the Hamilton-Jacobi equation.

The reason for the coined name is as follows. If we have a solution $u = u(t, x)$, then on considering

$$\overline{u} = u(\overline{t}, \overline{x}) \tag{23}$$

we observe that under equation (3), equation (23) gets transformed to

$$u + \delta v + \Theta = u(t + \delta\lambda + \Theta, x + \delta\mu + \Theta).$$

An expansion of the above gives us

$$u + \delta v + \Theta = u(t, x) + \delta(\lambda u_t + \mu u_x) + \Theta.$$

Equating the coefficients of δ on both sides, we see that we get equation (21).

7. Conclusion

Using the local inverse theorem, we have developed the invariance condition for PDEs of the first-order. We then use this to compute the symmetry algebra of a Hamilton-Jacobi type equation. We have found that it admits a non-solvable 10-dimensional Lie algebra. As such, we have discovered the radical of this Lie algebra. We then discussed two approaches to compute the exact solutions of this nonlinear differential equation under study. The solutions have also been represented graphically. Finally, we have found the invariant surface condition for this PDE.

References

[1] G. W. Bluman, and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York (1989).

[2] A. Chauhan, K. Sharma, and R. Arora, Lie Symmetry Analysis, Optimum System, and Generalized Group Invariant Solutions of the (2+1)-Dimensional Date-Jimbo-Kashiwara-Miwa Equation, *Mathematical Methods in Applied Sciences*, 43(15), pp. 8823-8840 (2020).

[3] L. E. Dickson, Differential Equations from the Group Standpoint, *Annals of Mathematics*, 25, pp. 287-378 (1924).

[4] L. Dressner, Applications of Lie's Theory of Ordinary and Partial Differential Equations, Institute of Physics Publishing, Bristol and Philadelphia (1999).

[5] K. Erdmann, and M. Wildon, Introduction to Lie Algebras, Springer-Verlag, United States of America (2006).

[6] J. Hill, Differential Equations and Group Methods for Scientists and Engineers, CRC Press, Florida (1992).

[7] N. H. Ibragimov, Handbook of Lie Group Analysis of Differential Equations, CRC Press, Boca Raton (1996).

[8] E. H. El Kinani, and A. Ouhadan, Lie Symmetry Analysis of Some Time Fractional Partial Differential Equations, *International Journal of Modern Physics: Conference Series*, 38, pp. 1-8 (2015).

[9] J. Z. Lobo, and Y. S. Valaulikar, Lie Symmetries for First Order Neutral Differential Equations, *Journal of Applied Mathematics and Computational Mechanics,* 18(1), pp. 29-40 (2019).

[10] A. M. Nass, Symmetry Analysis of space-time fractional Poisson equation With Delay, *Quaestiones Mathematicae*, 42(9), pp. 1221-1235 (2019).

[11] F. Oliveri, Lie Symmetries of Differential Equations: Classical Results and Recent Contributions, *Symmetry Journal,* 2(2), pp. 658-706 (2010).

[12] M. A. Rasheed, R. A. Hameed, S. Kareem, and A. Jameel, On Numerical Blow-Up Solutions of Semilinear Heat Equations, *Iraqi Journal of Science*, 61(8), pp. 2077-2086 (2020).

[13] M. A. Rasheed, and F. Ghaffoori, On Blow-Up Time and Growth Rate of a Reaction-Diffusion Equation, *Italian Journal of Pure and Applied Mathematics*, 44, pp. 805-813 (2020).

[14] R. Sahadevan, and P. Ponnusamy, Exact Solutions and Maximal Dimension of Invariant Subspaces of Time Fractional Coupled Nonlinear Partial Differential Equations, *Communications in Nonlinear Science and Numerical Simulation,* 42, (2016). DOI: https://doi.org/10.1016/j.cnsns.2016.05.017.

[15] R. Sahadevan, and G. Nagavigneshwari, Continuous Symmetries of Certain Nonlinear Partial Differential Equations and their Reductions, *Physics Letters A,* 378(43), pp. 3155-3160 (2014).

[16] K. Sharma, R. Arora, and A. Chauhan, Invariance Analysis, Exact Solutions and Conservation Laws of (2+1)-Dimensional Dispersive Long Wave Equations, *Physica Scripta*, 95(5), (2020). DOI: https://doi.org/10.1088/1402-4896/ab5eae.

[17] S. Tikare, and C. Tisdell, Nonlinear Dynamic Equations on Time Scales with Impulses and Nonlocal Conditions, *Journal of Classical Analysis*, 16(2), pp. 125-140 (2020).

[18] C. Tunç, and Y. Altun, On the global stability of a neutral differential equation with variable time lags, *Bulletin of Mathematical Analysis and Applications,* 9(4), pp. 31-41 (2017).