Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 1, pp. 95–119
DOI : 10.47974/JIM-1650

Dynamical analysis of effects of human behavior on cholera in context to specific strata's

Chetan Swarup
Department of Basic Science
College of Science and Theoretical Studies
Riyadh-Male Campus
Saudi Electronic University
Riyadh 13316
Saudi Arabia

Sudipa Chauhan *
Amulya Chaudhary
Mamta Barik
Amity Institute of Applied Science
Amity University
Noida 201301
Uttar Pradesh
India

Abstract

This paper focuses on the dynamics of the mathematical model by considering the effect of human conduct and the role of immunity on cholera contamination. Since, spreading of infection is commonly modeled with the help of classes of the SIR model, the model is formulated as a SIR model inaugurated with one more compartment B which showcases the concentration of bacteria in the polluted water. We have obtained the basic reproductive rate R0 and showcased local stability as well as global stability through geometrical approach. Without infection, equilibrium point (E_0) is stable locally in conjunction with globally whenever $R_0 < 1$. Similarly, the stability condition for endemic equilibrium E^* is $R_0 \geq 1$. Hence, our aim is basically to explore the outcome as to how the alertness of mankind can play an effective role in the dynamics of such infections.

Subject Classification: *(2000) 34D20, 34D23.*

Keywords: *Basic reproduction number, Discrete delay, Distributive delay, Local stability, Geometrical approach.*

* *E-mail:* sudipachauhan@gmail.com

1. Introduction

Disease epidemics are the outcomes that occur when microorganisms (bacterium, germ, plague, etc) interact with the healthy organism. From the past years of medical history, infectious diseases treatment has gained extraordinary attainments having a major impact on mankind. As we know that since 2013, India has been facing natural disaster like floods every year in different strata's of our country which not only disturb the lives of people, but it has also created circumstances that are mainly responsible for the spread of many waterborne diseases. Furthermore, it has become a major obstacle from the viewpoint of community health as these diseases can spread very quickly and can pave the way to extreme disease load. As a repercussion, the economy can be badly affected in such countries where groundwater pollution and open excretion are incorporated with poor sanitation, unhygienic practices lead to large incidents of these types of diseases. Cholera is one out of such diseases. Cholera is one of the intense, diarrheal infections brought about by Vibrio cholera O1 or O139 that is related to a high worldwide weight. The thickly populated seaside zones of Bangladesh and India, with around 35 and 200 million individuals separately (2), has a long history of cholera flare-ups and stay among the high-hazard endemic nations (4). The clinical range of V. cholera diseases goes from asymptomatic colonization, from mild to acute diarrhea which could be hazardous. Generally, most tainted in individuals are asymptomatic, but the microorganisms are available in their stool for certain days after contamination. These microbes are shed back into the climate, which can possibly contaminate others and can occur mostly in those places that are overpopulated in specific slum areas or those where the environmental conditions act as a catalyst in spreading the infection. A key to controlling cholera and lessening is, utilizing a multifaceted methodology like sterilization, cleanliness, reconnaissance, human awareness programs, Oral rehydration salts (ORS) arrangements, and cholera immunization. A major reason for people suffering due to this is lack of immunity. V. cholerae-specific Memory B Cells (MBCs) play a vital role in sustaining long-term immunity against V. cholerae antigens and persist even after vibriocidal antibody titer levels go down (1) which is generally extremely less in the slum areas. For a long time, mathematical modeling, investigation, and reenactment for infectious diseases have provided useful insights into disease elements that can be used to manage general well-being and create realistic scourge prevention and control methods (12; 16). Biological, ecological, and geological features such as regional heterogeneity and age were integrated into some mathematical models. Several studies have been done in the past few years, among which some of them have played a vital role in awareness programme and media as the main aspect to check its effect (6; 7; 24). Whereas some have examined the impact of self alertness considering discrete-time delay (31; 35; 37) and some other model includes distributed time delay (36; 8). In (13), it is articulated by segregating the epidemic models in the occupancy of observable changes which are credence-based and familiarity-based and how its upshot impacts us. Heffernan and Collinson, on the other hand, used their study to examine how the media function is chosen to produce a dominant result. They learned about the ultimate outcomes of an endemic model with mass out-turns from the role with whom they came into contact (10). In (28), Zindoga Mukandavire had taken past data of cholera outburst at Zimbabwe and then by using reproduction number R_0 concluded that mass vaccine could improve the health of the populace. In (3), X. Wang and (26) Movahedi diversified their model by introducing behavioural change in humankind. Furthermore, Weiss and Mummert identified and later investigated the social distancing methods with whom diseases might multiply and move, is essential for short-term outbreaks of diseases (27). Then (20) scrutinized the psychological influence on the illness dynamics that entangle the multiple outsets of sustained infections. Later on, (9) equipped a logistic growth model just to cumulative the publicized number of Ebola sufferers to attest to the changes in populace behaviour and involvement. (11) fitted a basic SIS model by amalgamating the impact of media coverage. They enhanced the chores by using a patch model with capricious transfer coefficients. Meanwhile, a cholera model was proposed with delay and from the reproduction number R_0 we came to know that delay doesn't escorts to the periodic oscillations. (25; 34; 29) included delay with the cholera model in the attempt of controlling the bacterial infection. Hence the motivation behind this paper is to acquaint immune response in cholera dynamic transmission models. This model is used when individuals infect one another directly (i.e. more than through a disease vector) and an individual who recovered from the sickness is also included in the model so that we can have better immunity afterwards. Similarly, contacts between persons is also incorporated to be erratic and the population size is taken as permanent(i.e. there is no birth, no death due to disease, or death by God's will). But, we know that a slight time lag is always present in the transmission phase of every infection. Therefore, if an epidemic model has time delay, then it is more consistent with the true world situation. Hence, it is required to include delay to study the rich dynamics of such models.

In reference to the above Section 1, we have formulated a SIR mathematical model with an extra compartment with both discrete as well as distributive time delay with the fact that its contact rate is a linear function considering human behavior as a key point. In Section 2, we have defined the model structure which includes description of model and formation with help of few assumptions. In Section 3, we have discussed about the equilibrium points which are infection-free and endemic equilibrium. Next in Section 4, we have also obtained reproduction Number (R_0) using Next generation matrix method. Section 5 deals with local stability for both equilibrium points followed by global stability in Section 6, in which a Lyapunov function is derived to show global stability for E_0 whereas geometric procedure is used to show the global stability of E^*.

2. Structure of the model

In the respective model, there are four compartments, $S(t)$ (uninfected populace), $I(t)$ (infected populace), $R(t)$ (recovered populace) and $B(t)$ which means the concentration of bacteria, dubbed as Vibrio-cholera in the tainted water as we are fixating on cholera infection in the presence of discrete and distributive time delay. But our main concern is to find out the effects of the behaviour of mankind on the dynamics of Cholera-infection. The model is suggested as follows:

$$\begin{aligned}
\frac{dS}{dt} &= b(S+I+R) - bS - \beta_1(I)\int_{-\infty}^{t} F(t-\tau)S(\tau)d\tau I(t-\tau) \\
\frac{dI}{dt} &= \beta_1(I)\int_{-\infty}^{t} F(t-\tau)S(\tau)d\tau I(t-\tau) - e^{-b\tau}\beta_2(I)S(t-\tau)I(t-\tau) - bI \\
\frac{dR}{dt} &= e^{-b\tau}\beta_2(I)S(t-\tau)I(t-\tau) - bR \\
\frac{dB}{dt} &= \beta_3(I)I - \delta B
\end{aligned} \tag{1}$$

Where $F(t)$ is considered as the delay kernel which is a weighting factor and by the definition, $F(t)$ indicates how much significance should be given to the population at the early stage to calculate the current upshot on resource availability. So, here we will normalize it to $\int_0^{\infty} F(\tau)d\tau = 1$ so that distributive delay should not affect the equilibrium points. Now, for further calculation, we will consider $F(t) = ae^{-at}, a > 0,$ which signifies a weak delay kernel and it represents the maximal weighted feedback of the growth rate of the total population which is due to ongoing populace density meanwhile past densities has exponential nature so they will decreasingly influence the situation.

In addition, here are some standard assumptions:

1. All the new born individuals are considered to be susceptible and the total populace is locked already, as $N = S + I + R = 1 \quad \forall t$.
2. The fertility rate and mortality rate are equal which we mentioned by b in N.
3. The transmission rate and shedding rate of bacteria entirely depend upon the aggregate of infectives.
4. Infected individuals after they get recovered can be transferred to the removed class R through the infection period τ is $e^{-b\tau}\beta_2(I)S(t-\tau)I(t-\tau)$.
5. Our supreme aspect of the model is the inclusion of the prevalence of disease depending on the contact rates as well as on the host shedding rate.

Next, we'll define,

$$\beta_i(I) = c_i - d_i m_i(I)$$

Where c_i represents the infection rate by not taking the effects of behaviour of mankind, d_i represents the maximum shortened contact rate because of change in behaviour and $m_i(I)$ represents the saturation

Table 1

Description of Parameters

Parameter	Description
N	Total Populace
b	Fertility and Mortality rate
β_i	Contact rate $\beta_i(I) = c_i - d_i m_i(I)$ where $i = 1,2,3$
c_1	Direct transmission rate
c_2	Indirect transmission rate
δ	Bacterial net death rate
c_3	Shedding rate
τ	Discrete time delay
$e^{-b\tau}$	Survival rate of humans

function which dispense the effect of the adumbrated number of infected personages rest (14).

Moreover, we can reduce the system (1) by using linear chain rule (21) by defining

$$Z(t)=\int_{-\infty}^{t}F(t-\tau)S(\tau)d\tau$$

By using Leibenitz integral rule,

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)}f(x,t)dt\right]=f(x,b(x))\frac{d}{dx}b(x)-f(x,a(x))\frac{d}{dx}a(x)+\int_{a(x)}^{b(x)}\frac{d}{dx}f(x,t)dt$$

where $F(t)=a.e^{-at}, a>0$ and $\int_0^{+\infty}F(\tau)d\tau=1$, we get

$$\frac{dZ}{dt}=a(S(t)-Z(t)) \tag{2}$$

Since, in the above system of equations (1), the three equations are independent of R, hence, the new model will be,

$$\begin{aligned}
\frac{dS}{dt}&=b-bS-\beta_1(I)ZI\\
\frac{dI}{dt}&=\beta_1(I)ZI-e^{-b\tau}\beta_2(I)S(t-\tau)I(t-\tau)-bI\\
\frac{dZ}{dt}&=a(S-Z)\\
\frac{dB}{dt}&=\beta_3(I)I-\delta B
\end{aligned} \tag{3}$$

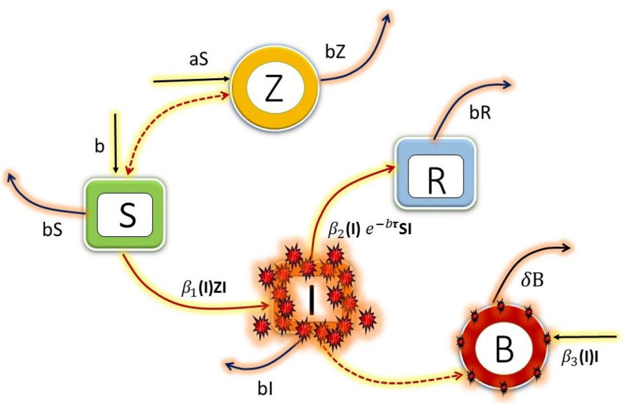

Figure 1

Flow Chart of Model

All parameters and variables are already mentioned in Table 1 and the schematic diagram is shown in the Figure 2.

2.1 *Reproduction Number*

We have calculated the fundamental reproduction number (R_0) using the next-generation approach. In the model, we define the F matrix for infection terms and the V matrix for viral production terms:

$$F = \begin{bmatrix} \beta_1(I)Z + \beta_1'(I)ZI & 0 \\ \beta_3(I) + \beta_3'(I)I & 0 \end{bmatrix} \tag{4}$$

$$V = \begin{bmatrix} b + e^{-b\tau}(\beta_2(I)S(t-\tau) + \beta_2'(I)S(t-\tau)I(t-\tau)) & 0 \\ 0 & \delta \end{bmatrix} \tag{5}$$

At E_0 (1,0,1,0),

$$F = \begin{bmatrix} c_1 & 0 \\ c_3 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} b + c_2 e^{-b\tau} & 0 \\ 0 & \delta \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{b+c_2.e^{-b\tau}} & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix}$$

$$K = F.V^{-1} = \begin{bmatrix} \frac{c_1}{b+c_2.e^{-b\tau}} & 0 \\ \frac{c_3}{b+c_2.e^{-b\tau}} & 0 \end{bmatrix} \tag{6}$$

Since, $\rho(K) = \frac{1}{2}\left[Trace(K) + \sqrt{(Trace(K))^2 - 4|K|}\right]$

Our basic reproduction number R_0 of model (3) is

$$R_0 = \left(\frac{c_1}{b + c_2.e^{-b\tau}}\right) \tag{7}$$

Whereas in the absence of delay (i.e., $\tau = 0$) it will be,

$$R_{01} = \left(\frac{c_1}{b + c_2}\right)$$

3. Equilibrium Points

The system has two equilibrium points:

- Infection free equilibrium point $E_0(1,0,1,0)$
- Endemic Equilibrium point

which exists when $\tau < \frac{1}{b}\ln\frac{\beta_2(I)}{\beta_1(I)}$.

4. Local Stability

Jacobian of system (3) is given by

$$J = [J_{ij}] = \begin{bmatrix} -b & J_{12} & -\beta_1(I)I & 0 \\ -e^{-b\tau}\beta_2(I)I(t-\tau) & J_{22} & \beta_1(I)I & 0 \\ a & 0 & -a & 0 \\ 0 & \beta_3(I)+\beta_3'(I)I(t-\tau) & 0 & -\delta \end{bmatrix} \tag{8}$$

where,

$$J_{12} = -(\beta_1(I)Z + \beta_1'(I)IZ)$$
$$J_{22} = \beta_1(I)Z + \beta_1'(I)IZ - b - S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2'(I)I(t-\tau))$$

4.1 *Local stability for infection free equilibrium*

Theorem 1 : *Disease free equilibrium point E_0 is locally asymptotically stable with $\tau > 0$ whenever $R_0 < 1$ and is unstable if $R_0 \geq 1$.*

Proof : Since we already know $N = S + I + R,$ then Jacobian matrix of (3) at E_0 will be,

$$J\,|_{E_0} = \begin{bmatrix} -b & -c_1 & 0 & 0 \\ 0 & c_1 - b - c_2 e^{-b\tau} & 0 & 0 \\ a & 0 & -a & 0 \\ 0 & c_3 & 0 & -\delta \end{bmatrix}$$

Characteristic equation of above matrix is ,

$$(-b-\lambda)(c_1 - b - c_2 e^{-b\tau} - \lambda)(-a-\lambda)(-\delta-\lambda) = 0 \tag{9}$$

Clearly (9) has three negative roots, which are $\lambda_1 = -a,$ $\lambda_2 = -b,$ $\lambda_3 = -\delta$ and $\lambda_4 = -(c_1 - b - c_2 e^{-b\tau}) = (b + c_2 e^{-b\tau})\,(R_0 - 1) < 0$ if $R_0 < 1$ and $\tau > 0$.

Hence, E_0 is asymptotically stable if $R_0 < 1$ and unstable if $R_0 \geq 1$.

4.2 *Local stability for endemic equilibrium*

The characteristic equation for E^* is $(\delta+\lambda)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0$.

Where, $a_1 = a + b + J_{22} > 0$, if $a + b + (\beta_1(I) + \beta_{1'}(I)I^*)Z^* > b + (\beta_2(I) + \beta_{2'}(I)I^*)S^* e^{-b\tau}$ if $2ab + a\beta_1(I)I^* + (a+b)(\beta_2(I) + \beta_{2'}(I)I^*)S^* e^{-b\tau} + b^2 > (\beta_1(I) + \beta_{1'}(I)I)Z^*(a + b + e^{-b\tau}\beta_2(I)I^*)$ if $(\beta_2(I) + \beta_{2'}(I)I^*)S^* e^{-b\tau}(b + \beta_1(I)I) + ab + \beta_1(I)(\beta_1(I) + \beta_{1'}(I)I^*)Z^* + b\beta_1(I)I^* > a(\beta_1(I) + \beta_{1'}(I)I^*)Z^*(b + (\beta_1(I) + \beta_2(I)e^{-b\tau})I^*)$

One of the roots is $\lambda = -\delta$ is stable, where as other three roots are stable by Routh-Hurwitz criterion $a_1 > 0, a_2 > 0, a_3 > 0$ and $a_1 a_2 > a_3$.

5. Global Stability

Here, we'll concentrate on the global stability of both the equilibrium points. So, here with the help of simple comparison theorem we get to know that,

$$\limsup_{x\to\infty} B(t) \leq \frac{c_3}{\delta}$$

Now, we'll presume the domain

$$D = \left\{ (S(t), I(t), Z(t), B(t)) \in \mathbb{R}_+^4 : S + I + R \text{ is fixed}, B \leq \frac{c_3}{\delta} \right\}$$

So, the feasible region $\mathfrak{M}$ is an invariant set. Also as $c_3 > 0$, $\delta > 0$, $\frac{c_3}{\delta} > 0$, i.e., the feasible region $\mathfrak{M}$ is positive. Hence all solutions of (3) will enter the field $\mathfrak{M}$ and will remain in $\mathfrak{M}$.

5.1 *Global stability for infection free equilibrium*

Theorem 2 : *The infection-free equilibrium E_0 of the system (3) is asymptotically stable in $\mathfrak{M}$ whenever $R_0 \leq 1$.*

Proof : Using Section (3.3) we'll consider

$$F^* = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V^* = \begin{bmatrix} b + c_2 e^{-b\tau} & 0 \\ -c_3 & \delta \end{bmatrix}$$

Now, let $\chi = (I, B)^T$ and here we'll presume that $\beta_i(I) \leq c_i$ for $i = 1, 2, 3$. Then, the system (3) will fulfill the equation,

$$\frac{d\chi}{dt} \le (F^* - V^*)\chi$$

Let us take $\upsilon = (c_1, 0)$ and since we know that $R_0 = \rho(F^* V^{*-1}) = \rho(V^{*-1} F^*)$ can be verified by $\upsilon V^{*-1} F^* = R_0\, \upsilon$. From (30), we can define a Lyapunov function as:

$$M = \upsilon V^{*-1} \chi \tag{10}$$

and its Lyapunov derivative,

$$\begin{aligned} M' &= \upsilon V^{*-1} \frac{d\chi}{dt} \\ &\le \upsilon V^{*-1}(F^* - V^*)\chi \\ &\le (V^{*-1}F^* - 1)\upsilon.\,\chi \\ &\le (R_0 - 1)\upsilon.\,\chi \qquad \text{for } (S, I, Z, B) \in \mathfrak{M} \end{aligned}$$

Therefore,

(a) if $R_0 < 1$, $M' \le 0$. As $M' = 0$ indicates that $\upsilon\chi = 0$ hence $I = 0, B = 0$ and from the 1^{st} and 3^{rd} equations of (3) we came to know that only invariant set where $M' = 0$ is (1, 0, 1, 0).

(b) If $R_0 = 1$, $M' = 0$ indicates that $\beta_1(I)ZI = c_1 I$ and $e^{-b\tau}\beta_2(I)S(t-\tau)I(t-\tau) = c_2 I(t-\tau)e^{-b\tau}$ by the presumption on β_i with $i = 1,2$ and this could only occur when $S = 1, Z = 1$ and $I = 0, B = 0$. Then, again we get the same largest invariant set where $M' = 0$ is $(1,0,1,0)$.

Now by LaSalle's Invariant Principle (17), E_0 is asymptotic stable globally whenever $R_0 \le 1$.

5.2 *Global stability for E^**

Theorem 3 : *When $R_0 > 1$, then endemic equilibrium $E^*(S^*, I^*, Z^*, B^*)$ is asymptotically stable globally in $\mathfrak{M} \setminus \mathfrak{M}_0$ if $(\beta_1(I)[I - Z] + b, -\beta_1(I)Z, e^{-b\tau}\beta_2(I)[S(t-\tau) - I(t-\tau)], -\beta_1(I)I) \le 0$.*

Proof: This approach for the global dynamics is modified by . For this, three conditions are required:

1. The chief condition is the presence of a distinctive locally stable balance point which we have effectively demonstrated in the previous segment that E^* is locally steady.

2. Next condition is the satisfaction of a conservative set in the inside of the space. So, our system (3) is feasible in the domain $\mathfrak{M}$.

3. Last condition is the fulfillment of the Bendixson rules (18). Since, we have already calculated the jacobian matrix of (3) in the equation (8) and its 2^{nd} additive compound matrix (defined in Appendix) is,

$$J^{[2]} = \begin{bmatrix} J_{22}-b & \beta_1(I)I & 0 & \beta_1(I)I & 0 & 0 \\ 0 & -(a+b) & 0 & J_{12} & 0 & 0 \\ \beta_3(I)+\beta_3'(I)I & 0 & -(b+\delta) & 0 & J_{12} & -\beta_1(I)I \\ -a & -\beta_2(I)I(t-\tau)e^{-B\tau} & 0 & J_{22}-a & 0 & 0 \\ 0 & 0 & J_{12} & 0 & J_{22}-\delta & \beta_1(I)I \\ 0 & 0 & a & -(\beta_3(I)+\beta_3'(I)I) & 0 & -(a+\delta) \end{bmatrix} \tag{11}$$

where,

$$J_{12} = -(\beta_1(I)Z + \beta_1'(I)IZ)$$
$$J_{22} = \beta_1(I)Z + \beta_1'(I)IZ - b - S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau))$$

Now, we take

$$W = \begin{bmatrix} \frac{1}{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I} & 0 & 0 \\ 0 & 0 & \frac{1}{B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{B} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{B} \end{bmatrix} \tag{12}$$

Now, we have $\mathfrak{S} = WJ^{[2]}W^{-1} + W_f W^{-1}$ whenever W_f is derivatives of W into direction of the vector field f. Therefore, the matrix $\mathfrak{S}$ will be,

$$\begin{bmatrix} \frac{1}{I}-(J_{22}-b) & -\beta_1(I)I & -\beta_1(I)I & 0 & 0 & 0 \\ 0 & \frac{1}{I}+a+b & -J_{12} & 0 & 0 & 0 \\ a & \beta_2(I)I(t-\tau)e^{-b\tau} & \frac{1}{I}-(J_{22}-a) & 0 & 0 & 0 \\ -\frac{I}{B}(\beta_3(I)+\beta_3'(I)I) & 0 & 0 & \frac{1}{B}+b+\delta & -J_{12} & \beta_1(I)I \\ 0 & 0 & 0 & -J_{12} & \frac{1}{B}-(J_{22}-\delta) & -\beta_1(I)I \\ 0 & 0 & \frac{I}{B}(\beta_3(I)+\beta_3'(I)I) & -a & 0 & \frac{1}{B}+a+\delta \end{bmatrix} \tag{13}$$

Now, we can apply the theorem in (19) through which we could find a sufficient condition on the above parameters to prove E_* is globally stable. From (19), let us assume that $\exists$ a compact absorbing sub-set $\mathfrak{M}_1$ in interior of set $\mathfrak{M}$, then $\exists$ $\eta > 0$ as well as Lozinski measure, such that $\overline{\zeta}(\mathfrak{S}) \le -\eta$, $\forall f \in \mathfrak{M}_1$, then each points of (36) in interior of $\mathfrak{M}$ which will become equilibrium point in $\mathfrak{M}$. So, here whenever $R_0 > 1$, then the E_0 will push back towards interior and as a matter of fact, there will always be a compact absorbing set which will rest in int $\mathfrak{M}$ as it will enchant all the orbits who bisects int $\mathfrak{M}$ and because of it we'll get the following outcomes.

Corollary 1 : *When* $R_0 > 1$, $\exists$ *a Lozinski measure* $\overline{\zeta}(\mathfrak{S})$ *such that* $\overline{\zeta}(\mathfrak{S}) < 0$ $\forall f \in int\, \mathfrak{M}$, *later every orbit of our system (36) who will bisect the limits of int* $\mathfrak{M}$ *to the* E^*.

As the norm $\|.\|$ on $\mathbb{R}^n$, our Lozinski measure $\overline{\zeta}$ is linked along the $\|.\|$ to analyzed the $m \times m$ matrix $\mathcal{L}$ such that,

$$\overline{\zeta}(\mathcal{L}) = \inf\{\nu : \mathfrak{M}_+ \| z \| \le \nu \| z \|, \forall\, z' = \mathcal{L}z\} \tag{14}$$

So, here $\mathfrak{M}_+$ is right-hand side derivatives (22). Thus, with presumption we could wangle a particular norm on $\mathbb{R}^6$ for whom the associated *Lozinskiĭ measure* must satisfy $(\mathfrak{S}) < 0$ $\forall$ $f \in int\, \mathcal{M}$ later, the E^* will be stable globally whenever $R_0 > 1$.

Now, we'll define the norm on the $\mathbb{R}^6$ (15), where the definition could differ from a single to another orthants. So, we take

$$\| v \| = \max\{\mathfrak{V}_1, \mathfrak{V}_2\} \tag{15}$$

here $v \in \mathbb{R}^6$ with v_i such that $i = 1$ to 6. So, let

$$\mathfrak{V}_1(v_1,v_2,v_3)=\begin{cases}\max\{|v_1|,|v_2|+|v_3|\}, & if\ \ sgn(v_1)=sgn(v_2)=sgn(v_3)\\ \max\{|v_2|,|v_1|+|v_3|\}, & if\ \ sgn(v_1)=sgn(v_2)=-sgn(v_3)\\ \max\{|v_1|,|v_2|,|v_3|\}, & if\ \ sgn(v_1)=-sgn(v_2)=sgn(v_3)\\ \max\{|v_1|+|v_3|,|v_2|+|v_3|\}, & if\ \ -sgn(v_1)=sgn(v_2)=sgn(v_3)\end{cases}$$

$$\mathfrak{V}_2(v_4,v_5,v_6)=\begin{cases}|v_4|+|v_5|+|v_6|, & if\ \ sgn(v_4)=sgn(v_5)=sgn(v_6)\\ \max\{|v_4|+|v_5|,|v_4|+|v_6|\}, & if\ \ sgn(v_4)=sgn(v_5)=-sgn(v_6)\\ \max\{|v_5|,|v_4|+|v_6|\}, & if\ \ sgn(v_4)=-sgn(v_5)=sgn(v_6)\\ \max\{|v_4|+|v_6|,|v_5|+|v_6|\}, & if\ \ -sgn(v_4)=sgn(v_5)=sgn(v_6)\end{cases}$$

Next, we'll define the $\|.\|$ on the interior of all orthants that is why here we have to expand $\|.\|$ to boundaries of all the orthants continuously and we could also express that $\|.\|$ is a non-negative comparable function for $\{v:\|v\|\leq 1\}$ will be real valued.

$$v'(t)=\mathfrak{S}(t)v(t)$$

But here we have to remember that time dependence *w.r.t* $\mathfrak{S}(t)$ go along with the fact $\mathfrak{S}$ depends on S,I,Z and B, which will successively depends on t.

$$\mathfrak{M}_+\|v\|<(\beta_1(I)[I-Z]+b,-\beta_1(I)Z,e^{-b\tau}\beta_2(I)[S(t-\tau)-I(t-\tau)],-\beta_1(I)I)\|v\| \tag{16}$$

where $\|.\|$ is a smooth function of z in the region and $\mathfrak{M}_+$ is the typical derivative *w.r.t* t. Whereas, points at which $\|.\|$ is non smooth (*i.e.*, on coordinate axes), but $\mathfrak{M}_+$ will still be well defined then all calculations to illustrate an equation (16) includes 8 cases and sub-cases which are hinged on these unlike Orthants and the definition of $\|.\|$ inside every of the Orthant. So, here we have mooted all the cases which will facilitate more apprehension.

Case I: $\mathfrak{V}_1(v)>\mathfrak{V}_2(v),\ v_1,v_2,v_3>0.$

Here, $\|v\|=\max\{|v_1|,|v_2|+|v_3|\}$.

Sub-case I(a) : If $\|v_1\|>\|v_2\|+\|v_3\|$, $\|v\|=|v_1|=v_1$ then $\mathfrak{V}_2(v)<|v_1|$. by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+\|v\| &= v_1' \\ &\leq\left[\frac{1}{I}-\beta_1(I)Z-\beta_1'(I)IZ+2b+S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2'(I)I(t-\tau))\right]v_1 \\ &\quad+|-\beta_1(I)I|v_2+|-\beta_1(I)I|v_3\end{aligned}$$

Since $|v_2| \le |v_1|$ and $|v_3| \le |v_1|$,

$$\mathfrak{M}_+ \|v\| \le \left[\frac{1}{I} - \beta_1(I)Z - \beta_1'(I)IZ + 2b + S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau))\right]|v_1|$$
$$+[\beta_1(I)I]|v_2| + |v_3|$$
$$\le \left[\frac{1}{I} - \beta_1(I)Z + 2b + \beta_2(I)S(t-\tau).e^{-b\tau} + \beta_1(I)I\right]|v_1|$$

Hence,

$$\mathfrak{M}_+ \|v\| \le \left[\frac{1}{I} + \beta_1(I)[I-Z] + 2b + \beta_2(I)S(t-\tau).e^{-b\tau}\right]\|v\|. \qquad (17)$$

Sub-case I(b): If $\|v_1\| < \|v_2\| + \|v_3\|$, $\|v\| = |v_2| + |v_3| = v_2 + v_3$ then $\mathfrak{V}_2(v) < |v_2| + |v_3|$. As $|v_4 + v_5 + v_6| \le \mathfrak{V}_2(v) < |v_2| + |v_3|$, by abducting derivative of $\|.\|$,

$$\mathfrak{V}_+ \|v\| = v_2' + v_3'$$
$$\le av_1 + \left[\frac{1}{I} + b + a - \beta_2(I)I(t-\tau)e^{-b\tau}\right]v_2$$
$$+\left[\frac{1}{I} + b + a + S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau))\right]v_3$$
$$\le \left[\frac{1}{I} + b + e^{-b\tau}\beta_2(I)(S(t-\tau) - I(t-\tau))\right](|v_2| + |v_3|)$$

Therefore,

$$\mathfrak{M}_+ \|v\| \le \left[\frac{1}{I} + b + e^{-b\tau}\beta_2(I)(S(t-\tau) - I(t-\tau))\right]\|v\|. \qquad (18)$$

Case II: $\mathfrak{V}_1(v) > \mathfrak{V}_2(v),.$

Here, $\|v\| = \max\{|v_1| + |v_3|, |v_2| + |v_3|\}$

Sub-case II(a): If $\|v_1\| > \|v_2\|$, $\|v\| = |v_1| + |v_3| = -v_1 + v_3$ then $\mathfrak{V}_2(v) < |v_1| + |v_3|$. So, by abducting the derivative of $\|.\|$,

$$\mathfrak{M}_+ \|v\| = -v_1' + v_3'$$
$$\le \left[-\frac{1}{I} + \beta_1(I)Z + \beta_1'(I)IZ + a - 2b - S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau))\right]v_1$$
$$+[\beta_1(I)I + \beta_2(I)I(t-\tau)e^{-b\tau}]v_2 + \left[\beta_1(I)I + \frac{1}{I} - \beta_1(I)Z - \beta_1'(I)IZ + a\right.$$

$$+b+S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2{}'(I)I(t-\tau))\Big]v_3$$

$$\le\left[\frac{1}{I}-\beta_1(I)Z+a-2b+\beta_2(I)S(t-\tau).e^{-b\tau}\right]|v_1|+[\beta_1(I)I+\beta_2(I)I(t-\tau)e^{-b\tau}]|v_2|$$

$$+\left[\beta_1(I)I+\frac{1}{I}-\beta_1(I)Z+a+b+\beta_2(I)S(t-\tau).e^{-b\tau}\right]|v_3|$$

$$\le[\beta_1(I)[I-Z]+b+2a+e^{-b\tau}\beta_2(I)(S(t-\tau)-I(t-\tau))].(|v_1|+|v_3|)$$

Therefore,

$$\mathfrak{M}_+\|v\|\le[\beta_1(I)[I-Z]+b+2a+e^{-b\tau}\beta_2(I)(S(t-\tau)-I(t-\tau))]\|v\| \tag{19}$$

Sub-case II(b): If $\|v_1\|<\|v_2\|$, $\|v\|=|v_2|+|v_3|=v_2+v_3$ then

$\mathfrak{V}_2(v)<|v_2|+|v_3|$. As $|v_4+v_5+v_6|\le\mathfrak{V}_2(v)<|v_2|+|v_3|$,

by abducting derivative of $\|.\|$,

$$\mathfrak{M}_+\|v\|=v_2'+v_3'\le av_1+\left[\frac{1}{I}+b+a-\beta_2(I)I(t-\tau)e^{-b\tau}\right]v_2$$

$$+\left[\frac{1}{I}+b+a+S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2{}'(I)I(t-\tau))\right]v_3$$

$$\le\left[\frac{1}{I}+b+e^{-b\tau}\beta_2(I)(S(t-\tau)-I(t-\tau))\right](|v_2|+|v_3|)$$

Therefore,

$$\mathfrak{M}_+\|v\|\le\left[\frac{1}{I}+b+e^{-b\tau}\beta_2(I)(S(t-\tau)-I(t-\tau))\right]\|v\|. \tag{20}$$

Case III: $\mathfrak{V}_1(v)>\mathfrak{V}_2(v)$, $v_3<0<v_1,v_2$.

Here, $\|v\|=\max\{|v_2|,|v_1|+|v_3|\}$

Sub-case III(a): If $\|v_2\|>\|v_1\|+\|v_3\|$, $\|v\|=|v_2|=v_2$ then

$\mathfrak{V}_2(v)<|v_2|$. So, by abducting the derivative of $\|.\|$,

Therefore,

$$\mathfrak{M}_+\|v\|\le\left[\frac{1}{I}+a+b-\beta_1(I)Z\right]\|v\| \tag{21}$$

Sub-case III(b) : If $\| v_2 \| < \| v_1 \| + \| v_3 \|$, $\| v \| = | v_1 | + | v_3 | = v_1 - v_3$ then

$$\mathfrak{V}_2(v) < | v_1 | + | v_3 | . \text{ As } | v_4 + v_5 + v_6 | \le \mathfrak{V}_2(v) < | v_1 | + | v_3 |,$$

by abducting derivative of $\| . \|$,

Therefore,

$$\mathfrak{M}_+ \| v \| \le [-\beta_1(I)I + a + b + \beta_2(I)I(t-\tau).e^{-b\tau}] \| v \| . \tag{22}$$

Case IV: $\mathfrak{V}_1(v) > \mathfrak{V}_2(v)$, $v_2 < 0 < v_1, v_3$.

Here, $\| v \| = \max\{ | v_1 |, | v_2 |, | v_3 | \}$

Sub-case IV(a): If $\| v_1 \| > \max\{\| v_2 \|, \| v_3 \|\}$, $\| v \| = | z_1 | = z_1$. So, by abducting the derivative of $\| . \|$,

$$\begin{aligned}
\mathfrak{M}_+ \| v \| &= v_1' \\
&\le \left[\frac{1}{I} - \beta_1(I)Z - \beta_1{}'(I)IZ + 2b + S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2{}'(I)I(t-\tau)) \right] v_1 \\
&\quad + | -\beta_1(I)I | v_2 + | -\beta_1(I)I | v_3 \\
&\le \left[\frac{1}{I} - \beta_1(I)Z - \beta_1{}'(I)IZ + 2b + S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2{}'(I)I(t-\tau) \right] | v_1 | \\
&\quad + [\beta_1(I)I].(| v_2 | + | v_3 |) \\
&\le \left[\frac{1}{I} - \beta_1(I)Z + 2b + \beta_2(I)S(t-\tau).e^{-b\tau} + \beta_1(I)I \right] . | v_1 |
\end{aligned}$$

Therefore,

$$\mathfrak{M}_+ \| v \| \le \left[\frac{1}{I} + \beta_1(I)[I - Z] + 2b + \beta_2(I)S(t-\tau).e^{-b\tau} \right] \| v \| \tag{23}$$

Sub-case IV(b): If $\| v_2 \| > \max\{\| v_1 \|, \| v_3 \|\}$, $\| v \| = | v_2 | = -v_2$. So, by abducting derivative of $\| . \|$,

$$\begin{aligned}
\mathfrak{M}_+ \| v \| = -v_2' &\le \left[-\frac{1}{I} - b - a]v_2 + [\beta_1(I)Z + \beta_2{}'(I)IZ \right] v_3 \\
&\le \left[\frac{1}{I} + a + b - \beta_1(I)Z \right] | v_2 |
\end{aligned}$$

Therefore,

$$\mathfrak{M}_{+} \| v \| \leq \left[\frac{1}{I} + b + e^{-b\tau} \beta_2(I)(S(t-\tau) - I(t-\tau)) \right] \| v \|. \tag{24}$$

Sub-case IV(c): If $\| v_3 \| > \max\{\| v_1 \|, \| v_2 \|\}$, $\| v \| = | v_3 | = v_3$. So, by abducting derivative of $\|.\|$,

Therefore,

$$\mathfrak{M}_{+} \| v \| \leq \left[\frac{1}{I} + b + e^{-b\tau} \beta_2(I)(S(t-\tau) - I(t-\tau)) - \beta_1(I)Z \right] \| v \|. \tag{25}$$

Case V: $\mathfrak{V}_2(v) > \mathfrak{V}_1(v)$, $v_6 < 0 < v_4, v_5$.

Here $\| v \| = \max\{ | v_4 | + | v_5 |, | v_4 | + | v_6 | \}$

Sub-case V(a): If $\| v_5 \| > \| v_6 \|$, $\| v \| = | v_4 | + | v_4 | = v_4 + v_5$ then $\mathfrak{V}_1(v) < | v_4 | + | v_5 |$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}
\mathfrak{M}_{+} \| v \| &= v_4' + v_5' \\
&\leq \left[-\frac{I}{B}(\beta_3(I) + \beta_3'(I)I) \right] v_1 + \left[\frac{1}{B} + b + \delta + \beta_1(I)Z + \beta_1'(I)IZ \right] v_4 \\
&\quad + \left[\frac{1}{B} + b + \delta + S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau)) \right] v_5 \\
&\leq \left[\frac{1}{B} + b + \delta \right] | v_4 + v_5 | + [-\beta_1(I)Z] | v_4 | + [b + \delta + \beta_2(I)S(t-\tau).e^{-b\tau}] | v_5 | \\
&\leq \left[\frac{1}{B}\beta_3(I) + b + \delta - \beta_1(I)Z + \beta_2(I)S(t-\tau).e^{-b\tau} \right] (| v_4 | + | \, | v_5 |)
\end{aligned}$$

Therefore,

$$\mathfrak{M}_{+} \| v \| \leq \left[\frac{1}{B}\beta_3(I) + b + \delta - \beta_1(I)Z + \beta_2(I)S(t-\tau).e^{-b\tau} \right] \| v \|. \tag{26}$$

Sub-case V(b): If $\| v_5 \| < \| v_6 \|$, $\| v \| = | v_4 | + | v_6 | = v_4 - v_6$ then $\mathfrak{V}_1(v) < | v_4 | + | v_6 |$. So, by abducting derivative of $\|.\|$,

Therefore,

$$\mathfrak{M}_{+} \| v \| \leq [\beta_1(I)[I - Z] + b] \| v \|. \tag{27}$$

Case VI: $\mathfrak{V}_2(v) > \mathfrak{V}_1(v)$, $v_5 < 0 < v_4, v_6$.

Here $\| v \| = \max\{ | v_5 |, | v_4 | + | v_6 | \}$

Sub-case VI(a): If $\|v_5\| > \|v_4\| + \|v_6\|$, $\|v\| = |v_5| = -v_5$ then $\mathfrak{U}_1(v) < |v_4| + |v_6|$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+ \|v\| = -v_5' &\le [-\beta_1(I)Z - \beta_1'(I)IZ]v_4 \\ &+ \left[-\frac{1}{B} - b - \delta + \beta_1(I)Z + \beta_1'(I)IZ - S(t-\tau).e^{-b\tau}(\beta_2(I) + \beta_2'(I)I(t-\tau))\right]z_5 \\ &+ [beta_1(I)I]|v_6| \le [\beta_1(I)[I-Z]](|v_4| + |v_6|) \\ &+ \left[\frac{1}{B} + b + \delta + S(t-\tau)e^{-b\tau}\beta_2(I)\right]|v_5| \\ &\le \left[\frac{1}{B} + b + \delta + \beta_1(I)[I-Z] + \beta_2(I)S(t-\tau).e^{-b\tau}\right]|v_5|\end{aligned}$$

Therefore,

$$\mathfrak{M}_+ \|v\| \le \left[\frac{1}{B} + b + \delta + \beta_1(I)[I-Z] + \beta_2(I)S(t-\tau).e^{-b\tau}\right]\|v\|. \tag{28}$$

Sub-case VI(b): If $\|v_5\| < \|v_4\| + \|v_6\|$, $\|v\| = |v_4| + |v_6| = v_4 + v_6$ then $\mathfrak{V}_1(v) < |v_4| + |v_6|$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+ \|v\| = v_4' + v_6' &\le \left[-\frac{I}{B}(\beta_3(I) + \beta_3'(I)I)\right]v_1 + \left[\frac{I}{B}(\beta_3(I) + \beta_3'(I)I)\right]z_3 \\ &+ \left[\frac{1}{B} + b + \delta - a\right]v_4 + [\beta_1(I)Z + \beta_1'(I)IZ]v_5 + \left[\beta_1(I)I + \frac{1}{B} + a + \delta\right]v_6 \\ &\le [\beta_1(I)Z]|v_5| + \left[\frac{1}{B} + b + \delta - \beta_1(I)Z\right](|v_4| + |v_6|)\end{aligned}$$

As $|v_1 + v_3| < \mathfrak{V}_1 < |v_4| + |v_6|$, $|v_5| < |v_6|$,

$$\mathfrak{M}_+ \|v\| \le [\beta_1(I)[I-Z] + b](|v_4| + |v_6|)$$

Therefore,

$$\mathfrak{M}_+ \|v\| \le [\beta_1(I)[I-Z] + b]\|v\|. \tag{29}$$

Case VII: $\mathfrak{V}_2(v) > \mathfrak{V}_1(v)$, $v_4 < 0 < v_5, v_6$.

Here $\|v\| = \max\{|v_4| + |v_6|, |v_5| + |v_6|\}$

Sub-case VII(a): If $\|v_5\| > \|v_4\|$, $\|v\| = |v_5| + |v_6| = v_5 + v_6$ then $\mathfrak{V}_1(v) < |v_5| + |v_6|$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+\|v\| &= v_5' + v_6' \\ &\le \left[\frac{I}{B}(\beta_3(I)+\beta_3{}'(I)I)\right]v_3 + [\beta_1(I)Z+\beta_1{}'(I)IZ-a]v_4 \\ &+\left[\frac{1}{B}-\beta_1(I)Z-\beta_1{}'(I)IZ+b+\delta+S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2{}'(I)I(t-\tau))\right]v_5 \\ &+\left[-\beta_1(I)I+\frac{1}{B}+a+\delta\right]v_6 \\ &\le \left[\frac{1}{B}\beta_3(I)+b+\delta+\beta_1(I)[I-Z]+\beta_2(I)S(t-\tau).e^{-b\tau}\right](|v_5|+|v_6|)\end{aligned}$$

Therefore,

$$\mathfrak{M}_+\|v\| \le \left[\frac{1}{B}\beta_3(I)+b+\delta+\beta_1(I)[I-Z]+\beta_2(I)S(t-\tau).e^{-b\tau}\right]\|v\|. \tag{30}$$

Sub-case VII(b): If $\|v_5\|<\|v_4\|$, $\|v\|=|v_4|+|v_6|=-v_4+v_6$ then $\mathfrak{V}_1(v)<|v_4|+|v_6|$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+\|v\| &= -v_4'+v_6' \le \left[\frac{I}{B}(\beta_3(I)+\beta_3{}'(I)I)\right]v_1 + \left[-\frac{I}{B}(\beta_3(I)+\beta_3{}'(I)I)\right]z_3 \\ &+\left[-\frac{1}{B}-b-\delta-a\right]v_4 + [\beta_1(I)Z+\beta_1{}'(I)IZ]v_5 + \left[-\beta_1(I)I+\frac{1}{B}-a-\delta\right]v_6 \\ &\le [-\beta_1(I)[I-Z]-b](|v_4|+|v_6|)\end{aligned}$$

Therefore,

$$\mathfrak{M}_+\|v\| \le [-\beta_1(I)[I-Z]-b]\|v\|. \tag{31}$$

Case VIII: $\mathfrak{V}_2(v)>\mathfrak{V}_1(v),\ v_4, v_5, v_6>0.$

Here $\|v\|=|v_4|+|v_5|+|v_6|=v_4+v_5+v_6$. So, by abducting derivative of $\|.\|$,

$$\begin{aligned}\mathfrak{M}_+\|v\| &= v_4'+v_5'+v_6' \le \left[-\frac{I}{B}(\beta_3(I)+\beta_3{}'(I)I)\right]z_1 + \left[\frac{I}{B}(\beta_3(I)+\beta_3{}'(I)I)\right]v_3 \\ &+\left[\frac{1}{B}+b+\delta+\beta_1(I)Z+\beta_1{}'(I)IZ-a\right]v_4 \\ &+\left[\frac{1}{B}+b+\delta+S(t-\tau).e^{-b\tau}(\beta_2(I)+\beta_2{}'(I)I(t-\tau))\right]v_5\end{aligned}$$

$$+\left[\frac{1}{B}+a+\delta\right]v_6 \leq [2b+\beta_1(I)Z-S(t-\tau)\beta_2(I)e^{-b\tau}](|v_4|+|v_5|)$$
$$+\left[\frac{1}{B}+\delta\right](|v_4|+|v_5|+|v_6|)$$
$$\leq\left[\frac{1}{B}+\delta+2b-\beta_1(I)Z+S(t-\tau)\beta_2(I)e^{-b\tau}\right](|v_4|+|v_5|+|v_6|)$$

Therefore,

$$\mathfrak{M}_+ \|v\| \leq \left[\frac{1}{B}+\delta+2b-\beta_1(I)Z+S(t-\tau)\beta_2(I)e^{-b\tau}\right]\|v\|. \tag{32}$$

Now, by unifying the results of all cases shown above from equation (17) to (32), we get

$$\mathfrak{M}_+ \|v\| \leq \max\{\beta_1(I)[I-Z]+b, -\beta_1(I)Z, e^{-b\tau}\beta_2(I)[S(t-\tau)-I(t-\tau)], -\beta_1(I)I\}\|v\| \tag{33}$$

Therefore, from equation (14),we get

$$\bar{\zeta}(\mathfrak{S}) \leq \max\{\beta_1(I)[I-Z]+b, -\beta_1(I)Z, e^{-b\tau}\beta_2(I)[S(t-\tau)-I(t-\tau)], -\beta_1(I)I\}$$

Hence, the Lozinski measure $\bar{\zeta}(\mathfrak{S})<0$ in int $\mathfrak{M}$. Thus, from Corollary 1 we know that when $R_0>1$, then the E_* of (3) is globally stable.

Theorem 4 : *If $R_0>1$, then for I^{st} compartment, our endemic equilibrium E^* of (3), (I_u) is decreasing strictly in maximal reduced transmission coefficient due to the change in behavior, d_i for $i=1,2$ in the presence of $\tau>\tau^*$ where $\tau^*=\frac{1}{b}ln\frac{\beta_1(I)}{\beta_2(I)}$.*

Proof : We have presumed that if $R_0>1$, then $\exists$ a distinctive endemic equilibrium (S_u, I_u, Z_u, B_u) of the model (3). Then, from the presumption $N=S+I+R$ we get,

$$\psi(I_u)=\frac{1}{\zeta}(N-I) \quad \text{where} \quad \zeta=1-\frac{e^{-b\tau}\beta_2(I)I}{b}$$

And whenever $\mathcal{R}_0>1$ from 3^{rd} eq of system (3) we get,

$$\phi(I_u)=\frac{bI}{\beta_1(I)-\beta_2(I)e^{-b\tau}I}$$

Therefore, by unifying both the functions,

$$\psi(I_u) = \phi(I_u) \quad \Leftrightarrow \quad \frac{1}{\zeta}(N - I) = \frac{bI}{\beta_1(I) - \beta_2(I)e^{-b\tau}I} \tag{34}$$

Since, we know $\beta_1(I_u) = c_1 - d_1 m_1(I_u)$. Differentiating both sides of equation (39) w.r.t d_1, we get

$$\Leftrightarrow -\frac{1}{\zeta}\frac{\partial I_u}{\partial d_1} = \frac{b}{(\beta_1(I) - e^{-b\tau}\beta_2(I)I)^2}$$

$$\left((\beta_1(I) - e^{-b\tau}\beta_2(I)I)\frac{\partial I_u}{\partial d_1} + m_1(I_u) + d_1 m_1'(I_u)\frac{\partial I_u}{\partial d_1} + e^{-b\tau}\beta_2(I)\frac{\partial I_u}{1} + e^{-b\tau}\beta_2'(I)I_u \right)$$

$$\Leftrightarrow -\frac{b}{(\beta_1(I) - e^{-b\tau}\beta_2(I)I)^2}(m_1(I_u) + e^{-b\tau}\beta_2'(I)I_u)$$

$$= \frac{\partial I_u}{\partial d_1}\left(\frac{1}{\zeta} + \frac{b}{(\beta_1(I) - e^{-b\tau}\beta_2(I)I)} + d_1 b I_u m_1'(I_u) + e^{-b\tau}\beta_2(I)bI_u \right)$$

Thus, $\frac{\partial I_u}{\partial d_1} < 0.$ Similarly, we can show that $\frac{\partial I_u}{\partial d_2} < 0$ which clearly indicates that the change in human behavior cannot extinguish the disease, yet it can glaringly dilute the epidemic level and ginormous changed behavior directs to faintly infections.

6. Conclusion

In this study, we assembled the local as well as global stability analysis of modified SIR model with delay. Since we have considered this model more especially for cholera because of which our model discriminates itself from the SIR epidemiological models and by adding delay the approach of our model is more practical than other cholera models. From our results, it is clearly evident that reproduction number R_0 satisfies a foremost mathematical property as whenever $R_0 < 1$, it is been evinced that the infection-free equilibrium E_0 is locally as well globally stable under adequate condition and, when $R_0 \geq 1$, then only endemic equilibrium E^* is globally stable which means disease persists in the environment. From the last theorem, we can conclude that changed behavior can really reduce the rate of infection spreading whenever delay is greater than a particular value. This reveals that cholera-like disease can be diminished from the populace if mankind is well cognizant about the expansion and seriousness of the disease outburst and role of immunity, required for protection against cholera. It can be achieved through media attention or the government can use smartphones to inform more people about the disease and how to take relevant safety measures or can organize alertness camps in which we can teach people as to how they can dilute contact with other people or from the infected environment or how to have a well prepared meal and can recommend them ways of safe stool disposal.

Appendix

A. Sample Appendix Section

Appendix A.1. 2^{nd} *Additive Compound Matrix*

If $C = [c_{ij}]$ is an matrix of order $m \times m$, then its 2^{nd} additive compound matrix $C^{[2]}$ is an $\binom{m}{2} \times \binom{m}{2}$ matrix which is delineated in the hereinafter manner. As for every integer $i = 1, \ldots, \binom{m}{2}$, we suppose $(i) = (i_1, i_2)$ prevails the i^{th} paradigmate in the lexicographic ordering of integer couple (i_1, i_2) $s.t.$ $1 \le i_1 < i_2 \le m.$ Later, the item into $i-row$ and the $j-column$ of $C^{[2]}$ will be

$$\begin{cases} c_{i_1 i_1} + c_{i_2 i_2}, & \text{if } (i) = (i) \\ (-1)^{p+q} c_{i_p j_q}, & \text{if precisely one noting } i_p \text{ of } (i) \text{ doesn't} \\ & \text{transpires in } (j) \text{and } j_q \text{ doesn't transpires in } (i) \\ 0, & \text{if not either noting from } (i) \text{ transpires in } (j) \end{cases}$$

Here, in some exceptional case, whenever $m = 2,$ then $C^{[2]}_{2\times 2} = c_{11} + c_{22}$ $= tr\, C.$ So generally, per noting of $C^{[2]}$ is the linear expression of C. Like whenever $m = 3,$ later, the 2^{nd} additive compound matrix of $C = (c_{ij})$ will be

$$C^{[2]} = \begin{bmatrix} c_{11} + c_{22} & c_{23} & -c_{13} \\ c_{32} & c_{11} + c_{33} & c_{12} \\ -c_{31} & c_{21} & c_{22} + c_{33} \end{bmatrix}$$

And likewise whenever $m = 4,$ then the 2^{nd} additive matrix of C will be,

$$C^{[2]} = \begin{bmatrix} c_{11}+c_{22} & c_{23} & c_{24} & -c_{13} & -c_{14} & 0 \\ c_{32} & c_{11}+c_{33} & c_{34} & c_{12} & 0 & -c_{14} \\ c_{42} & c_{43} & c_{11}+c_{44} & 0 & c_{12} & c_{13} \\ -c_{31} & c_{21} & 0 & c_{22}+c_{33} & c_{34} & -c_{24} \\ -c_{41} & 0 & c_{21} & c_{43} & c_{22}+c_{44} & c_{23} \\ 0 & -c_{41} & c_{31} & -c_{42} & c_{32} & c_{33}+c_{44} \end{bmatrix}$$

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