

$\mathcal{C}$ -consistent line signed graphs of Cayley signed graphs

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Abstract

The Cayley graph, signed graph and line graph are well known in the literature. Now, our focus is to study line signed graphs of Cayley signed graphs. In this paper, we have characterized the Cayley sets and generating sets for which line signed graphs of Cayley signed graphs are canonically consistent($\mathcal{C}$ -consistent).

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1. Introduction

For terminology and notations from group theory or graph theory, the reader is referred to [7] and [6], respectively. Cayley graph comes into picture in 1878 when the concept was introduced by A. Cayley. It is well known graph for further study the reader is referred to [2, 3, 4, 8, 9, 13]. “A graph Γ equipped with a signature σ is called a *signed graph*, denoted by $\Sigma := (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is an *underlying graph* and $\sigma : E \rightarrow \{+, -\}$ is the signature that labels each edge of Γ either by ‘+’ or ‘-’. The edges which receive the ‘+’ (–) sign is called positive(negative) edge. A signed graph is an *all-positive (all-negative)* if all of its edges are positive (negative); further,

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it is said to be *homogeneous* if it is either an all-positive or an all-negative and *heterogeneous* otherwise. The *negative degree* $d^-(v)$ of a vertex v is the number of negative edges incident at v in Σ and the *positive degree* $d^+(v)$ is defined similarly. F. Harary [5] introduced the notion of balanced signed graph - "A signed graph is *balanced* if every cycle has an even number of negative edges". For detailed study of signed graphs the reader is referred to [15]. Due to vast use of these graphs in social psychology, combinatorial and geometric graph theory everyone is interested in these two graphs. Canonically marked signed graph is a marked signed graph with specific marking. In a signed graph, a vertex receive the positive sign if the product of sign of edges incident to that vertex is positive, and receive negative sign if product of sign of edges incident to that vertex is negative, then such a marking is called canonical marking or $\mathcal{C}$ -marking and signed graph along with such marking is called Canonically marked signed graph. If a cycle in a canonically marked signed graph contains an even number of negatively marked vertices, then cycle is said to be consistent and if every cycle in signed graph is consistent, then canonically marked signed graph is called $\mathcal{C}$ -consistent. For more study of $\mathcal{C}$ -consistent marked signed graphs the reader is referred to [1]. Now, turning towards the derived graph in which line graph is one of them. Line graph $L(G)$ of a graph G which is defined as a graph in which edges of G represent vertices of $L(G)$ and two vertices in $L(G)$ are adjacent if and only if two edges in G have a common vertex. For detailed study of properties of line graph, the reader is referred to [12]. A line signed graph of a signed graph is a derived graph where line graph of a graph is an underlying graph with specific rule signature. Depending upon the rule of signature, there are three types of line signed graph of signed graph in the literature, namely, line signed graph, product line signed graph and dot line signed graph denoted by $L_\Sigma(S)$, $L_{\times\Sigma}(S)$, and $L_{\bullet\Sigma}(S)$ respectively.

In $L_\Sigma(S)$, an edge is negative if and only if both edges are negative in S .

In $L_{\times\Sigma}(S)$, an edge is negative if and only if product of sign of edges in S is negative.

In $L_{\bullet\Sigma}(S)$, an edge is negative if and only if the sign of common vertex of edges in S is negative.

2. Basic Preliminaries

In this section, we recall some basic definitions which are required here.

Definition 2.1 : Cayley set is a non-empty subset S of Γ containing elements with its inverse except identity element. For example: Let $\Gamma \cong \mathbb{Z}_4 = \{0, 1, 2, 3\}$. Then $S_1 = \{2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{1, 2, 3\}$ are three Cayley sets.

Definition 2.2 : If group Γ generated by Cayley set S , then set S is called generating set. For example: let $\Gamma \cong \mathbb{Z}_4 = \{0, 1, 2, 3\}$. Then $S_2 = \{1, 3\}$ and $S_3 = \{1, 2, 3\}$ are both generating sets.

The following definition of Cayley signed graph was established in [11].

Definition 2.3 : Let S be a subset of a finite group Γ . The *Cayley signed graph*, denoted by $Cay_\Sigma(\Gamma, S)$ has vertex set Γ and two distinct vertices $x, y \in \Gamma$ are joined by an edge from x to y if and only if there exists $s \in S$ such that $x = sy$, where Σ is a signed graph whose vertex set is Γ and σ is a function defined as:

$$\sigma(xy) = \begin{cases} +, & \text{if } x \in S \text{ or } y \in S; \\ -, & \text{otherwise.} \end{cases}$$

Cayley graphs and signed graphs are used to solve the problems in extremal graph theory. Due to importance of Cayley graphs, signed graphs, and line graphs, we are interested to study these three graphs together in one notion. The line signed graph of Cayley signed graph is a signed graph having line graph of Cayley graph as its underlying graph whose edges are assigned the signs according to the rule of signature by looking the sign of edges of Cayley signed graph. In this paper, we have discussed the $\mathcal{C}$ -consistent line signed graphs of Cayley signed graphs. Throughout the paper, we mean by ' Γ ' the finite abelian group and by $\mathbb{Z}_n$, the group of integers modulo n , the sets $Z(\mathbb{Z}_n)$ and $U(\mathbb{Z}_n)$ are defined as; $U(\mathbb{Z}_n) = \{x : \gcd(x, n) = 1\}$ and $Z(\mathbb{Z}_n) = \{y : \gcd(y, n) \neq 1\}$. Also, $U(\mathbb{Z}_m \times \mathbb{Z}_n)$ is defined as; $U(\mathbb{Z}_m \times \mathbb{Z}_n) = \{(x, y) : \gcd(x, m) = 1 \text{ \& } \gcd(y, n) = 1\}$. $L_\Sigma(Cay_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(Cay_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(Cay_\Sigma(\Gamma, S))$ denotes line signed graph of Cayley signed graph, product line signed graph of Cayley signed graph and dot line signed graph of Cayley signed graph with respect to Γ and S respectively.

Now, we will take an example of $\mathcal{C}$ -consistent line signed graphs of Cayley signed graphs.

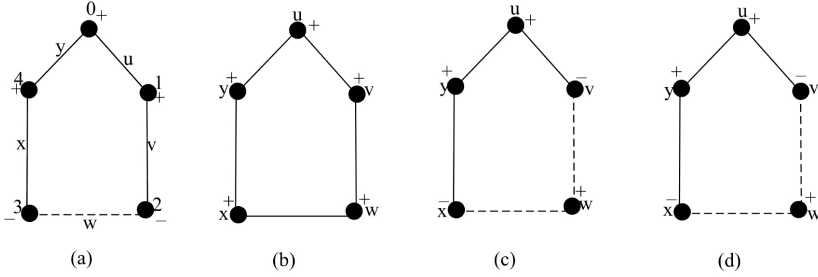


Figure 1

The Canonically marked Cayley signed graph and its line signed graphs

Example 2.1 : Consider $\Gamma \cong \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and let $S = \{1, 4\}$ be it's generating set. Here, u, v, w, x and y are edges in Cayley signed graph which becomes vertices in its line signed graph. Figure 1(a), Figure 1(b), Figure 1(c), Figure 1(d) show canonically marked $\text{Cay}_\Sigma(\mathbb{Z}_5, S)$, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ respectively. It can easily be observed from Figure 1 that each marked signed graph contains an even number of negative vertices under $\mathcal{C}$ -marking. Hence, $\text{Cay}_\Sigma(\mathbb{Z}_5, S)$, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are all $\mathcal{C}$ -consistent.

Now, we discuss some results which have been established in [10] and [14] and will be useful to derive new results.

Lemma 2.1 : [14] Let $\Gamma \cong \mathbb{Z}_n$ be a cyclic group of order n and let $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ be line signed graph of Cayley signed graph associated with Cayley set ' S '. Then, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is an all positive if one of following conditions hold:

- (i) $|S| = n - 1$;
- (ii) $|S| = n - 2$;
- (iii) $|S| = n - 3$ and no self inverse element;
- (iv) $S = U(\Gamma)$ where, $n = p^k$, p is prime number and k is positive integer;
- (v) $S = U(\Gamma) \cup \{p\}$ where, $n = 2p$ and p is prime number.

Lemma 2.2 : [14] Let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ be a finite abelian group. Let $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ be line signed graph of Cayley signed graph associated with Cayley set ' S '. Then, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is balanced if one of following conditions hold:

- (i) $|S| = 1$ or 2 or 5 or 6 or 7 ;
- (ii) S contains all non-zero elements except all self inverse elements;

(iii) S contains all non-zero elements except all self inverse elements.

Lemma 2.3 : [10] Let Γ be a finite cyclic group with $|\Gamma| = 2p$, $p > 3$, p is prime and S be a Cayley set such that $S \subseteq Z^*(\Gamma) \setminus \{p\}$ and $|S| = 4$. Then $\text{Cay}_\Sigma(\Gamma, S)$ is $\mathcal{C}$ -Consistent.

Lemma 2.4 : [10] Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ be a finite abelian group, where p_i 's are primes, k_i 's, i and t are positive integers. Assume that $S = U(\Gamma)$ be a generating set of Γ . If at least one of $\mathbb{Z}_{p_i}$'s ($\not\cong \mathbb{Z}_2$) has $\mathbb{Z}_2$ as a quotient, then $\text{Cay}_\Sigma(\Gamma, S)$ is $\mathcal{C}$ -Consistent.

Lemma 2.5 : [10] Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ be a finite abelian group, p_i 's are prime, k_i 's, i and t are positive integers. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ . Then Cayley signed graph $\text{Cay}_\Sigma(\Gamma, S)$ is $\mathcal{C}$ -Consistent if and only if one of following conditions hold:

- (i) for $t = 1$, Γ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_{p^k}$, $p > 2$;
- (ii) for $t > 1$, none of $\mathbb{Z}_{p_i^{k_i}}$ has $\mathbb{Z}_2$ as a quotient;
- (iii) for $t > 1$, $\Gamma \cong \mathbb{Z}_2^t$.

3. Main Results

Now onwards we shall characterize the Cayley set and generating set for which $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are $\mathcal{C}$ -consistent. Now there two basic questions arises, what are those ' S ' for which line signed graph of Cayley signed graph is $\mathcal{C}$ -consistent? or what is the impact of ' Γ ' for which line signed graph of Cayley signed graph is $\mathcal{C}$ -consistent? The following observations will help us to determine $\mathcal{C}$ -consistency trivially in line signed graph of Cayley signed graph.

Observation 3.1 : If Cayley signed graph has all positive edges, then its line signed graph is always $\mathcal{C}$ -consistent.

Observation 3.2 : If Cayley signed graph has exactly one negative edge, then its line signed graph is always $\mathcal{C}$ -consistent.

In the next section, we shall attempt the above questions.

3.1 $\mathcal{C}$ -Consistent $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$

In this section, we shall derive some results on $\mathcal{C}$ -consistency of line signed graph $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ of Cayley signed graph. To answer the above two questions, we have established following results:

Theorem 3.1 : Let Γ be a finite abelian group and let $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ be line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent if $|S| = t$ and $t \in \{1, 2, n-1, n-2\}$.

Proof: First let $|S| = 1$ this implies that $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ does not have edges. Hence, $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent trivially.

If $|S| = 2$, then $\text{Cay}_{\Sigma}(\Gamma, S)$ is either cycle or copies of cycle in which a cycle have all positive edges or a cycle containing one negative edge or a cycle containing negative section containing all negative edges or a cycle of all negative edges in which all vertices are incident with an even number of negative edges. This implies that every cycle in $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ have an even number of negative vertices under $\mathcal{C}$ -marking. Hence, $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = n-1$ and $|S| = n-2$, then $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is all-positive due to Lemma 2.1. This implies that all vertices receive positive sign in $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ under $\mathcal{C}$ -marking. Therefore $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.2 : Let $\Gamma \cong \mathbb{Z}_n$ be a cyclic group of order n and let $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ be line signed graph of Cayley signed graph with respect to Cayley set ' S '. Then $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent if one of following conditions hold:

- (i) S is of the form as appeared in Theorem 3.1;
- (ii) $|S| = n-3$ and no self-inverse element;
- (iii) $S = U(\Gamma)$, where $n = p^k$, p is prime number and k is positive integer;
- (iv) $S = \mathbb{Z}_n^* \setminus \{n/2\}$ and n is an even positive integer;
- (v) $S = U(\Gamma) \cup \{p\}$, where $n = 2p$ and p is prime number.

Proof: (i) Since, we have already proved that $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite abelian group, so $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite cyclic group as well.

For (ii), (iii), (iv) and (v) $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is all-positive due to Lemma 2.1. In this way, all vertices are positive in $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ is $\mathcal{C}$ -consistent.

For the next result; let $A = \{\{1, 9, 3, 7\}, \{3, 7, 4, 6\}, \{1, 9, 2, 8\}\}$ and $B = \{\{3, 7, 4, 6, 5\}, \{1, 9, 5, 2, 8\}\}$.

Theorem 3.3 : Let Γ be a finite cyclic group of order n , ($1 < n < 11$). Let $L_{\Sigma}(\text{Cay}_{\Sigma}(\Gamma, S))$ be line signed graph of Cayley signed graph associated with

Cayley set ' S '. Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if and only if one of following conditions hold:

- (i) Any ' S ' for which $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is balanced;
- (ii) $|S| = 2$;
- (iii) $S = Z^0(\Gamma)$;
- (iv) For $n = 6$, any ' S ';
- (v) For $n = 8$, $|S| = 4$;
- (vi) For $n = 10$, $|S| = 4$ and $S \notin A$; $|S| = 5$ and $S \notin B$; $|S| = 6$.

Proof: Necessity: Let $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent and each of the above listed conditions are false. In order to violate all above conditions, let $\Gamma \cong \mathbb{Z}_8$ and $S = \{1, 4, 7\}$. Then in $\text{Cay}_\Sigma(\Gamma, S)$ there is a negative edge incident at vertex 5 and it is also adjacent to another negative edge incident at vertex 6. So, corresponding to edges incident at vertex 5 there exist a triangle in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ in which one vertex is negative and another two vertices are positive under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is not $\mathcal{C}$ -consistent. This contradicts to the fact that $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent. Therefore, if $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent, then one of the above listed condition is true.

Sufficiency:

- (i) If $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is balanced and Γ be a finite cyclic group of order $n(1 < n < 11)$, then every cycle contain an even number of negative edges adjacent to each other in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$. Now under $\mathcal{C}$ -marking this implies that every cycle contains even number of negative vertices in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (ii) If $|S| = 2$, then by Theorem 3.1(ii), $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (iii) If $S = Z^0(\Gamma)$, then $\text{Cay}_\Sigma(\Gamma, S)$ is 1-regular or have one negative edge or have cycles of all positive edges or have cycles of all negative edges which are adjacent to even number of negative edges. This implies that every cycle in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ have an even number of negative vertices under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (iv) For $\Gamma \cong \mathbb{Z}_6$, if $S = \{3\}$, then $\text{Cay}_\Sigma(\Gamma, S)$ is 1-regular or if $S = \{2, 4\}$, then $\text{Cay}_\Sigma(\Gamma, S)$ is $C_3 \cup C_3$ in which one C_3 formed by all negative edges and another C_3 formed by all positive edges or if $S = \{1, 5\}$, then $\text{Cay}_\Sigma(\Gamma, S)$ is C_6 in which a negative section contains two

negative edges or if $S = \{2, 3, 4\}$, then $\text{Cay}_\Sigma(\Gamma, S)$ have only one negative edge between 1 and 5 or in rest three S , an $\text{Cay}_\Sigma(\Gamma, S)$ have all positive edges. This implies that every cycle in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ have an even number of negative vertices under $\mathcal{C}$ -marking which shows that with respect to each ' S ', $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

- (v) For $\Gamma \cong \mathbb{Z}_8$, if $|S| = 4$, then either $\text{Cay}_\Sigma(\Gamma, S)$ have all positive edges or every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with even number of negative edges. In this way, every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ receive positive sign under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (vi) For $\Gamma \cong \mathbb{Z}_{10}$, if $|S| = 4$ and $S \notin A$, then every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with an even number of negative edges. In this way, every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ receives positive sign under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

For $\Gamma \cong \mathbb{Z}_{10}$, if $|S| = 5$ and $S \notin B$, then either $\text{Cay}_\Sigma(\Gamma, S)$ have all positive edges or every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with an even number of negative edges. In this way, all vertices are positive in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

For $\Gamma \cong \mathbb{Z}_{10}$, if $|S| = 6$, then every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with an even number of negative edges. Therefore all the vertices are positive in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Proposition 3.1 : Let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for any Cayley set ' S '.

Proof: Consider a finite abelian group $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, if $|S| \in \{1, 2, 5, 6, 7\}$ implies $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is balanced due to Lemma 2.2. Then every cycle contain an even number of negative edges in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$. This implies that every cycle contains an even number of negative vertices under $\mathcal{C}$ -marking in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Next if $|S| = 3$ or $|S| = 4$, then either $\text{Cay}_\Sigma(\Gamma, S)$ have all positive edges or every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with an even number of negative edges. In this way, all vertices receive positive sign in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Thus, the result holds.

Proposition 3.2 : Let $\Gamma \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for any Cayley set ' S '.

Proof: Consider $\Gamma \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then corresponding to any Cayley set ' S ' either $\text{Cay}_\Sigma(\Gamma, S)$ have all positive edges or every negative edge in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of negative edges. This implies that every vertex in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ incident with an even number of negative edges. Therefore all vertices receive positive sign in $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.4: Let $\Gamma \cong \mathbb{Z}_p^k$, p is prime, $k \geq 1$, and $S = U(\Gamma)$. Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Proof: Let $\Gamma \cong \mathbb{Z}_p^k$, p is prime, $k \geq 1$, and $S = U(\Gamma)$. Now, we have to prove that $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent with respect to $S = U(\Gamma)$. Since in this case $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is an all-positive signed graph. Therefore all vertices receive positive sign under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.5 : Let $\Gamma \cong \mathbb{Z}_n$ be a finite cyclic group of order n , ($1 < n < 8$), then with respect to each Cayley set, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Proof: Let us consider $\mathbb{Z}_n$ ($1 < n < 8$). Now, we have to prove that $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ with respect to each Cayley set is $\mathcal{C}$ -consistent. For $n = 2$, $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_2, S))$ is a trivial graph. For $n = 3$ or 5 , $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_3, S))$ and $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_5, S))$ are all-positive signed graph, so trivially $\mathcal{C}$ -consistent. For $n = 4$, $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_4, S))$ either does not have edges or have all positive edges, so again $\mathcal{C}$ -consistent trivially. For $n = 6$, $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_6, S))$ is either all vertices receive positive sign under $\mathcal{C}$ -marking or graph is null. For $n = 7$, $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_7, S))$ is a cycle graph containing two negative edges corresponding to $|S| = 2$ and for other S , $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_7, S))$ is all-positive signed graph. Therefore again each vertex receive positive sign under $\mathcal{C}$ -marking. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$ with respect to each Cayley set is $\mathcal{C}$ -consistent.

3.2 $\mathcal{C}$ -consistent $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$

This section is devoted to product line signed graph $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ of Cayley signed graph. Here, we shall find some conditions on ' S ' so that $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.6 : *Let Γ be an abelian group and let $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ be product line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if $|S| = t$ and $t \in \{1, 2, n-1, n-2\}$.*

Proof: Given $|S| = 1$ implies that graph is edgeless. Hence, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = 2$, then $\text{Cay}_\Sigma(\Gamma, S)$ is a cycle graph or copies of cycle graph in which cycle have all positive edges or have one negative edge or have a negative section of negative edges. Therefore every cycle contains an even number of negative vertices under $\mathcal{C}$ -marking in $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$. Hence, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = n-1$, then all non-zero elements are in S . In this way, all vertices receive positive sign in $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = n-2$, then all edges are positive in graph $\text{Cay}_\Sigma(\Gamma, S)$. Therefore all vertices receive positive sign in $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.7 : *Let $\Gamma \cong \mathbb{Z}_n$ be a cyclic group and let $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ be product line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if one of following conditions hold:*

- (i) S is of the form as appeared in Theorem 3.6;
- (ii) $S = U(\Gamma)$, where $n = p^k$, k is a positive integer and p is prime number;
- (iii) $S = U(\Gamma) \cup \{p\}$, where $n = 2p$ and p is prime.

Proof:

- (i) Since, we have already proved that $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite abelian group in Theorem 3.6. This implies that $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite cyclic group as well.
- (ii) If $S = U(\Gamma)$, where $n = p^k$, k is a positive integer and p is prime, then all edges are positive in $\text{Cay}_\Sigma(\Gamma, S)$. Hence, all vertices receive positive sign in $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

(iii) If $S = U(\Gamma) \cup \{p\}$, where $n = 2p$ and p is prime, then all edges are positive in graph $\text{Cay}_\Sigma(\Gamma, S)$. Hence, all vertices receive positive sign in $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Thus, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

3.3 $\mathcal{C}$ -consistent $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$

In this section, we shall study dot line signed graph $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ of Cayley signed graph.

Theorem 3.8 : Let Γ be a finite abelian group and let $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ be dot line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if $|S| = t$ and $t \in \{1, 2, n-1, n-2\}$.

Proof: If $|S| = 1$, then this implies that $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is edgeless. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent trivially.

If $|S| = 2$, then $\text{Cay}_\Sigma(\Gamma, S)$ is all-positive or have one negative edge or have a negative section of negative edges. In this way, every cycle contains an even number of negative vertices under $\mathcal{C}$ -marking in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = n-1$, then all non-zero elements are in S . In this way, all vertices are positive in graph $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

If $|S| = n-2$, then all edges are positive in $\text{Cay}_\Sigma(\Gamma, S)$. Therefore all vertices receive positive sign in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.9 : Let $\Gamma \cong \mathbb{Z}_n$ be a cyclic group and let $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ be dot line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if one of following conditions hold:

- (i) S is of form as appeared in Theorem 3.8;
- (ii) $S = U(\Gamma)$, where $n = p^k$, k is a positive integer and p is prime number;
- (iii) $S = U(\Gamma) \cup \{p\}$, where $n = 2p$ and p is prime number.

Proof:

- (i) In Theorem 3.8, we have proved $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite abelian group, this implies that $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent for a finite cyclic group as well.

- (ii) If $S = U(\Gamma)$, where $n = p^k$, k is a positive integer and p is prime, then all edges are positive in graph $\text{Cay}_\Sigma(\Gamma, S)$. In this way, all vertices receive positive sign in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (iii) If $S = U(\Gamma) \cup \{p\}$, where $n = 2p$ and p is prime, then all edges are positive in graph $\text{Cay}_\Sigma(\Gamma, S)$. In this way, all vertices receive positive sign in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.10 : Let Γ be a finite cyclic group of order n , ($1 < n < 11$). Let $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ be dot line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if and only if one of following conditions hold:

- (i) Any ' S ' for which $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is balanced;
- (ii) For $n = 8$, $|S| = 3$ or $|S| = 5$;
- (iii) For $n = 10$, $|S| = 3$ or $|S| = 5$ or $|S| = 7$;
- (iv) For $n = 6$, any ' S '.

Proof: Necessity: Let $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent and each of the above listed conditions are false. In order to violate all the above conditions, let $\Gamma \cong \mathbb{Z}_7$ and $|S| = 4$. Then $\text{Cay}_\Sigma(\Gamma, S)$ have only one negative edge. This implies that $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ have a cycle containing odd number of negative vertices under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is not $\mathcal{C}$ -consistent. This contradict to our fact that $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent. Therefore, if $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent then one of the above listed condition must hold.

Sufficiency: In order to prove sufficiency part we shall tackle each case separately.

- (i) If $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is balanced, then every cycle contain an even number of negative edges in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$. This implies that every cycle contains even number of negative vertices under $\mathcal{C}$ -marking in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (ii) For $\Gamma \cong \mathbb{Z}_8$, if $|S| = 3$ or $|S| = 5$, then every edge incident at negative vertex in $\text{Cay}_\Sigma(\Gamma, S)$ adjacent to even number of edges under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (iii) For $\Gamma \cong \mathbb{Z}_{10}$, if $|S| = 3$ or $|S| = 5$ or $|S| = 7$, then either all edges positive in $\text{Cay}_\Sigma(\Gamma, S)$ or every edge incident at negative vertex in

$Cay_\Sigma(\Gamma, S)$ adjacent to even number of edges under $\mathcal{C}$ -marking. Hence, $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

- (iv) For $\Gamma \cong \mathbb{Z}_6$, $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is balanced for each S except $S = \{2, 3, 4\}$. This implies that $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent. If $S = \{2, 3, 4\}$, then every edge incident at negative vertex in $Cay_\Sigma(\Gamma, S)$ adjacent to even number of edges under $\mathcal{C}$ -marking. Hence, $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent. Therefore for each Cayley set ' S ', $L_{\bullet, \Sigma}(Cay_\Sigma(\mathbb{Z}_6, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.11 : Let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ be a finite abelian group and let $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ be dot line signed graph of Cayley signed graph associated with Cayley set ' S '. Then $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if and only if one of following conditions hold:

- (i) Any ' S ' for which $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is balanced;
- (ii) $|S| = 3$ or $|S| = 5$.

Proof: Necessity: Let $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent and each of the above listed conditions are false. In order to violate all the above conditions, let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and S contains 4 non-zero elements including both $(1, 0)$ and $(1, 2)$. Then $Cay_\Sigma(\Gamma, S)$ have only one negative edge. This implies that $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ have a cycle containing odd number of negative vertices under $\mathcal{C}$ -marking. Hence, $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is not $\mathcal{C}$ -consistent. This contradict to our fact that $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent. Therefore, if $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent then one of the above listed condition must hold.

Sufficiency: To show the sufficiency part, we shall deal with each case.

- (i) If $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is balanced, then every cycle contain an even number of negative edges in $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$. This implies that every cycle contains an even number of negative vertices under $\mathcal{C}$ -marking in $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$. Therefore $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.
- (ii) For $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, either $|S| = 3$ or $|S| = 5$, then every edge incident at negative vertex in $Cay_\Sigma(\Gamma, S)$ adjacent to even number of edges under $\mathcal{C}$ -marking. Hence, $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.12 : Let $\Gamma \cong \mathbb{Z}_n$ be a finite cyclic group of order n , ($1 < n < 7$). Then $L_{\bullet, \Sigma}(Cay_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent with respect to each Cayley set.

Proof: Let us consider $\Gamma \cong \mathbb{Z}_n$ ($1 < n < 7$). Now, we have to prove $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ with respect to each Cayley set is $\mathcal{C}$ -consistent. For $n = 2$, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\mathbb{Z}_2, S))$ is a trivial graph. For $n = 3$ or 4 , $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\mathbb{Z}_3, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\mathbb{Z}_4, S))$ are either edgeless graph or all-positive signed graph. For $n = 5$, $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_5, S))$ is either cycle containing even number of negative vertices under $\mathcal{C}$ -marking or all positive edge graph. $L_\Sigma(\text{Cay}_\Sigma(\mathbb{Z}_6, S))$ is either edgeless or cycle containing an even number of negative vertices under $\mathcal{C}$ -marking or all-positive signed graph. Hence, for $\Gamma = \mathbb{Z}_n$ ($1 < n < 7$), $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent with respect to each Cayley sets.

In this section, we derived some results for $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ with respect to specific Cayley set.

Remark 3.1 : If $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be a finite abelian group, then corresponding to each Cayley set ' S ', $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are all $\mathcal{C}$ -consistent.

Theorem 3.13 : Let Γ be a finite cyclic group of order $|\Gamma| = 2p$, $p > 3$, p is a prime number and S be a Cayley set such that $S \subseteq Z^*(\Gamma) \setminus \{p\}$ and $|S| = 4$. Then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Proof: Let Γ be a finite cyclic group with $|\Gamma| = 2p$, $p > 3$, p is a prime number and S be a Cayley set such that $S \subseteq Z^*(\Gamma) \setminus \{p\}$ and $|S| = 4$. Then $\text{Cay}_\Sigma(\Gamma, S)$ is an all-positive vertices graph in the view of Lemma 2.3. This implies that all vertices receives positive sign in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.14 : Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ be a finite abelian group, where p_i 's are primes, k_i 's, i and t are positive integers. Assume that $S = U(\Gamma)$ be a generating set of Γ . If at least one of $\mathbb{Z}_{p_i}$'s ($\not\cong \mathbb{Z}_2$) has $\mathbb{Z}_2$ as a quotient, then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Proof: Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ be a finite abelian group, where p_i 's are primes, k_i 's, i and t are positive integers. Assume that $S = U(\Gamma)$ be a generating set of Γ . If at least one of $\mathbb{Z}_{p_i}$'s ($\not\cong \mathbb{Z}_2$) has $\mathbb{Z}_2$ as a quotient, then in $\text{Cay}_\Sigma(\Gamma, S)$ each vertex receive positive sign under $\mathcal{C}$ -marking due to Lemma 2.4. This implies that all vertices in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ receives positive sign under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

Theorem 3.15 : Let $\Gamma \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_2$; $t > 1$ be a finite abelian group and $S = U(\Gamma)$ be a generating set of Γ . Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are all $\mathcal{C}$ -consistent.

Proof: Here, $|S|=1$ and for each line signed graph $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$, it's underlying graph is null. Hence, $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are all $\mathcal{C}$ -consistent trivially.

Theorem 3.16 : Let $\Gamma \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$; $t > 1$ be a finite abelian group and $S = U(\Gamma)$ be a generating set of Γ . Then $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are all $\mathcal{C}$ -consistent.

Proof: Here, $|S|=2$ and invoking Theorem 3.1, Theorem 3.6, and Theorem 3.8 we get the desired result.

Theorem 3.17 : Let $\Gamma \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$ be a finite abelian group, p_i 's are prime, k_i 's, i and t are positive integers. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ . Then $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent if one of following conditions hold:

- (i) for $t = 1$, Γ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_{p^k}$, $p > 2$;
- (ii) for $t > 1$, none of $\mathbb{Z}_{p_i^{k_i}}$ has $\mathbb{Z}_2$ as a quotient;
- (iii) for $t > 1$, $\Gamma \cong \mathbb{Z}_2^t$.

Proof: In the above listed cases, $\text{Cay}_\Sigma(\Gamma, S)$ is $\mathcal{C}$ -consistent due to Lemma 2.5 as each vertex in $\text{Cay}_\Sigma(\Gamma, S)$ receive positive sign under $\mathcal{C}$ -marking. This implies that all the edges in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are positive and hence each vertex receive positive sign in $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ under $\mathcal{C}$ -marking. Hence, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ is $\mathcal{C}$ -consistent.

4. Conclusion

In this paper, we have established several sufficient conditions on the Cayley set ' S ' in order to determine the canonically consistent line signed graphs of Cayley signed graphs $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$, and $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$. One of the highlights of this paper is a complete constructive characterization of Cayley sets and generating sets for which $L_\Sigma(\text{Cay}_\Sigma(\Gamma, S))$, $L_{\bullet\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ and $L_{\times\Sigma}(\text{Cay}_\Sigma(\Gamma, S))$ are $\mathcal{C}$ -consistent.

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