

Using the GML decomposition method for solving nonlinear differential equations

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Abstract

In this manuscript, we will introduce a new technique known by the GML decomposition technique. This technique has been used to various types of (NODEs) and as

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well as to derive the exact solutions of theorems. The foundation of the GMLDM is (ADM) and the GML Transform Method. The method that has been presented has made it easy to solve nonlinear ordinary differential equations. A quick convergence series solution is provided by the suggested approach. One excellent mathematical tool for solving NODEs and NODEs is the GML Decomposition Method (GMLDM). The GMLDM is incredibly simple and effective.

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1. Introduction

NODEs have attracted a significant amount of attention due to their numerous examples. NODEs find various applications in different areas of engineering mathematics and its applications in other fields such as engineering, mechanics, nuclear physics, chemistry, astronomy, and biology [1-5], [7-10].

We give an integral technique called the GML decomposition method [29], and use it in finding the exact solutions of the NODEs. There exist many integrated transform methods [13–19].

The most commonly used method is the (LT) [30]. Other techniques are being adopted to tackle PDEs as well as ODEs, including the integral technique [6, 14-19], the reduced differential technique [25-28], and other techniques.

In this manuscript, we present the development of an efficient algorithm for numerical computation based on the GML decomposition method in order to solve and apply as theorems and problems [32-36].

The GML Decomposition Method delivers for solving from the form of rapidly series of convergent.

2. Definitions and Properties of the GML-Technique

In this following partition, we concentrate definition for the GML technique, we assume that we have the function The new transform of time mapping μ denoted as the operator $G\{k(\tau)\}$ or simply $T(s)$ is:

$$G[\rho(\tau)] = T(s) = \int_{\tau=0}^{\infty} \rho(\tau)[g(s)]^{-s\tau} d\tau \quad (1)$$

Where $g(s) > 0$, $t \in [0, \infty)$ and $s \in \mathbb{C}$.

The parameter(s) is independent of time (τ) and has values in terms of either real or complex values. The definition and essential details of GML Integral Transform will be described in this work. This transform has a large role in a number of solving problems in nuclear physics, engineering, and other practical problems by converting ODEs to a simple algebraic equation. The exact solution for the problem is then found by inverting the transformation, which is called GML technique (Table 1).

Table 1
Different between GML-Transforms and Laplace Transforms [11-12] [36].

$\rho(\tau)$	$G[\rho(\tau)]$	$[L\rho(\tau)]$
1	$\frac{1}{\text{sln}g(s)}$	$\frac{1}{s}$
τ	$\frac{1}{(\text{sln}g(s))^2}$	$\frac{1}{s^2}$
$e^{m\tau}$	$\frac{1}{\text{sln}g(s) - m}$	$\frac{1}{s - m}$
$\frac{\tau^{k-1}}{(k-1)!}, k = 1, 2,$	$\frac{1}{(\text{sln}g(s))^k}$	$\frac{1}{s^k}$
$\sin(\tau)$	$\frac{1}{(\text{sln}g(s))^2 + 1}$	$\frac{1}{1 + s^2}$

Now we present some efficient properties of the GML–Techniques are given as like as [11-12], [20- 21]:

Theorem 2.1:

- (i) $G\{\rho'(\tau)\} = \text{sln}g(s)G\{\rho(\tau)\} - f(0)$
- (ii) $G\{\rho''(\tau)\} = [\text{sln}g(s)]^2 G\{\rho(\tau)\} - \text{sln}g(s)f(0) - \rho'(0)$
- (iii) $G\{\rho'''(\tau)\} = [\text{sln}g(s)]^3 G\{\rho(\tau)\} - (s \ln g(s))^2 \rho(0) - \text{sln}g(s)\rho'(0) - \rho''(0)$
- (iv) $G\{\rho^{(n)}(\tau)\} = [\text{sln}g(s)]^n G\{\rho(\tau)\} - [\text{sln}g(s)]^{n-1} \rho(0) - [\text{sln}g(s)]^{n-2} \rho'(0) - \dots - \rho^{(n-1)}(0)$

Proofs:

- (i) By using the condition (1) we get:

$$G\{\rho'(\tau)\} = \int_0^\infty \rho'(\tau)g(s)^{-st} dt = \lim_{B \rightarrow \infty} \left[\int_0^B \rho'(\tau)g(s)^{-st} d\tau \right]$$

Integrating by partition, we get:

$$G\{\rho'(\tau)\} = \text{sln}g(s)G\{\rho(\tau)\} - \rho(0)$$

- (ii) Let $\rho'(\tau) = z(\tau)$

Also, by the definition, we have:

$$G\{\rho''(\tau)\} = G\{z'(\tau)\} = \text{sln}g(s)G\{z(\tau)\} - z(0)$$

Then we get

$$G\{\rho''(\tau)\} = [\text{sln}g(s)]^2 G\{\rho(\tau)\} - \text{sln}g(s)\rho(0) - \rho'(0)$$

- (iii) and (iv) are proved as (i) and (ii).

3. The GML Decomposition Method

Assume the general NODEs of the form:

$$M\varphi + C(\varphi) + V(\varphi) = g(\tau), \quad (2)$$

subject to the I.C

$$\varphi(0) = h(\tau), \quad (3)$$

M is the operator of the higher order differentiation, C is the remaining part of the differential operator, $g(\tau)$ is the non-homogeneous term, and $V(\varphi)$ is the nonlinear source.

If L is the differential operator of the first order, then by using the GML-Technique of Eq. (2), we have

$$s\ln g(s)F(s, u) - \varphi(0) + G[N(\varphi)] + G[V(\varphi)] = G[g(\tau)] \quad (4)$$

By substituting from (3) into (4), we obtain:

$$\varnothing(s) = \frac{h(t)}{s\ln g(s)} + \frac{1}{s\ln g(s)} G[g(t)] - \frac{1}{s\ln g(s)} G[N(\varphi) + V(\varphi)] \quad (5)$$

On taking the inverse of the GML-Transform of (5), we have:

$$\varphi(\tau) = K(\tau) - G^{-1} \left[\frac{1}{s\ln g(s)} G[N(\varphi) + V(\varphi)] \right] \quad (6)$$

where $K(\tau)$ is the extra term.

Now we present the infinite series solution of the infamous function $\varphi(t)$ of the following formula:

$$\varphi(\tau) = \sum_{n=0}^{\infty} \varphi_n(\tau) \quad (7)$$

By using (7), we rephrase (6) in the form:

$$\sum_{n=0}^{\infty} \varphi_n(\tau) = K(\tau) - G^{-1} \left[\frac{1}{s\ln g(s)} G[N \sum_{n=0}^{\infty} \varphi_n(t) + \sum_{n=0}^{\infty} A_n(t)] \right] \quad (8)$$

where $A_n(\tau)$ is the a domian polynomial which introduces the nonlinear source.

The comparison for both sides of Eq (8), we obtain the recursive formula,

In the following

$$\begin{aligned} \varphi_0(\tau) &= K(\tau) \\ \varphi_1(\tau) &= -G^{-1} \left[\frac{1}{s\ln g(s)} G[N\varphi_0(\tau) + A_0(\tau)] \right] \\ \varphi_2(\tau) &= -G^{-1} \left[\frac{1}{s\ln g(s)} G[N\varphi_1(\tau) + A_1(\tau)] \right] \\ \varphi_3(\tau) &= -G^{-1} \left[\frac{1}{s\ln g(s)} G[N\varphi_2(\tau) + A_2(\tau)] \right] \end{aligned}$$

Therefore, we get the generalized recursive formula on the form:

$$\varphi_{n+1}(\tau) = -\varphi^{-1} \left[\frac{1}{s\ln g(s)} G[N\varphi_n(\tau) + A_n(\tau)] \right], \quad n \geq 0 \quad (9)$$

Since, the approximation solution is obtained by:

$$\varphi(\tau) = \sum_{n=0}^{\infty} \varphi_n(\tau) \quad (10)$$

4. Worked Examples is applied in this partition

We will present theorem and some worked applications for the GMLDM to some examples and the comparison our solutions with the exact solutions.

Theorem 4.1:

- i) consider that the NSODE of the general form:

$$a \frac{d^2\varphi}{dt^2} + b \left(\frac{d\varphi}{dt} \right)^2 + c\varphi^2(t) = d + ef(t), \quad (11)$$

$$a = b = c = d = 1, e = -1 \text{ and } f(t) = \sin t$$

Subject to the I.Cs

$$\varphi(0) = 1, \quad \varphi'(0) = 0,$$

$$\text{Then the analytic solution is } = f(t) = \sin t \quad (12)$$

- ii) consider the NSODE of the general form:

$$a \frac{d^2\varphi}{dt^2} + b \left(\frac{d\varphi}{dt} \right)^2 + c\varphi^2(t) = d + ef(t), \quad (13)$$

$$a = b = c = d = 1, e = -1 \text{ and } f(t) = \cos t$$

Subject to the I. Cs

$$\varphi(0) = 1, \quad \varphi'(0) = 0,$$

$$\text{Then the exact solution is } = f(x) = \cos t \quad (14)$$

Applied Example for Theorem 4.1 for proving (i)

Assume that the first order (NODE) of the form:

$$\frac{d^2x}{d\tau^2} + \left(\frac{dx}{d\tau} \right)^2 + x^2(\tau) = 1 - \sin(\tau) \quad (15)$$

Subject to the I. Cs

$$x(0) = 0, \quad x'(0) = 1 \quad (16)$$

We begin by using the GML–technique in order to both the sides of Eq. (15), we have:

$$\begin{aligned} & (\sin g(s))^2 X(s) - \sin g(s)x(0) - x'(0) + G \left[\left(\frac{dx}{d\tau} \right)^2 \right] + \\ & G[x^2(\tau)] = \frac{1}{\sin g(s)} - \frac{1}{(\sin g(s))^2 + 1} \end{aligned} \quad (17)$$

By substituting from (16) into (17) we have:

$$\begin{aligned} X(s) = & \frac{1}{(\sin g(s))^2} + \frac{1}{(\sin g(s))^3} - \frac{1}{(\sin g(s))^2(\sin g(s))^2 + 1} - \\ & \frac{1}{(\sin g(s))^2} G \left[\left(\frac{dx}{d\tau} \right)^2 + x^2(\tau) \right] \end{aligned} \quad (18)$$

Then by taking the inverse GML–Transform of (18), we have:

$$x(\tau) = \frac{\tau^2}{2!} + \sin(\tau) - G^{-1} \left[\frac{1}{(\sin g(s))^2} G \left[\left(\frac{dx}{d\tau} \right)^2 + x^2(\tau) \right] \right] \quad (19)$$

We now present an infinite series of solutions of the infamous function $x(t)$ of the form:

$$x(\tau) = \sum_{n=0}^{\infty} x_n(\tau) \quad (20)$$

By utilizing equation (20), we can rewrite equation (19) to have:

$$\sum_{n=0}^{\infty} x_n(\tau) = \frac{\tau^2}{2!} + \sin(\tau) - G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G^+ [\sum_{n=0}^{\infty} A_n(\tau) + \sum_{n=0}^{\infty} B_n(\tau)] \right] \quad (21)$$

Where A_n and B_n are a domain polynomials of nonlinear sources $\left(\frac{dx}{d\tau}\right)^2$, and $x^2(\tau)$ respectively.

Therefore, the comparison for both sides of Eq. (21), we get the generalized recursive formula of the forms:

$$\begin{aligned} x_0(\tau) &= \frac{\tau^2}{2!} + \sin(\tau) \\ x_1(\tau) &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[A_0 + B_0] \right], \\ x_2(\tau) &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[A_1 + B_1] \right] \\ x_3(\tau) &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[A_2 + B_2] \right] \end{aligned}$$

Then, the generalized recursive formula is the following the form:

$$x_{n+1}(\tau) = -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[A_n + B_n] \right], \quad n \geq 0. \quad (22)$$

Applying the recursive formula got in Eq. (22), we can simplify to derive the remaining components of the function $x(\tau)$ on the forms:

$$\begin{aligned} x_1(\tau) &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[A_0 + B_0] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[(x'_0)^2 + x_0^2] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[(x'_0)^2 + x_0^2] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} G[1] \right] + \dots \\ &= -G^{-1} \left[\frac{1}{(\text{stng}(s))^2} \right] + \dots \\ &= -\frac{\tau^2}{2!} + \dots \end{aligned}$$

Thus, cancelling out the noise terms in $x(\tau)$ and $x_1(\tau)$, one can observe that a term of $x_0(t)$ still fulfills the differential equation and thus an exact solution of the form: $\varphi(\tau) = \sin(\tau)$

The closed solution matches exactly to the solution given by (ADM) [31].

Applied Example for theorem 4.1 for proving (ii)

Consider the first order (NODE) of the form:

$$\frac{d^2 z}{dt^2} + \left(\frac{dz}{dt}\right)^2 + z^2(t) = 1 - \cos(t) \quad (23)$$

Subject to the I.C

$$z(0) = 0, \quad z'(0) = 1 \quad (24)$$

Using the GML–technique in order to both the sides of Eq. (24), we obtain:

$$\begin{aligned} (\text{sln}g(s))^2 Z(s) - \text{sln}g(s)x(0) - z'(0) + G \left[\left(\frac{dz}{dt}\right)^2 \right] + G[z^2(t)] = \\ \frac{1}{\text{sln}g(s)} - \frac{\text{sln}g(s)}{(\text{sln}g(s))^2 + 1} \end{aligned} \quad (25)$$

By substituting Eq. (24) into Eq. (25) we obtain:

$$Z(s) = \frac{1}{(\text{sln}g(s))^3} + \frac{1}{\text{sln}g(s)} - \frac{1}{\text{sln}g(s)(\text{sln}g(s))^2 + 1} - \frac{1}{(\text{sln}g(s))^2} G \left[\left(\frac{dz}{dt}\right)^2 + z^2(t) \right] \quad (26)$$

By taking the GML–Transform for the inverse of Eq. (26), we obtain:

$$z(t) = \frac{t^2}{2!} + \cos(t) - G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G \left[\left(\frac{dz}{dt}\right)^2 + z^2(t) \right] \right] \quad (27)$$

We now present an infinite series of solution of the infamous function $z(t)$ of the form:

$$z(t) = \sum_{n=0}^{\infty} z_n(t) \quad (28)$$

Applying Eq. (28), we rephrase (27) as follows:

$$\sum_{n=0}^{\infty} z_n(t) = \frac{t^2}{2!} + \cos(t) - G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G [\sum_{n=0}^{\infty} A_n(t) + \sum_{n=0}^{\infty} B_n(t)] \right] \quad (29)$$

where A_n and B_n are a domain polynomials of the nonlinear sources

$\left(\frac{dz}{dt}\right)^2$ and $z^2(t)$ respectively.

The comparison for both the sides of Eq. (29), we get the general recursive formula on the forms:

$$\begin{aligned} z_0(t) &= \frac{t^2}{2!} + \cos(t) \\ z_1(t) &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G [A_0 + B_0] \right], \\ z_2(t) &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G [A_1 + B_1] \right] \end{aligned}$$

$$z_3(t) = -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[A_2 + B_2] \right]$$

Then, the generalized recursive formula is obtained by:

$$z_{n+1}(t) = -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[A_n + B_n] \right], \quad n \geq 0. \quad (30)$$

Applying the recursive formula deduced in (30), we simplify derive the remaining components of the following function $x(t)$ on the following form:

$$\begin{aligned} z_1(t) &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[A_0 + B_0] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[(z'_0)^2 + z_0^2] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[(z'_0)^2 + z_0^2] \right] \\ &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} G[1] \right] + \dots \\ &= -G^{-1} \left[\frac{1}{(\text{sln}g(s))^2} \right] + \dots \\ &= -\frac{t^2}{2!} + \dots \end{aligned}$$

Thus, having eliminated noise terms by-cancelled noise terms that appears between $z(t)$ and $z_1(t)$, it is possible to observe that non-cancelled noise terms from $z_0(t)$ still satisfies obtained differential equation leading to an exact solution given by: $z(t) = \cos(t)$

That is satisfied [31].

Example 4.2: Solve (NODE) [31]:

$$\frac{du}{d\tau} - 1 = u^2(\tau) \quad (31)$$

subject to the I.C

$$u(0) = 0 \quad (32)$$

Taking the GML transform to both sides of (31), we have:

$$\text{ln}g(s)U(s) - u(0) - 1 = G[u^2(\tau)] \quad (33)$$

Substituting in (33), we have:

$$U(s) = \frac{1}{(\text{sln}g(s))^2} + \frac{1}{\text{sln}g(s)} [G[u^2(\tau)]] \quad (34)$$

Taking the inverse GML transform of (34), we get:

$$u(\tau) = \tau + G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[u^2(\tau)]] \right] \quad (35)$$

By assuming an infinite solution of the infamous function $u(\tau)$ of the next form

$$u(t) = \sum_{n=0}^{\infty} u_n(\tau) \quad (36)$$

Using (36), we rephrase (35) as:

$$\sum_{n=0}^{\infty} u_n(\tau) = \tau + G^{-1} \left[\frac{1}{s\ln g(s)} [G[\sum_{n=0}^{\infty} A_n(\tau)]] \right] \quad (37)$$

where $A_n(t)$ is the Adomian polynomial representing the nonlinear source $u^2(\tau)$.

And from (37), we can get the generalized recursive formula as:

$$\begin{aligned} u_0(\tau) &= \tau \\ u_1(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_0(\tau)]] \right] \\ u_2(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_1(\tau)]] \right] \\ u_3(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_2(\tau)]] \right] \end{aligned}$$

Thus, the generalized recursive formula is:

$$u_{n+1}(\tau) = G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_n(\tau)]] \right], \quad n \geq 0 \quad (38)$$

Applying Eq. (38), we can simplify derive the remaining components of the infamous function $u(t)$ in the forms:

$$\begin{aligned} u_1(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_0(\tau)]] \right] = G^{-1} \left[\frac{1}{s\ln g(s)} [G[u_0^2(\tau)]] \right] \\ &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[\tau^2]] \right] = G^{-1} \left[\frac{1}{s\ln g(s)} \frac{2}{(s\ln g(s))^3} \right] = \frac{1}{3} \tau^3 \\ u_2(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_1(\tau)]] \right] = G^{-1} \left[\frac{1}{s\ln g(s)} [G[2u_0(\tau)u_1(\tau)]] \right] \\ &= G^{-1} \left[\frac{1}{s\ln g(s)} \left[G \left[\frac{2\tau^4}{3} \right] \right] \right] = G^{-1} \left[\frac{48u^5}{3s^6} \right] = \frac{2\tau^5}{15} \\ u_3(\tau) &= G^{-1} \left[\frac{1}{s\ln g(s)} [G[A_2(\tau)]] \right] = G^{-1} \left[\frac{1}{s\ln g(s)} [G[2u_0(\tau)u_2(\tau) + u_1^2(\tau)]] \right] \\ &= G^{-1} \left[\frac{1}{s\ln g(s)} \left[G^+ \left[\frac{17\tau^6}{45} \right] \right] \right] = G^{-1} \left[\frac{12240u^7}{45s^8} \right] = \frac{17\tau^7}{315} \end{aligned}$$

Therefore, the approximation solution of the infamous function $u(t)$ is obtained by:

$$\begin{aligned} u(\tau) &= \sum_{n=0}^{\infty} u_n(\tau) \\ &= u(\tau) + u_1(\tau) + u_2(\tau) + u_3(\tau) + \dots \end{aligned}$$

$$= \tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \frac{17\tau^7}{315} + \dots$$

Hence, the analytic solution of Eq. (38) is obtained by:

$$u(\tau) = \tan(\tau)$$

So, the result obtained by (ADM) is the exact solution [31].

Example 4.3: Solve [31]:

$$\frac{dz}{dt} = 1 - t^2 + z^2(t) \quad (39)$$

subject to the I.C

$$z(0) = 0 \quad (40)$$

Using the GML–Technique to both the sides of Eq (39), we have:

$$\text{sln}g(s)Z(s) - z(0) = \frac{1}{\text{sln}g(s)} - \frac{2}{(\text{sln}g(s))^3} + G[z^2(t)] \quad (41)$$

By substituting from (40) into (41), we have:

$$z(s) = \frac{1}{(\text{sln}g(s))^2} - \frac{2}{(\text{sln}g(s))^4} + \frac{1}{\text{sln}g(s)} [G[z^2(t)]] \quad (42)$$

Taking the inverse GML technique of (42), we have:

$$z(t) = t - \frac{2t^3}{3!} + G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[z^2(t)]] \right] \quad (43)$$

By assuming that an infinite series solution of the following function $z(t)$ of the form:

$$z(t) = \sum_{n=0}^{\infty} z_n(t) \quad (44)$$

Then by using Eq. (44), we rephrase Eq. (43) as:

$$\sum_{n=0}^{\infty} z_n(t) = t - \frac{t^3}{3} + G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[z^2(t)]] \right] \quad (45)$$

where A_n is a domain polynomial which represent the nonlinear source $z^2(t)$.

The comparison of the sides of (45), we simplify and decide the generalized recursive formula in the next form:

$$\begin{aligned} z_0(t) &= t - \frac{t^3}{3} \\ z_1(t) &= G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[A_0(t)]] \right] \\ z_2(t) &= G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[A_1(t)]] \right] \\ z_3(t) &= G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[A_2(t)]] \right] \end{aligned}$$

Then the generalized recursive formula is obtained by:

$$z_{n+1}(t) = G^{-1} \left[\frac{1}{\text{sln}g(s)} [G[A_n(t)]] \right] \quad (46)$$

By using (46), we derive the remaining components of the infamous function $z(t)$ in the next form:

$$\begin{aligned}
 z_1(t) &= G^{-1} \left[\frac{1}{\text{slng}(s)} G[A_0(t)] \right] \\
 &= G^{-1} \left[\frac{1}{\text{slng}(s)} G[v_0^2(t)] \right] \\
 &= G^{-1} \left[\frac{1}{\text{slng}(s)} G \left[\left(t - \frac{t^3}{3} \right)^2 \right] \right] \\
 &= G^{-1} \left[\frac{1}{\text{slng}(s)} G^+[t^2] \right] - \frac{2}{3} G^{-1} \left[\frac{1}{\text{slng}(s)} G^+[t^4] \right] + \frac{1}{9} G^{-1} \left[\frac{1}{\text{slng}(s)} G^+[t^6] \right] \\
 &= G^{-1} \left[\frac{2}{(\text{slng}(s))^4} \right] - \frac{2}{3} G^{-1} \left[\frac{4!}{(\text{slng}(s))^6} \right] + \frac{1}{9} G^{-1} \left[\frac{6!}{(\text{slng}(s))^8} \right] \\
 &= \frac{t^3}{3} - \frac{2t^5}{15} + \frac{t^7}{63}
 \end{aligned}$$

From $z_1(t)$ it is clear that there is one noise term that appears in the components $z_0(t)$. And by subtracting the noise source from $z_0(t)$ the remaining non-canceled source $z_0(t)$ In the equation, the analytic or the approximation solution is

$$z(t) = t \quad (47)$$

5. Conclusion

By applying the GMLDM to obtain for solving of two NODEs and the Riccati -differential equations in this work, we managed to get the exact solutions for each of the three examples. When compared to earlier techniques, the GMLDM significantly enhances all areas. Later we would like to use our GMLDM for other the (LODEs) and the (NODEs) for all from the PDEs and the ODEs.

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