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Two-step iterative method for nonlinear equations using Caputo-Fabrizio fractional derivatives

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Abstract

The main objective of this paper is to develop a two-step fractional Newton-Raphson method for solving non-linear equations using the Caputo-Fabrizio fractional derivative. In further detail, we discuss the convergence analysis for the proposed method. In addition to it, we provide an example to compare the method we proposed with the existing approaches.

Subject Classification: *47H10, 47J26.*

Keywords: *Nonlinear equations, Convergence analysis, Numerical analysis, Caputo-Fabrizio fractional derivative.*

1. Introduction

Fractional calculas and nonlinear equations play significant role in arriving of the branches of science and engineering, including applied mathematics, physics, biology, and economics. Instead of classical calculas, fractional derivatives include non-local and memory-dependent properties [1, 2, 3, 4, 5, 6, 7, 8, 9]. Among the various fractional derivatives proposed, the Caputo-Fabrizio derivative has gained significant attention due to its nonsingular kernel [10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In this paper, we present a two-step fractional Newton-Raphson approach for solving nonlinear equations based on Caputo-Fabrizio fractional derivative. The method proposed here involves an extension of the traditional Newton-Raphson method by utilizing the properties of Caputo-Fabrizio derivative. The prime contribution is the derivation of the two-step fractional

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Newton-Raphson method based on the Caputo-Fabrizio derivative and to check its convergence analysis at a local convergence rate of atleast 3α and also a numerical example is provided with its comparison with existing numerical methods.

In Section 2 we state existing results and Section 3 provides the main results that we use to formulate the two-step fractional Newton-Raphson method using the Caputo-Fabrizio derivative that we introduce in section 4. Section 5 provides an example of the application of the proposed method in comparison to previous existing approahes.

2. Preliminaries

We recall some existing results of Caputo fractional derivative and Caputo–Fabrizio fractional derivative in this paper that will be used in the development of our method.

Definition 2.1: (Caputo Fractional Derivative) [14, 15]

$$ {}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} (x-\tau)^{-\alpha} f'(s)\, ds, \tag{1} $$

in which $0 < \alpha < 1$ is the fractional order, and $\Gamma(s) = \int_{0}^{+\infty} e^{-t} t^{s-1} dt$ is the Gamma function.

Definition 2.2: (Caputo–Fabrizio Fractional Derivative)[8]. Let $f \in H^{1}(a,b)$ and $0 < \alpha < 1$. The Caputo–Fabrizio fractional derivative of order α is defined as

$$ {}^{CF}D_{a}^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_{a}^{x} f'(s) \exp\left(-\frac{\alpha}{1-\alpha}(x-s)\right) ds, \tag{2} $$

where $M(\alpha)$ is a normalization factor having $M(0) = M(1) = 1$.

Lemma [Two-step fractional Newton-Raphson method 1 (TSFNRM1)] [12, 16, 17].

Algorithm: Two-step Caputo fractional Newton-Raphson Method (TSFNRM).

Initially x_0 is sufficiently close to $x_{(i+1)}$ by the following Caputo type two-step scheme:

$$ y_i = x_i - \Gamma(\alpha+1) \frac{f(x_i)}{{}_{C}D_{x}^{\alpha} f(x_i)}, $$

$$ x_{i+1} = y_i - \Gamma(\alpha+1) \frac{f(x_i)}{{}_{CF}D_{x}^{\alpha} f(x_i)}, \quad i = 0,1,2,\ldots $$

3. Main Results

Theorem 1: *Suppose that* $f \in C^{n+1}[a,b]$ *and* ${}^{CF}D_{a}^{\alpha} f(x)$ *be the Caputo–Fabrizio fractional derivative of order* $0 < \alpha \leq 1$, *and let* ${}^{CF}D_{a}^{j\alpha} f(x) \in C[a,b]$ *for all* $j = 1,2,\ldots,n+1$. *Then,*

$$f(x) = \sum_{k=0}^{n} {}^{CF}D_a^{k\alpha} f(a) \cdot \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} + R_n(x), \tag{3}$$

where $R_n(x) = {}^{CF}D_a^{(n+1)\alpha} f(\xi) \cdot \frac{(x-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}, \forall \xi$ *such that* $a \le \xi \le b$ *is the remainder term.*

Proof: Let $n = 0$. Then, equation (3) reduces to the Caputo-Fabrizio fractional mean value theorem:

$$f(x) = f(a) + {}^{CF}D_a^{\alpha} f(\xi) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{4}$$

Using (2) and integrating both sides from a to x, we have:

$$\int_a^x {}^{CF}D_a^{\alpha} f(s)\, ds = \frac{M(\alpha)}{1-\alpha} \int_a^x \int_a^s f'(\tau) \exp\left(-\frac{\alpha(s-\tau)}{1-\alpha}\right) d\tau\, ds. \tag{5}$$

Here,

$$\int_a^x \int_a^s f'(\tau) \exp\left(-\frac{\alpha(s-\tau)}{1-\alpha}\right) d\tau\, ds = \int_a^x f'(\tau) \frac{1-\alpha}{\alpha} \left[1 - \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right)\right] d\tau \tag{6}$$

Now (5) implies that

$$\int_a^x {}^{CF}D_a^{\alpha} f(s)\, ds = \frac{M(\alpha)}{1-\alpha} \int_a^x f'(\tau) \frac{1-\alpha}{\alpha} \left[1 - \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right)\right] d\tau \tag{7}$$

As

$$f(x) - f(a) = \int_a^x f'(\tau) d\tau \tag{8}$$

Using (8), (7) becomes

$$\int_a^x {}^{CF}D_a^{\alpha} f(s)\, ds = \frac{M(\alpha)}{1-\alpha} \left[ ] f(x) - f(a) - \int_a^x f'(\tau) \exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right) d\tau\right] \tag{9}$$

On simplifying and by applying mean value theorem, we have,

$$f(x) - f(a) = {}^{CF}D_a^{\alpha} f(\xi) \frac{1-\alpha}{M(\alpha)} \int_a^x \exp\left(-\frac{\alpha(x-s)}{1-\alpha}\right) ds \tag{10}$$

Upon simplification the integral $\int_a^x \exp\left(-\frac{\alpha(x-s)}{1-\alpha}\right) ds$ we have,

$$\int_a^x \exp\left(-\frac{\alpha(x-s)}{1-\alpha}\right) ds = \frac{1-\alpha}{\alpha} \left[1 - \exp\left(-\frac{\alpha(x-a)}{1-\alpha}\right)\right] \tag{11}$$

For $\alpha \approx 1, \exp\left(-\frac{\alpha(x-a)}{1-\alpha}\right) \approx 0$, we have, $f(x) - f(a) \approx {}^{CF}D_a^{\alpha} f(\xi) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$ and now we assume that it holds for n=m and for n=m+1 we have to expand f(x) to m terms

$$f(x) = \sum_{k=0}^{m} {}^{CF}D_a^{k\alpha} f(a) \cdot \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} + R_m(x) \tag{12}$$

where,

$$R_m(x) = \frac{1}{\Gamma(m\alpha+1)} \int_a^x (x-\tau)^{m\alpha} {}^{CF}D_a^{(m+1)\alpha} f(\tau) d\tau \tag{13}$$

Now, we have to express (13) as an integral. In order to do so let m=0 in (12) which implies that $f(x) = f(a) + R_0(x)$ where $R_0(x)$ is obtained by integrating Caputo-Fabrizio derivative,

$$R_0(x) = \frac{1-\alpha}{\alpha}\int_a^x f'(\tau)\left[1 - exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right)\right]d\tau \quad (14)$$

Now, for $\alpha \approx 1, exp\left(-\frac{\alpha(x-\tau)}{1-\alpha}\right) \approx 0$ (14) become

$$R_m(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\,{}^{CF}D_a^{\alpha} f(\tau)d\tau \quad (15)$$

By assuming that it holds for m and considering for m+1 we get that

$$R_m(x) = \frac{1}{\Gamma(m+1)\alpha}\int_a^x (x-\tau)^{(m+1)\alpha-1}\,{}^{CF}D_a^{\alpha} f(\tau)d\tau \quad (16)$$

This implies that it is true for m+1. Now by applying mean value theorem and after simplifying it we have,

$$R_m(x) = \frac{{}^{CF}D_a^{(n+1)\alpha} f(\xi)}{\Gamma((m+1)\alpha+1)}(x-a)^{(m+1)\alpha} \quad (17)$$

which is our remainder term and (12) and (17) proves our theorem.

Theorem 2: *Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function in a domain D that contains the root $\bar{x}$ of $f(x)$. For any positive integers k and α, $\alpha \in (0,1]$, it possesses Caputo-Fabrizio fractional derivatives of order 2α. Also, assume that ${}^{CF}D_x^{\alpha} f(x)$ is a continuous, nonzero fractional derivative of Caputo-Fabrizio type in a neighbourhood of $\bar{x}$ and the initial guess x_0 is sufficiently close to $\bar{x}$, then*

$$x_{k+1} = x_k - M_{\alpha}\frac{f(x_k)}{{}^{CF}D_x^{\alpha} f(x_k)}, \qquad k = 0,1,\ldots, \quad (18)$$

converges locally at a rate of at least 2α,where M_{α} is a normalization constant with Caputo-Fabrizio kernel and the error equation is given by

$$e_{k+1} = M(\alpha)\frac{{}^{CF}D_x^{2\alpha} f(\bar{x})[\Gamma(1+2\alpha)-\Gamma(1+\alpha)^2]}{{}^{CF}D_x^{\alpha} f(\bar{x})[\Gamma(1+\alpha)^2-\Gamma(1+2\alpha)]}e_k^{2\alpha} + O(e_k^{3\alpha}). \quad (19)$$

Proof: By using Caputo-Fabrizio based Taylor series expansion of a function f(x) centered at $a = \bar{x}$

$$f(x_k) = f(\bar{x}) + \frac{{}^{CF}D_x^{2\alpha} f(\bar{x})}{\Gamma(1+\alpha)}e_k^{\alpha} + \frac{{}^{CF}D_x^{2\alpha} f(\bar{x})}{\Gamma(1+2\alpha)}e_k^{2\alpha} + O(e_k^{3\alpha}).$$

Since $f(\bar{x}) = 0$, then

$$f(x_k) = \frac{{}^{CF}D_x^{\alpha} f(\bar{x})}{\Gamma(1+\alpha)}e_k^{\alpha} + \frac{{}^{CF}D_x^{2\alpha} f(\bar{x})}{\Gamma(1+2\alpha)}e_k^{2\alpha} + O(e_k^{3\alpha}) \quad (20)$$

Similarly, by expanding Caputo-Fabrizio derivative at x_k around $\bar{x}$, we have

$${}^{CF}D_x^{\alpha} f(x_k) = {}^{CF}D_x^{\alpha} f(\bar{x}) + \frac{{}^{CF}D_x^{\alpha} f(\bar{x})}{\Gamma(1+\alpha)}e_k^{\alpha} + O(e_k^{2\alpha}) \quad (21)$$

Subtracting $\bar{x}$ on both sides of (18), we have the following error equation

$$e_{k+1} = e_k - M(\alpha)\frac{f(x_k)}{{}^{CF}D_x^{\alpha} f(x_k)} \quad (22)$$

Substituting (20) and (21) in (22)

$$e_{k+1} = e_k - M(\alpha)\frac{\frac{{}^{CF}D_x^{\alpha} f(\bar{x})}{\Gamma(1+\alpha)} e_k^{\alpha} + \frac{{}^{CF}D_x^{2\alpha} f(\bar{x})}{\Gamma(1+2\alpha)} e_k^{2\alpha} + O(e_k^{3\alpha})}{{}^{CF}D_x^{\alpha} f(\bar{x}) + \frac{{}^{CF}D_x^{2\alpha} f(\bar{x})}{\Gamma(1+\alpha)} e_k^{\alpha} + O(e_k^{2\alpha})}$$

To attain a convergence rate of 2α, it is necessary that Caputo-Fabrizio fractional Newton's method leads to the following resulting error equation:

$$e_{k+1} = M(\alpha)\frac{{}^{CF}D_x^{2\alpha} f(\bar{x})[\Gamma(1+2\alpha) - \Gamma(1+\alpha)^2]}{{}^{CF}D_x^{\alpha} f(\bar{x})[\Gamma(1+\alpha)^2 - \Gamma(1+2\alpha)]} e_k^{2\alpha} + O(e_k^{3\alpha}) \tag{23}$$

This completes the proof.

4. New Fractional Numerical Techniques

In this section, we introduce two-step fractional Newton-Raphson method using Caputo-Fabrizio fractional derivative which is modified version of Newton-Raphson technique.

Algorithm 1: Two-step Caputo-Fabrizio fractional Newton-Raphson method (TSFCFNRM).

For an initial approximation x_0 sufficiently close to $x_{(k+1)}$ by the following Caputo-Fabrizio type two-step scheme:

$$y_k = x_k - \Gamma(\alpha+1)\frac{f(x_k)}{{}^{C}D_x^{\alpha} f(x_k)}, \tag{24}$$

$$x_{k+1} = y_k - \Gamma(\alpha+1)\frac{f(x_k)}{{}^{CF}D_x^{\alpha} f(x_k)}, \quad k = 0,1,\ldots \tag{25}$$

Now, the foolowing result gives the convergence analysis of Algorithm 1.

Theorem 3: *Let us suppose that $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous function in a domain D containing the root $\bar{x}$ of $f(x)$. For any positive integers k and α, $\alpha \in (0,1]$, suppose that it possesses fractional derivatives of order $k\alpha$ in the Caputo-Fabrizio. Also, assume that ${}^{CF}D_x^{\alpha} f(x)$ is a continuous, nonzero fractional derivative of Caputo-Fabrizio type at $\bar{x}$. If the initial guess x_0 is sufficiently close to $\bar{x}$, then Algorithm 1 has a local convergence rate of at least 3α, and the error equation is:*

$$e_{k+1} = \frac{\left({}^{CF}D_x^{2\alpha} f(x_k)\right)^2 \left(-M(\alpha)\Gamma(\alpha+1)\cdot {}^{CF}D_x^{\alpha} f(x_k) + \Gamma(2\alpha+1)\right)}{\left({}^{CF}D_x^{\alpha} f(x_k)\right)^4 \Gamma(\alpha+1)^2 \Gamma(2\alpha+1)} e_k^{3\alpha} + O(e_k^{4\alpha}) \tag{26}$$

Proof: Let $\{x_k\}_{k\geq 0}$, obtained by (18), be a sequence of iterations produced by $\bar{x}$ of $f(x)$. Expanding $f(x)$ and its Caputo-Fabrizio derivative at x_k, around $\bar{x}$ yields as (20) and (21).

Thus, a Caputo-Fabrizio Newton iteration can be derived and represented in terms of the approximation error $e_k = x_k - \bar{x}$ at the k-th iteration as

$$\frac{f(x_k)}{{}^{CF}D_{\bar{x}}^{\alpha} f(x_k)} = \frac{\frac{{}^{CF}D_{\bar{x}}^{\alpha} f(\bar{x})}{\Gamma(1+\alpha)} e_k^{\alpha} + \frac{{}^{CF}D_{\bar{x}}^{2\alpha} f(\bar{x})}{\Gamma(2\alpha+1)} e_k^{2\alpha} + O(e_k^{3\alpha})}{{}^{CF}D_{\bar{x}}^{\alpha} f(\bar{x}) + \frac{{}^{CF}D_{\bar{x}}^{2\alpha} f(\bar{x})}{\Gamma(\alpha+1)} e_k^{\alpha} + O(e_k^{2\alpha})} \tag{27}$$

So, from (24), we have,

$$y_k = \bar{x} + e_k - \Gamma(\alpha+1)\frac{f(x_k)}{{}^{CF}D_{\bar{x}}^{\alpha} f(x_k)} =$$

$$\frac{{}^{CF}D_{\bar{x}}^{2\alpha} f(\bar{x})[\Gamma(2\alpha+1) - \Gamma(\alpha+1)]}{{}^{CF}D_{\bar{x}}^{\alpha} f(\bar{x})[\Gamma(\alpha+1)^2 - \Gamma(2\alpha+1)]} e_k^{2\alpha} + O(e_k^{3\alpha}) \tag{28}$$

From (20) and (28), we have,

$$f(y_k) = \frac{{}^{CF}D_{\bar{x}}^{2\alpha} f(\bar{x})[\Gamma(\alpha+1) - \Gamma(2\alpha+1)]}{\Gamma(\alpha+1)^2 - \Gamma(2\alpha+1)} e_k^{2\alpha} + O(e_k^{3\alpha}) \tag{29}$$

From (21) and (29), we have,

$$\frac{f(y_k)}{{}^{CF}D_{\bar{x}}^{\alpha} f(x_k)} = \frac{{}^{CF}D_{\bar{x}}^{2\alpha} f(\bar{x})[-\Gamma(\alpha+1) + \Gamma(2\alpha+1)]}{({}^{CF}D_{\bar{x}}^{\alpha} f(\bar{x}))^2 \Gamma(\alpha+1)^2 \Gamma(2\alpha+1)} e_k^{2\alpha} + O(e_k^{3\alpha}) \tag{30}$$

So, to attain a convergence rate of 3α, it is evident that the two step Caputo-Fabrizio fractional derivative must give $M(\alpha)$ as a normalization factor, and the error equation is:

$$e_{k+1} = y_k - M(\alpha)\frac{f(y_k)}{{}^{CF}D_{\bar{x}}^{\alpha} f(x_k)}, \tag{31}$$

which gives us

$$e_{k+1} = \frac{\left({}^{CF}D_{\bar{x}}^{2\alpha} f(x_k)\right)^2 \left(-M(\alpha)\Gamma(\alpha+1) \cdot {}^{CF}D_{\bar{x}}^{\alpha} f(x_k) + \Gamma(2\alpha+1)\right)}{\left({}^{CF}D_{\bar{x}}^{\alpha} f(x_k)\right)^4 \Gamma(\alpha+1)^2 \Gamma(2\alpha+1)} e_k^{3\alpha} + O(e_k^{4\alpha}) \tag{32}$$

This completes the proof.

5. Numerical example of proposed method

In this section, we provided the numerical example of the proposed method. The system that we used for numerical results is Matlab R2014b. Here, Table 1 provides the numerical outcomes of TSFNRM1 and TSFCFNRM for different values of α, x_n is the root root of function, iter is for iteration count, $|x_n - x_{n-1}|$ for absolute error and $|f(x_n)|$ is the functional value.

Table 1

Numerical outcomes of TSFNRM1 and TSFCFNRM for $f(x) = x^3 - 10$.

TSFNRM1					TSFCFNRM		
α	x_n	Iter	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	Iter	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$
0.2	2.154434689989885277	7	1.06×10^{-10}	4.18×10^{-9}	7	3.72×10^{-10}	5.62×10^{-11}
0.3	2.154434690018297478	6	5.26×10^{-10}	6.48×10^{-10}	6	3.92×10^{-11}	5.49×10^{-12}

Contd...

0.4	2.154434690030900574	5	2.43×10^{-10}	5.55×10^{-11}	5	8.39×10^{-11}	1.16×10^{-13}
0.5	2.154434690032031676	4	3.68×10^{-11}	5.54×10^{-11}	4	7.01×10^{-11}	9.76×10^{-13}
0.6	2.154434690032108962	5	4.57×10^{-11}	2.59×10^{-12}	5	5.15×10^{-12}	7.18×10^{-14}
0.7	2.154434690035180481	5	4.32×10^{-11}	6.02×10^{-12}	5	1.40×10^{-11}	1.95×10^{-13}
0.8	2.154434690034441470	5	3.71×10^{-11}	5.24×10^{-13}	5	1.28×10^{-11}	1.78×10^{-13}
0.9	2.154434690037226093	5	3.08×10^{-12}	1.95×10^{-14}	5	1.09×10^{-12}	1.52×10^{-14}

6. Conclusion

This study presents two-step fractional Newton-Raphson method using Caputo-Fabrizio fractional derivative. The convergence properties of the proposed method were established to show that this method have a higher convergence rate as compared to classical Newton- Raphson method and it have a convergence rate of at least 3α and a numerical example is provided to check the accuracy of the proposed method. This result confirms that Caputo-Fabrizio method have potential in solving nonlinear equations. Future research may explore the higher order extensions and multi dimensional system with its applications.

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