

Theoretical and numerical aspects for the second kind of generalized nonlinear Volterra-Fredholm integral equations

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Abstract

We investigate the existence and uniqueness of solutions to the second kind of generalized nonlinear Volterra-Fredholm integral equations (GNV-FIEs). These equations are analyzed using the Banach fixed point theorem and Picard's approach, particularly in situations where Picard's method might not work. We also present a new numerical method which incorporates quadrature principles, including Simpson's 3/8 and Romberg, with the modified Adomian decomposition method (MADM). This method can be extremely effective when calculating definite integrals becomes challenging. We begin our paper with

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a brief overview of the topic and then go into great detail into the suggested approach. We then illustrate the applicability of the theoretical results through numerical examples, which are presented in tables and figures for clarity.

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1. Introduction

Many physical and biological models give birth to the nonlinear integral equations (NIEs) [1, 2, 3, 4]. Estimating the approximate solutions of various NIE types is hence of tremendous interest [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Numerous academics have published a variety of articles to assess the numerical solutions of IEs over the past two decades. The Taylor expansion approach was invented and implemented by Yalcinbas [16] to solve the second kind of nonlinear Volterra-Fredholm integral equations (NV-FIEs). Abdou et al. [17, 18, 19] offered a number of strategies for resolving various NIEs kinds. Mahmoudi [20] solved the nonlinear Volterra and Fredholm IEs of the second class with the continuous Legendre wavelets. Additionally, Badr [21] solved the two-dimensional NV-FIEs using a block-by-block approach. Berenguer et al. [22] focused on approximating the solutions for the NVIEs of the second kind utilizing the Banach fixed point theorem and the characteristics of a biorthogonal system in Banach space. The Newton-Kantorovich and quadrature methods are combined in a new method devised by Saberi-Nadjafi and Heidari [23]. Mirzaee and Hoseini [24] used a mix of block pulse functions and Taylor series to solve the NV-FIEs numerically. Also, Zarebnia [25] provided a numerical method for solving the class of NV-FIEs that was based on globally determined sinc basis functions. Using multiquadric radial basis functions, Almasieh and Meleh [26] presented a numerical technique for solving the NV-FIEs in two-dimensional space. Using the Haar wavelet collocation method, Shiralashetti and Mundewadi [27] found numerical solutions of NV-FIEs. Berenguer and Gámez [28] also covered the numerical solutions of several two-dimensional IE types and the calculation of error bound. Additionally, Tofigh and Ezzati [29] introduced a novel category of orthogonal basis functions for resolving the second-kind NV-FIEs.

This article's primary goal is to investigate the solvability of second-kind GNV-FIEs. Furthermore, we devise a practical method that combines the MADM and the quadrature (Simpson $\frac{3}{8}$, Romberg) rules to approximate the solution of the GNV-FIEs of the second type. This method is employed in situations when calculating the definite integral may prove to be quite challenging.

2. Solvability for the second kind of the GNV-FIE

Consider the essence about the following the second kind of GNV-FIEs

$$\begin{aligned} \mu\phi(x, t) = f(x, t) - \Gamma_1(x, t, \phi(x, t)) + \lambda \int_0^t \int_a^b F(t, \tau) K(x, y) \\ \Gamma_2(y, \tau, \phi(y, \tau)) dy d\tau, \end{aligned} \quad (2.1)$$

where $\mu \neq 0, \lambda \neq 0$ are constants.

In the space $L_2[a, b] \times C[0, T]$, $T < 1$, where $[a, b]$ is the domain of integration with regard to the location x and the time $t \in [0, T]$, Eq. (2.1) is examined. So, assume the following conditions:

1. The positional kernel $K(x, y) \in L_2([a, b] \times [a, b])$ fulfills what follows: $|K(x, y)| \leq E_1, \forall x, y \in [a, b]$; on the other hand, the time kernel satisfies the following: $|F(t, \tau)| \leq E_2, \forall t, \tau \in [0, T]$.
2. The function $f(x, t)$ satisfies

$$\|f\|_{L_2[a, b] \times C[0, T]} = \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_a^b f^2(x, \tau) dx \right)^{\frac{1}{2}} dt \right| \leq E_3.$$

3. The functions $\Gamma_i(x, t, \phi(x, t)), i = 1, 2$ satisfy:

(a) The Lipschitz condition

$$|\Gamma_i(x, t, \phi_2(x, t)) - \Gamma_i(x, t, \phi_1(x, t))| \leq \mathcal{N}_i(x, t) |\phi_2(x, t) - \phi_1(x, t)|,$$

where

$$\|\mathcal{N}_i\| = \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_a^b \mathcal{N}_i^2(x, \tau) dx \right)^{\frac{1}{2}} d\tau \right| \leq l_i.$$

(b) The following inequality

$$\max_{0 \leq t \leq T} \left| \int_0^t \left\{ \int_a^b \Gamma_i^2(x, \tau, \phi(x, \tau)) dx \right\}^{\frac{1}{2}} d\tau \right| \leq \sigma_i \|\phi\|,$$

where $E_1 - E_3, l_1, l_2, \sigma_1$ and σ_2 are positive constants.

Theorem 2.1: Under the assumption that criteria (1)–(3.a) are satisfied, if

$$|\lambda| < \frac{|\mu| - l_1}{E_1 E_2 l_2 T}, \quad (2.2)$$

then, the equation (2.1) admits a unique solution $\phi(x, t)$ within the space $L_2[a, b] \times C[0, T]$.

Proof: The solution of Eq (2.1) is assumed to be constructible as a sequence of functions $\{\phi_n(x, t)\}_0^\infty$ utilizing Picard's method. So, we get

$$\begin{aligned} \mu\phi_m(x, t) = f(x, t) - \Gamma_1(x, t, \phi_{m-1}(x, t)) + \lambda \int_0^t \int_a^b K(x, y) F(t, \tau) \\ \Gamma_2(y, \tau, \phi_{m-1}(y, \tau)) dy d\tau, \end{aligned} \quad (2.3)$$

with $\phi_0(x, t) = f(x, t)$. Let

$$u_n(x, t) = \phi_n(x, t) - \phi_{n-1}(x, t), u_0(x, t) = f(x, t). \quad (2.4)$$

with

$$\phi_n(x, t) = \sum_{i=0}^n u_i(x, t), n = 1, 2, 3, \dots, \quad (2.5)$$

where the functions $u_i(x, t)$, $i = 0, 1, 2, \dots$, are continuous functions.

If an additional continuous solution $\tilde{\Phi}(x, t)$ exists for Eq. (2.1), applying the norm properties results in

$$|\mu| \|\Phi(x, t) - \tilde{\Phi}(x, t)\| \leq \|\tilde{\Gamma}_1(x, t)\| + |\lambda| \left\| \int_0^t \int_a^b F(t, \tau) K(x, y) \tilde{\Gamma}_2(y, \tau) dy d\tau \right\|, \quad (2.6)$$

where

$$\tilde{\Gamma}_i(x, t) = \Gamma_i(x, t, \Phi(x, t)) - \Gamma_i(x, t, \tilde{\Phi}(x, t)), i = 1, 2.$$

With the aid of the specified conditions (1)–(3.a), we obtain

$$\|\Phi(x, t) - \tilde{\Phi}(x, t)\| \leq \beta_1 \|\Phi(x, t) - \tilde{\Phi}(x, t)\|, \quad \beta_1 = \frac{l_1 + |\lambda| E_1 E_2 l_2 T}{|\mu|}. \quad (2.7)$$

When $\|\Phi(x, t) - \tilde{\Phi}(x, t)\| \neq 0$, $\beta_1 \geq 1$ is obtained, which is incompatible with (2.7). It follows that $\|\Phi(x, t) - \tilde{\Phi}(x, t)\| = 0$ and that $\Phi(x, t) = \tilde{\Phi}(x, t)$ is indicated.

Remark 1: Incase of $\Gamma_1(x, t, \Phi(x, t)) = 0$ in (2.1), Theorem 2.1 was proved with $|\lambda| < \frac{|\mu|}{E_1 E_2 l_2 T}$ El-Borai, Abdou, and El-Kojok [30].

3. Normality and continuity of the integral operators

We will now examine the Banach fixed point theorem as a source of existence and distinctive solution for the first and second kind of NV-FIE (2.1). So, the NV-FIE (2.1) can be expressed in the integral operator form

$$\bar{\Lambda}\Phi = f(x, t) + \Lambda\Phi, \bar{\Lambda}\Phi = \mu\Phi \quad (3.1)$$

where $\Lambda\Phi = -\Gamma_1(x, t, \Phi(x, t)) + \lambda \int_0^t \int_a^b K(x, y) F(t, \tau) \Gamma_2(y, \tau, \Phi(y, \tau)) dy d\tau$.

Theorem 3.1: *If conditions (1) to (3) are fulfilled, the operator $\bar{\Lambda}$ in (3.1) is normal and continuous. This ensures that the NV-FIE (2.1) admits a unique solution within the function space $L_2[a, b] \times C[0, T]$, as long as $|\lambda| < \frac{1-l_1}{E_1 E_2 l_2 T}$.*

Proof: Firstly for the normality

Employing $L_2[a, b]$ and $C[0, T]$ norm properties whenever (1), (2), and (3. b) meet the criteria, we acquire

$$\begin{aligned} \|\Lambda\Phi\| &\leq \|\Gamma_1(x, t, \Phi(x, t))\| + |\lambda| \left\| \int_0^t \int_a^b K(x, y) F(t, \tau) \Gamma_2(y, \tau, \Phi(y, \tau)) dy d\tau \right\| \\ &\leq \beta_2 \|\Phi\|, \beta_2 = \sigma_1 + |\lambda| E_1 E_2 \sigma_2 T. \end{aligned} \quad (3.2)$$

The operator Λ is contraction if $|\lambda| < \frac{1-\sigma_1}{E_1 E_2 \sigma_2 T}$. This is indicated by the result $\beta_2 < 1$. As a result, the normality of the integral operator Λ leads, by condition (2), to the normality of the operator $\bar{\Lambda}$.

Secondly for the continuity

Assuming that $\phi_1(x, t)$ and $\phi_2(x, t) \in L_2[a, b] \times C[0, T]$, we obtain the following by applying the norm properties under the constraints (1)-(3.a),

$$\begin{aligned} \|\bar{\Lambda}\phi_1 - \bar{\Lambda}\phi_2\| &= \|\Lambda\phi_1 - \Lambda\phi_2\| \\ &\leq \|\Gamma_1(x, t, \phi_1(x, t)) - \Gamma_1(x, t, \phi_2(x, t))\| \\ &\quad + |\lambda| \left\| \int_0^t \int_a^b K(x, y) F(t, \tau) \Gamma_2(y, \tau, \phi_1(y, \tau)) \right. \\ &\quad \left. - \Gamma_2(y, \tau, \phi_2(y, \tau)) dy d\tau \right\| \\ &\leq \beta_3 \|\phi_1 - \phi_2\|, \beta_3 = l_1 + |\lambda| E_1 E_2 l_2 T. \end{aligned} \quad (3)$$

Here $\beta_3 < 1$ implies that $\bar{\Lambda}$ is a contraction operator within the space $L_2[a, b] \times C[0, T]$ and that $\bar{\Lambda}$ is continuous if $|\lambda| < \frac{1-l_1}{E_1 E_2 l_2 T}$. In accordance to the Banach fixed point theorem, there is just a single fixed point for $\bar{\Lambda}$. The existence and uniqueness of the NV-FIE (2.1) solution are established if we apply the continuity and normality of the integral operator.

Remark 2: If $\Gamma_1(x, t, \phi(x, t)) = 0$ arise throughout (2.1), El-Borai, Abdou, and El-Kojok [30] employed $|\lambda| < \frac{1}{E_1 E_2 l_2 T}$ to verify Theorem 3.1.

4. Modified Adomian decomposition method for GNV-FIEs

Applications of MADM [31] have drawn a lot of attention because it is recognized as an energetic tool for addressing NIEs. As a result, the solution of Eq. (2.1) can be written as

$$\phi(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \quad (4.1)$$

additionally, the approximate solution is displayed as

$$\phi_N(x, t) = \sum_{i=0}^N u_i(x, t), N = 1, 2, \dots \quad (4.2)$$

Consequently, an infinite series of Adomian polynomials can be obtained by decomposing the nonlinear terms of Eq. (2.1) as

$$\Gamma_i(x, t, \phi(x, t)) = \sum_{j=0}^{\infty} A_{ij}(x, t), i = 1, 2, \quad (4.3)$$

where

$$A_{in}(x, t) = \frac{1}{n!} \left(\frac{d^n}{d\eta^n} \Gamma(x, t, \sum_{i=0}^{\infty} \eta^i u_i(x, t)) \right)_{\eta=0}, i = 1, 2 \quad (4.4)$$

and the free term can be expressed as

$$f(x, t) = \sum_{j=0}^{\infty} f_j(x, t). \quad (4.5)$$

Accordingly,

$$\begin{aligned} u_0(x, t) &= f_0(x, t) \\ u_n(x, t) &= f_n(x, t) - A_{1n-1}(x, t) + \lambda \int_0^t \int_a^b F(t, \tau) K(x, y) A_{2n-1}(y, \tau) dy d\tau, \\ n &= 1, 2, \dots, N. \end{aligned} \quad (4.6)$$

If obtaining the definite integral in (4.6) seems to be extremely challenging, we may employ one of the subsequent quadrature rules.

4.1 Simppson $\frac{3}{8}$ rule (SR)

Let $[a, b]$ be split into $3m$ subintervals of width $h_s = \frac{b-a}{3m}$. With the sample points equally spaced $x_k = h_s k + x_0$, $k = 0, 1, 2, \dots, 3m$, we obtain

$$\begin{aligned} \mathcal{S}_{n-1}(x, \tau) &= \int_a^b K(x, y) A_{2n-1}(y, \tau) dy \\ &= \frac{3h_s}{8} \sum_{k=1}^m K(x, y_{3k-3}) A_{2n-1}(x_{3k-3}, \tau) \\ &\quad + \frac{9h_s}{8} \sum_{k=1}^m K(x, y_{3k-2}) A_{2n-1}(x_{3k-2}, \tau) \\ &\quad + \frac{9h_s}{8} \sum_{k=1}^m K(x, y_{3k-1}) A_{2n-1}(x_{3k-1}, \tau) \\ &\quad + \frac{3h_s}{8} \sum_{k=1}^m K(x, y_{3k}) A_{2n-1}(x_{3k}, \tau). \end{aligned} \quad (4.7)$$

Hence, (4.6) becomes

$$\begin{aligned} u_0(x, t) &= f_0(x, t) \\ u_n(x, t) &= f_n(x, t) - A_{1n-1}(x, t) + \lambda \int_0^t F(t, \tau) \mathcal{S}_{n-1}(x, \tau) d\tau, \\ n &= 1, 2, \dots, N. \end{aligned} \quad (4.8)$$

4.2 Romberg rule (RR)

Consider the equally spaced sample points defined as $x_k = h_r k + x_0$, where $k = 0, 1, 2, \dots, 4m$, and partitioning the interval $[a, b]$ into $4m$ subintervals of equal length $h_r = \frac{b-a}{4m}$, it follows that

$$\begin{aligned} \mathcal{R}_{n-1}(x, \tau) &= \int_a^b K(x, y) A_{2n-1}(y, \tau) dy \\ &= \frac{14h_r}{45} (K(x, y_0) A_{2n-1}(y_0, \tau) + K(x, y_{4m}) A_{2n-1}(y_{4m}, \tau)) \\ &\quad + \frac{28h_r}{45} \sum_{k=1}^{m-1} K(x, y_{4k}) A_{2n-1}(x_{4k}, \tau) \\ &\quad + \frac{24h_r}{45} \sum_{k=1}^m K(x, y_{4k-2}) A_{2n-1}(x_{4k-2}, \tau) \\ &\quad + \frac{64h_r}{45} \sum_{k=1}^{m-1} K(x, y_{4k-1}) A_{2n-1}(x_{4k-1}, \tau) \\ &\quad + \frac{64h_r}{45} \sum_{k=1}^{m-1} K(x, y_{4k-3}) A_{2n-1}(x_{4k-3}, \tau). \end{aligned} \quad (4.9)$$

Hence, (4.6) becomes

$$\begin{aligned} u_0(x, t) &= f_0(x, t) \\ u_n(x, t) &= f_n(x, t) - A_{1n-1}(x, t) + \lambda \int_0^t F(t, \tau) \mathcal{R}_{n-1}(x, \tau) d\tau, \\ n &= 1, 2, \dots, N. \end{aligned} \quad (4.10)$$

4.3 Convergence analysis

We here offer the necessary condition for the series under consideration to converge.

Theorem 4.1: Assuming that $\Gamma_i(x, t, \phi(x, t))$ for $i = 1, 2$ satisfies the Lipschitz condition (3.a), and that condition (1) holds, the series solution (4.1) of equation (2.1) converges if and only if there exists a constant $\beta_1 = \frac{l_1 + |\lambda|E_1E_2Tl_2}{|\mu|}$ such that $\beta_1 \in (0, 1)$.

Proof: Let $\phi_N(x, t)$ is the approximate solution of Eq.(2.1), then we have

$$\begin{aligned}
 \|\phi - \phi_N\|_{L_2[a,b] \times C[0,T]} &= \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_a^b (\phi(x, \tau) - \phi_N(x, \tau))^2 dx \right)^{\frac{1}{2}} d\tau \right| \\
 &\leq \frac{1}{|\mu|} \|\Gamma_1(y, \tau, \phi(y, \tau)) - \Gamma_1(y, \tau, \phi_N(y, \tau))\| \\
 &\quad + \frac{|\lambda|}{|\mu|} \left\| \int_0^t \int_a^b |F(t, \tau)| |K(x, y)| (\Gamma_2(y, \tau, \phi(y, \tau)) \right. \\
 &\quad \left. - \Gamma_2(y, \tau, \phi_N(y, \tau))) dy d\tau \right\| \quad (4.11) \\
 &\leq \frac{1}{|\mu|} (l_1 \|\phi - \phi_N\| + |\lambda|E_1E_2Tl_2 \|\phi - \phi_N\|) \\
 &\leq \frac{1}{|\mu|} (l_1 + |\lambda|E_1E_2Tl_2) \|\phi - \phi_N\| \\
 &\leq \beta_1 \|\phi - \phi_N\|, \beta_1 = \frac{l_1 + |\lambda|E_1E_2Tl_2}{|\mu|}.
 \end{aligned}$$

In the event that $\|\phi - \phi_N\| \neq 0$, $\beta_1 > 1$, which is incompatible, occurs. Thus, as $N \rightarrow \infty$, $\|\phi - \phi_N\| = 0$, completing the proof.

5. Numerical results and discussion

In this section, the solution error behavior for the second kind of GNV-FIEs will be clarified for some applications using the methods that were previously described.

Example 1: Assess the following second kind GNV-FIE

$$\begin{aligned}
 \phi(x, t) &= \frac{t^7(4x - e^4(4x+1)+5)}{96e^4} + t^2e^{-2x} + \sqrt{t^2e^{-2x}} - \sqrt{\phi(x, t)} \\
 &\quad + \int_0^t \int_0^1 \tau \phi(x+y) \phi^2(y, \tau) dy d\tau. \quad (5.1)
 \end{aligned}$$

This IE has an exact solution of $\phi(x, t) = t^2e^{-2x}$. For $N = 4$ and $T = 0.9$, the behavior of the error and a comparison of the approaches offered at various values of x and t are displayed in Figures (1-3). The maximum error $E_{max}(t) = \max_i |\phi(x_i, t) - \phi_N(x_i, t)| \forall x_i \in [0, 1]$ and $t \in [0, T]$ is also listed in Table (1).

Table 1

$E_{max}(t)$, the maximum error, for various values of t .

t	MADM	MADM-SR	MADM-RR
0.3	1.13×10^{-14}	1.17×10^{-12}	1.17×10^{-14}
0.6	7.39×10^{-14}	1.37×10^{-11}	7.81×10^{-14}
0.9	1.78×10^{-13}	2.32×10^{-10}	2.53×10^{-13}

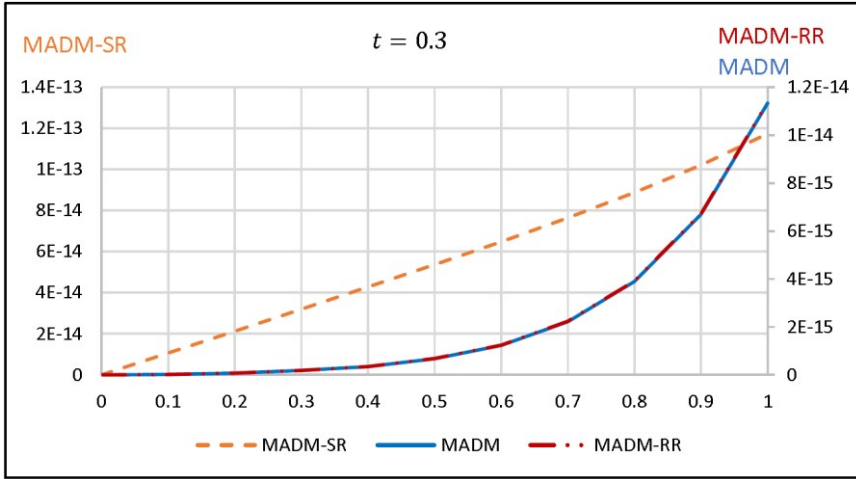


Figure 1

The errors behavior at $t = 0.3$ are indicated, as derived from the previously discussed methods.

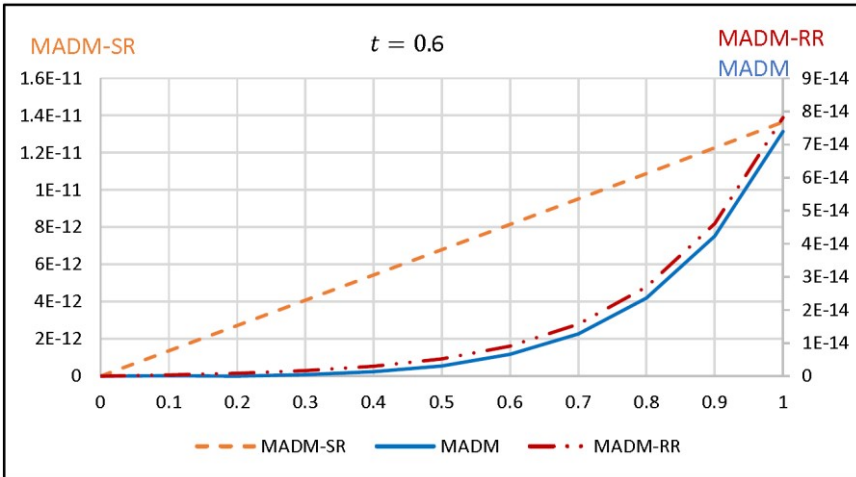
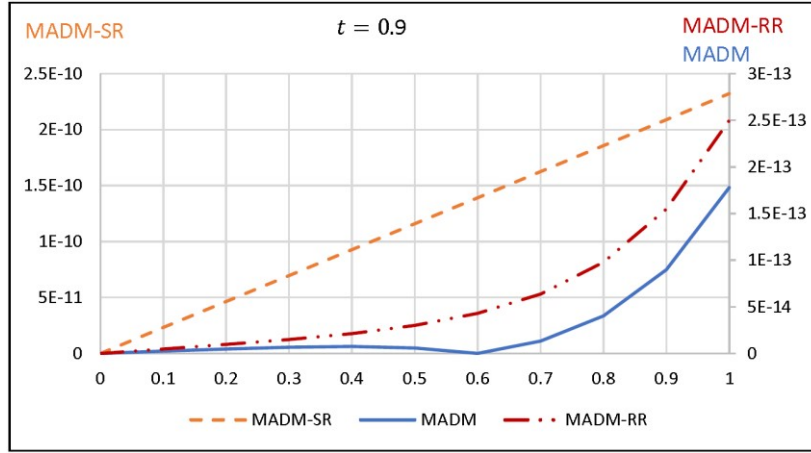


Figure 2

The errors behavior at $t = 0.6$ are indicated, as derived from the previously discussed methods.

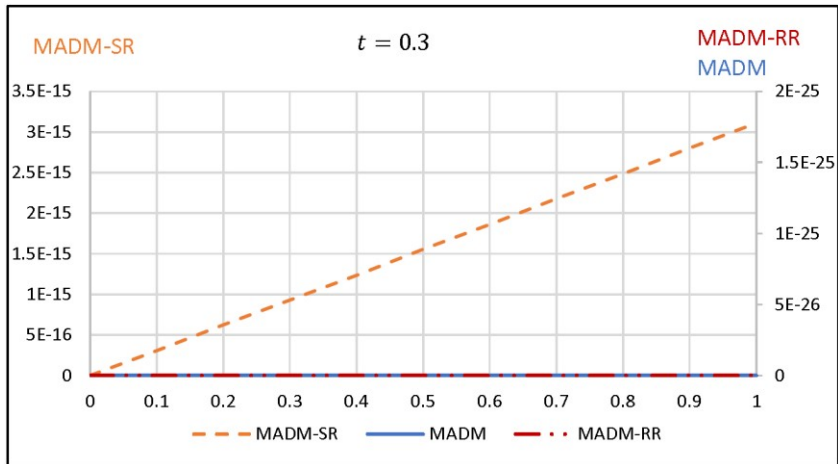
**Figure 3**

The errors behavior at $t = 0.9$ are indicated, as derived from the previously discussed methods.

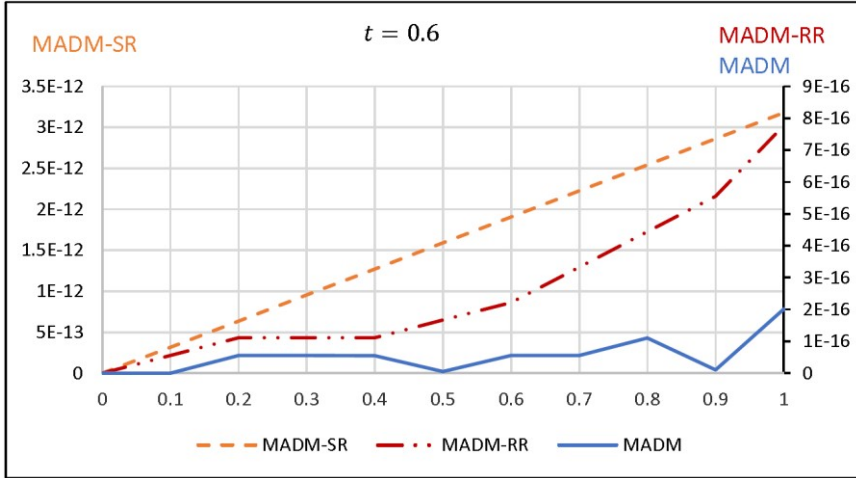
Example 2: Assess the following second kind GNV-FIE

$$\begin{aligned} \phi(x, t) = & -0.0263535t^{10}x + t^4 \cosh^2(x) + t^2 \cosh(x) - \phi^2(x, t) \\ & + 0.3 \int_0^t \int_0^1 t\tau^2 xy^2 \phi^3(y, \tau) dy d\tau. \end{aligned} \quad (5.2)$$

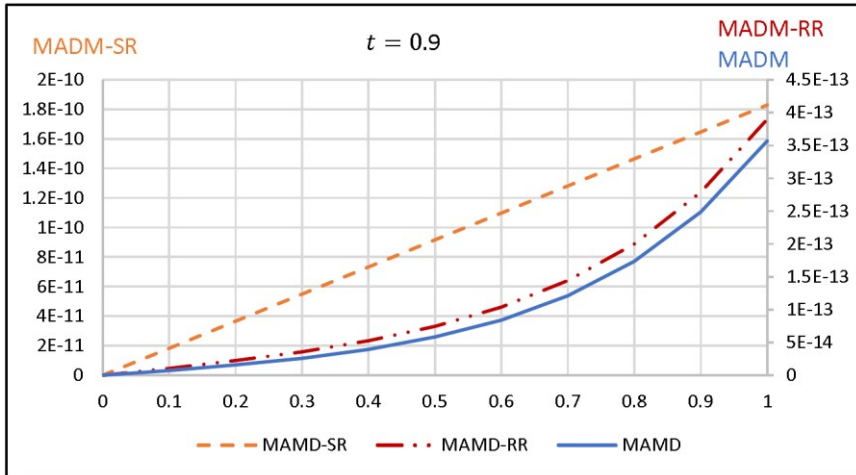
Here, $\phi(x, t) = t^2 \cosh(x)$ is the exact solution of this equation. In the case of $N = 4$ and $T = 0.9$, Figures (4-6) illustrate the error's behavior and compare the various approaches. Moreover, the maximum error $E_{max}(t)$ for some $t \in [0, T]$ is listed in Table (2).

**Figure 4**

The errors behavior at $t = 0.3$ are indicated, as derived from the previously discussed methods.

**Figure 5**

The errors behavior at $t = 0.6$ are indicated, as derived from the previously discussed methods.

**Figure 6**

The errors behavior at $t = 0.9$ are indicated, as derived from the previously discussed methods.

Table 2

$E_{max}(t)$, the maximum error, for various values of t .

t	MADM	MADM-SR	MADM-RR
0.3	0	3.11×10^{-15}	0
0.6	7.77×10^{-16}	3.18×10^{-12}	2.01×10^{-16}
0.9	3.9×10^{-13}	1.83×10^{-10}	3.57×10^{-13}

Example 3: Assess the following second kind GNV-FIE

$$\begin{aligned} \phi(x, t) = & 0.000643768t^7x^2 + t^3\log^3\left(\frac{2-x}{x+1}\right) + t\log\left(\frac{2-x}{x+1}\right) - \phi^3(x, t) \\ & + 0.1 \int_0^t \int_0^1 t^2\tau x^2 y \phi^3(y, \tau) dy d\tau. \end{aligned} \quad (5.3)$$

This IE has the accurate solution $\phi(x, t) = t\log\left(\frac{2-x}{1+x}\right)$. For $N = 4$ and $T = 0.3$, Figures (7-9) examine the behavior of the error and compare the various approaches that are given for various values of x and t . Furthermore, for some $t \in [0, T]$, Table (3) displays the greatest error $E_{max}(t)$.

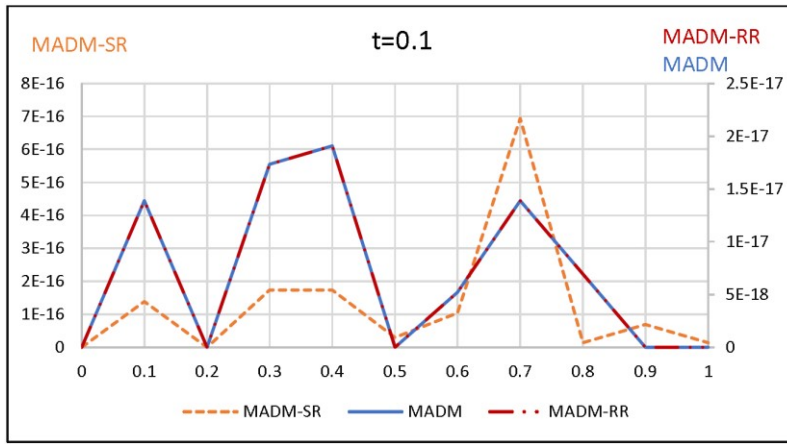


Figure 7

The errors behavior at $t = 0.1$ are indicated, as derived from the previously discussed methods.

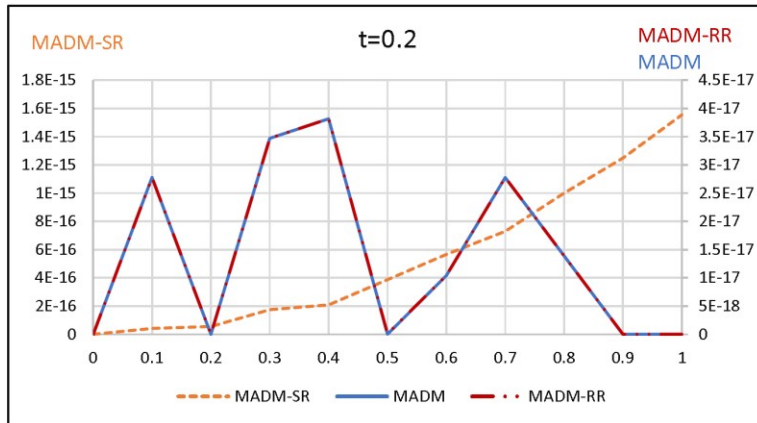
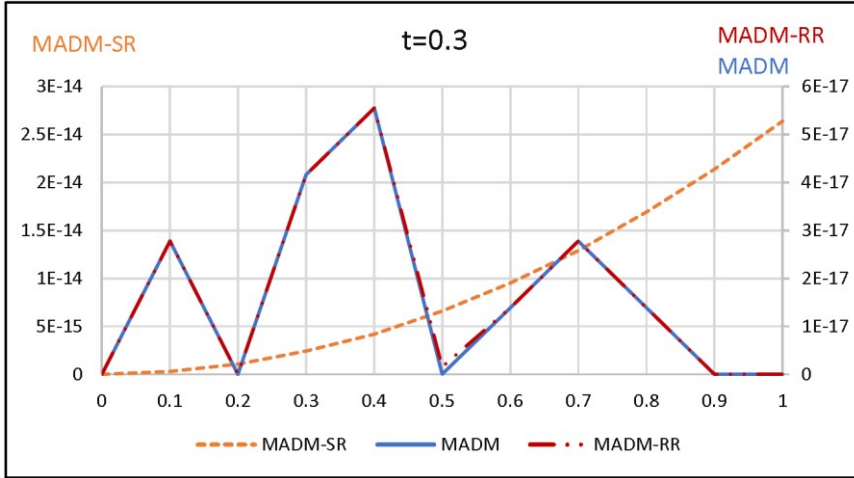


Figure 8

The errors behavior at $t = 0.2$ are indicated, as derived from the previously discussed methods.

**Figure 9**

The errors behavior at $t = 0.3$ are indicated, as derived from the previously discussed methods.

Table 3

$E_{max}(t)$, the maximum error, for various values of t .

t	MADM	MADM-SR	MADM-RR
0.1	1.91×10^{-17}	6.01×10^{-16}	1.91×10^{-17}
0.2	3.82×10^{-17}	1.55×10^{-15}	3.82×10^{-17}
0.3	5.55×10^{-17}	2.64×10^{-14}	5.55×10^{-17}

Example 4 Assess the following second kind GNV-FIE

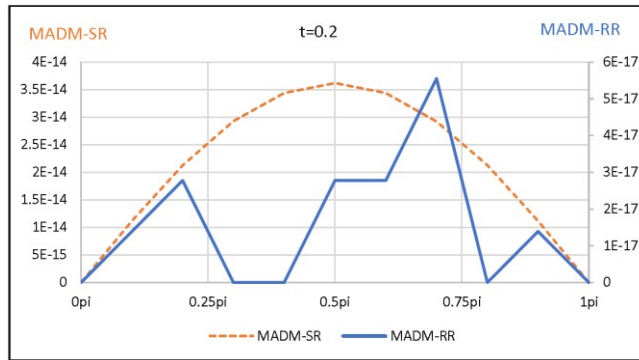
$$\begin{aligned} \phi(x, t) = & 0.0533333t^7 \sin(x) + t^2 \sin^2(x) + t \sin(x) - \phi^2(x, t) \\ & + 0.2 \int_0^t \int_0^\pi t^2 \tau^2 \cos(x+y) \phi^2(y, \tau) dy d\tau. \end{aligned} \quad (5.4)$$

This IE has an exact solution of $\phi(x, t) = t \sin(x)$. Since it was difficult to compute the finite integral in this example, we simply employed the MADM-SR and MADM-RR to examine how the error behaved at various values of x and t for $N = 3$ and $T = 0.6$, as shown in Figures (10-12). Additionally, for some $t \in [0, T]$, Table (4) examines the maximum value of the error $E_{max}(t)$ in the obtained results.

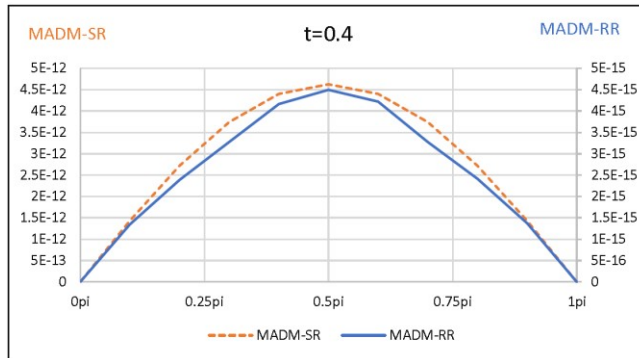
Table 4

$E_{max}(t)$, the maximum error, for various values of t .

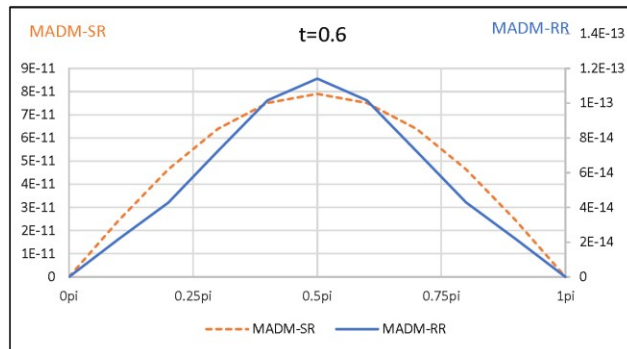
t	MADM-SR	MADM-RR
0.2	3.62×10^{-14}	5.55×10^{-17}
0.4	4.63×10^{-12}	4.51×10^{-15}
0.6	7.91×10^{-11}	1.14×10^{-13}

**Figure 10**

The errors behavior at $t = 0.2$ are indicated, as derived from the previously discussed methods.

**Figure 11**

The errors behavior at $t = 0.4$ are indicated, as derived from the previously discussed methods.

**Figure 12**

The errors behavior at $t = 0.6$ are indicated, as derived from the previously discussed methods.

6. Conclusion

The uniqueness and existence of the solution to Eq. (2.1) in the space $L_2[a, b] \times C[0, T]$ has been addressed in this study using Picard's technique. Additionally, in order to demonstrate the existence and uniqueness of the solution, we looked into the normality and continuity of GNV-FIE (2.1) using the Banach fixed point theorem. Furthermore, a modified Adomian decomposition approach based on the quadrature (Simpson $\frac{3}{8}$, Romberg) fundamentals has been developed to estimate the solution of the GNV-FIEs. This method is employed when the definite integral becomes extremely challenge to calculate. To demonstrate the accuracy and validity of the suggested ways displaying the error behavior for each instance, illustrative examples were provided. We can infer from the results that MADM-Romberg rule is more accurate than MADM-Simpson $\frac{3}{8}$ rule due to the precision of MADM with the quadrature formulae. Furthermore, compared to other rules, it is clear that the MADM-Romberg rule provides the same relative accuracy as MADM and is a superior way for obtaining the approximation solutions of GNV-FIEs.

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