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Solvability and stability of nonlinear differential problems with two fractional Caputo derivatives

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Abstract

In the current paper, we study a nonlinear fractional differential problems with two arbitrary orders. The existence and uniqueness of solutions for the considered problem by applying a number of fixed point theorems. The Ulam-Hyers stability of the proposed fractional problem is also discussed. Finally, some illustrative examples are treated.

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1. Fractional calculus

In differential equations, the fractional calculus have shown its wings in the modeling of many problems in science, technology and engineering. Several works on fractional calculus and fractional differential equations have appeared recently, see [1, 5, 7, 9, 10, 22, 23]. In quite recent times, there has been a very important progress in the study of the theory of differential problems involving fractional derivatives, see [2, 6, 8, 11, 12, 14, 17, 25]. Recently, many scholars, have studied the Ulam-Hyers stability of solutions for some differential equations with fractional calculus. See the research works [3, 4, 9, 13, 15, 16, 20, 21, 24]. So, in this manuscript, we study the propoesd fractional problem:

$$\begin{cases} D^{\beta}(D^{\alpha}+\lambda)z(\tau)-\theta f\big(\tau,z(t)\big)=\sum_{i=1}^{k}K_i I^{\delta_i}h_i\big(\tau,z(\tau)\big),\tau\in\Theta, \\ z(0)=\int_0^T \varphi(s)z(s)ds, D^{p}z(T)=AD^{p}z(\eta),0<\eta<T, \end{cases} \tag{1}$$

where $\Theta=[0,T], D^{q}$, $q=\alpha,\beta,p$ denote the Caputo fractional derivative, with $0<\alpha,\beta\leq 1,\alpha+\beta\in\,]1,2]$, $p<\alpha+\beta$ and $0<\delta_i\leq 1$, K_i,θ,λ, are real constant, f, $h_i\colon\Lambda\times\mathbb{R}^2\to\mathbb{R}, i\in\{1,\dots,k\}$ are continuous functions on Λ. The operator I^{ϑ} is defined as

$$I^{\vartheta}[f(t)]=\frac{1}{\Gamma(\vartheta)}\int_0^{\tau}(\tau-s)^{\vartheta-1}f(s)ds,\alpha>0,\tau\geq 0,$$

where $\Gamma(\vartheta):=\int_0^{\infty}e^{-u}u^{\vartheta-1}du$. The operator D^{q} is defined as

$$D^{q}[y(t)]=\frac{1}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-s)^{n-q-1}y^{(n)}(s)ds,a\leq\tau\leq b.$$

We need also the following auxiliary lemmas [18, 19]:

Lemma 1: *For* $n-1<q<n,n\in N^{*}$, *the general solution of* $D^{q}[z(\tau)]=0$ *is*

$$z(\tau)=e_0+e_1\tau+e_2\tau^2+\dots+e_{n-1}\tau^{n-1},$$

where $e_i\in\mathbb{R}$.

Lemma 2: *Suppose that* $n\in N^{*}$, *and* $n-1<q<n$. *Then, we have*

$$I^{q}\big[D^{q}[z(\tau)]\big]=z(\tau)+e_0+e_1\tau+e_2\tau^2+\dots+e_{n-1}\tau^{n-1},$$

for some $c_i\in\mathbb{R}$.

In order to define the solution of (1), we need the following auxiliary result:

Lemma 3: *Suppose that* $g\in C(\Lambda,\mathbb{R})$, *then, the problem*

$$D^{\beta}(D^{\alpha}+\lambda)z(\tau)=g(\tau),\tau\in\Lambda,<\beta,\alpha\in\,]1,2], \tag{2}$$

subject

$$z(0)=\int_0^T\varphi(s)z(s)ds,D^{p}z(T)=AD^{p}z(\eta),0<\eta<T,$$

admits the following solution

$$\begin{aligned} z(\tau) \;=\; & \int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s)ds - \lambda \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ & -\Lambda\tau^\alpha \int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} g(s)ds + \lambda\Lambda\tau^\alpha \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} x(s)ds \\ & +\Lambda A\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} g(s)ds - \lambda\Lambda A\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} x(s)ds \\ & + \int_0^T \varphi(s)x(s)ds, \end{aligned} \tag{3}$$

where

$$\Lambda := \frac{\Gamma(\alpha-p+1)}{\Gamma(\alpha+1)(T^{\alpha-p}-A\eta^{\alpha-p})}, T^{\alpha-p} \neq A\eta^{\alpha-p}.$$

Proof: Let z be a solution for the problem (1), by using lemma 3, we can write

$$z(\tau) = I^{\alpha+\beta}[g(\tau)] - \lambda I^{\alpha}[z(\tau)] - \frac{c_0}{\Gamma(\alpha+1)}\tau^\alpha - c_1, \tag{4}$$

where e_0 and e_1 are arbitrary constants. Since $z(0) = \int_0^T \varphi(s)z(s)ds$, we obtain $e_1 = -\int_0^T \varphi(s)z(s)ds$.

Thanks to lemma 2, we get

$$D^p z(\tau) = I^{\alpha+\beta-p}[g(\tau)] - \lambda I^{\alpha-p}[z(\tau)] - \frac{c_0}{\Gamma(\alpha-p+1)}\tau^{\alpha-p}.$$

By $D^p z(T) = AD^p z(\eta)$, we can write

$$c_0 = \frac{\Gamma(\alpha-p+1)}{(T^{\alpha-p}-A\eta^{\alpha-p})}(I^{\alpha+\beta-p}[g(T)] - \lambda I^{\alpha-p}[z(T)] - AI^{\alpha+\beta-p}[g(\eta)] + \lambda AI^{\alpha-p}[z(\eta)]).$$

Substituting the value of c_0 and c_1 in (4), we obtain the solution (3).

2. Main Results

Now, we show the existence and stability results for problem (1), to reach that we place the problem in the space $Z = C(\Theta, \mathbb{R})$, and we define a supremum norm $\|z\|$ such that $\|z\| = \sup_{\tau\in\Theta}|z(\tau)|$.

Now, we define $\phi: Z \to Z$ by:

$$\begin{aligned} \phi z(\tau) := & \theta \int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(s, x(x)\big)ds + \sum_{i=1}^k K_i \int_0^\tau \frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)} h_i\big(s, z(x)\big)ds \\ & -\lambda \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds - \Lambda\theta\tau^\alpha \int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} f\big(s, z(s)\big)ds \\ & -\Lambda\tau^\alpha \sum_{i=1}^k K_i \int_0^T \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} h_i\big(s, z(s)\big)ds \\ & +\lambda\Lambda\tau^\alpha \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} z(s)ds + \Lambda A\theta\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} f\big(s, z(s)\big)ds \\ & +\Lambda A\tau^\alpha \sum_{i=1}^k K_i \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} h_i\big(s, z(s)\big)ds \\ & -\lambda\Lambda A\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} z(s)ds + \int_0^T \varphi(s)z(s)ds. \end{aligned} \tag{5}$$

We give the notations:

$$\begin{aligned}\Delta_1 :=& \frac{|\theta|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda\theta|\left(T^{2\alpha+\beta-p}+|A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)} \\ &+ \sum_{i=1}^{k}|K_i|\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)} + \sum_{i=1}^{k}|K_i|\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)},\end{aligned} \tag{6}$$

and

$$\Delta_2 := \frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda\Lambda|\left(T^{2\alpha-p+1}+|A|T^{\alpha}\eta^{\alpha-p}\right)}{\Gamma(\alpha-p+1)} + MT, \tag{7}$$

where

$$M = \sup_{\tau\in\Theta}|\varphi(\tau)|.$$

2.1 *Existence and uniqueness results*

Now, we prove the uniqueness results for the studied problem:

Theorem 4: *If* $f,\ h_i\colon \Theta\times\mathbb{R}\to\mathbb{R},\ i\in\{1,\dots,k\}$ *are continuous functions. Also, let*

(H_1) $\exists\ \omega_i, i\in\{1,\dots,k+1\}$ so that for any $\tau\in\Theta$ and all $z,w\in\mathbb{R}$,

$$|f(\tau,z)-f(\tau,w)|\le\omega_1|z-w|, |h_i(\tau,z)-h_i(\tau,w)|\le\omega_{i+1}|z-w|, \quad \text{for } i=1,\dots,k.$$

Then (1) has a unique solution provided by $\omega\Delta_1+\Delta_2<1$, *where* $\omega = max\{\omega_i\colon i=1,\dots,k+1\}$, Δ_1 *and* Δ_2 *are defined by (6) and (7), respectively.*

Proof: Let $N=max\{N_i\colon i=1,\dots,k+1\}$, where N_i are finite numbers such that $N_1=\sup_{t\in[0,T]}|f(\tau,0)|$, $N_{i+1}=\sup_{\tau\in\Theta}|h_i(\tau,0)|$. Setting $r\ge\frac{N\Delta_1}{1-(\Delta_1\omega+\Delta_2)}$, we show that $\phi B_r\subset B_r$, where $B_r=\{z\in Z\colon\|z\|\le r\}$.

For $z\in B_r$ and for all $\tau\in\Theta$, from the hypothesis (H_1), we obtain:

$$\begin{aligned}&|\phi z(\tau)| \\ &\le \sup_{\tau\in\Theta}\Big\{|\theta|\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big|f\big(s,z(s)\big)\big|ds + \sum_{i=1}^{k}|K_i|\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)}\big|h_i\big(s,z(s)\big)\big|ds \\ &+|\lambda|\int_0^{\tau}\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}|z(s)|ds + |\Lambda\theta|t^{\alpha}\int_0^{T}\frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\big|f\big(s,z(s)\big)\big|ds \\ &+|\Lambda|\tau^{\alpha}\sum_{i=1}^{k}|K_i|\int_0^{T}\frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\big|h_i\big(s,z(s)\big)\big|ds \\ &+|\lambda\Lambda|\tau^{\alpha}\int_0^{T}\frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds + |\Lambda A\theta|t^{\alpha}\int_0^{\eta}\frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\big|f\big(s,z(s)\big)\big|ds \\ &+|\Lambda A|\tau^{\alpha}\sum_{i=1}^{k}|K_i|\int_0^{\eta}\frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\big|h_i\big(s,z(s)\big)\big|ds \\ &+|\lambda\Lambda A|\tau^{\alpha}\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds + \int_0^{T}|\varphi(s)z(s)|ds\Big\}\end{aligned}$$

$$\leq \sup_{\tau\in\Theta}\Big\{|\theta|\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big(\Big|f\big(s,z(s)\big)-|f(s,0)|\Big|+|f(s,0)|\Big)ds$$
$$+\sum_{i=1}^k |K_i|\int_0^\tau \frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)}\Big(\Big|h_i\big(s,z(s)\big)-|h_i(s,0)|\Big|+|h_i(s,0)|\Big)ds$$
$$+|\lambda|\int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}|z(s)|ds$$
$$+|\Lambda\theta|\tau^\alpha\int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\Big(\Big|f\big(s,z(s)\big)-|f(s,0)|\Big|+|f(s,0)|\Big)ds$$
$$+|\Lambda|\tau^\alpha\sum_{i=1}^k |K_i|\int_0^T \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\Big(\Big|h_i\big(s,z(s)\big)-|h_i(s,0)|\Big|+|h_i(s,0)|\Big)ds$$
$$+|\lambda\Lambda|\tau^\alpha\int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds$$
$$+|\Lambda A\theta|\tau^\alpha\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\Big(\Big|f\big(s,z(s)\big)-|f(s,0)|\Big|+|f(s,0)|\Big)ds$$
$$+|\Lambda A|\tau^\alpha\sum_{i=1}^k |K_i|\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\Big(\Big|h_i\big(s,z(s)\big)-|h_i(s,0)|\Big|+|h_i(s,0)|\Big)ds$$
$$+|\lambda\Lambda A|\tau^\alpha\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds+\int_0^T |\varphi(s)z(s)|ds\Big\}.$$
$$\leq \left[\frac{|\theta|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\Lambda\theta|\big(T^{2\alpha+\beta-p}+|A|T^\alpha\eta^{\alpha+\beta-p}\big)}{\Gamma(\alpha+\beta-p+1)}\right](\omega_1 r+N_1)$$
$$+\left[\sum_{i=1}^k |K_i|\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}+\sum_{i=1}^k |K_i|\frac{|\Lambda|\Big(T^{2\alpha+\beta-p+\delta_i}+|A|T^\alpha\eta^{\alpha+\beta-p+\delta_i}\Big)}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right]$$
$$(\omega_{i+1}r+N_{i+1})$$
$$+\left[\frac{|\lambda|T^\alpha}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|\big(T^{2\alpha-p+1}+|A|T^\alpha\eta^{\alpha-p}\big)}{\Gamma(\alpha-p+1)}+MT\right]r$$
$$=\Delta_1(\omega r+N)+\Delta_2 r\leq r,$$

so $\phi B_r\subset B_r$. Also, for $z,w\in B_r$ and $\tau\in\Theta$, we obtain:

$$|\phi z(\tau)-\phi w(\tau)|$$
$$\leq \sup_{\tau\in\Theta}\Big\{|\theta|\int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big|f\big(s,z(s)\big)-f\big(s,w(s)\big)\Big|ds$$
$$+\sum_{i=1}^k |K_i|\int_0^\tau \frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)}\Big|h_i\big(s,z(s)\big)-h_i\big(s,w(s)\big)\Big|ds$$
$$+|\lambda|\int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}|z(s)-w(s)|ds$$
$$+|\Lambda\theta|\tau^\alpha\int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\Big|f\big(s,z(s)\big)-f\big(s,w(s)\big)\Big|ds$$
$$+|\Lambda|\tau^\alpha\sum_{i=1}^k |K_i|\int_0^T \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\Big|h_i\big(s,z(s)\big)-h_i\big(s,w(s)\big)\Big|ds$$
$$+|\lambda\Lambda|\tau^\alpha\int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)-w(s)|ds$$
$$+|\Lambda A\theta|\tau^\alpha\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\Big|f\big(s,z(s)\big)-f\big(s,w(s)\big)\Big|ds$$

$$+|\Lambda A|\tau^{\alpha}\sum_{i=1}^{k} K_i \int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} \left|h_i\big(s,z(s)\big)-h_i\big(s,w(s)\big)\right| ds$$

$$+|\lambda\Lambda A|\tau^{\alpha}\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)-w(s)|ds+\int_0^{T}|\varphi(s)||z(s)-w(s)|ds\Big\}$$

$$\leq \left[\frac{|\theta|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\Lambda\theta|\left(T^{2\alpha+\beta-p}+|A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}\right]\omega_1\|z-w\|$$

$$+\left[\sum_{i=1}^{k}|K_i|\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}+|\Lambda|\sum_{i=1}^{k}|K_i|\frac{\left(T^{2\alpha+\beta-p+\delta_{i+}}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right]\omega_{i+1}\|z-w\|$$

$$+\left[\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|\left(T^{2\alpha-p+1}+|A|T^{\alpha}\eta^{\alpha-p}\right)}{\Gamma(\alpha-p+1)}+MT\right]\|z-w\|$$

$$=(\omega\Delta_1+\Delta_2)\|z-w\|,$$

which leads to $\|\phi z-\phi w\|\leq(\omega\Delta_1+\Delta_2)\|z-w\|$. Since $\omega\Delta_1+\Delta_2<1$, ϕ is a contraction mapping.

Theorem 5: *Suppose that the continuous functions* f, $h_i:\Theta\times\mathbb{R}\to\mathbb{R}$, $i=1,\dots,k$. *Assume that:*

(H_2) $|f(\tau,z)-f(\tau,w)|\leq a(\tau)|z-w|$, $|h_i(\tau,z)-h_i(\tau,w)|\leq b_i(\tau)|z-w|$, for each $t\in\Theta$, $z,w\in\mathbb{R}$, where $a,b_i\in L^{\frac{1}{\sigma}}(\Theta,\mathbb{R}^+)$, $i=1,\dots,k$, and $\sigma\in(0,1)$.

Denote $\|\vartheta\|=\left(\int_0^{T}|\vartheta(s)|^{\frac{1}{\sigma}}ds\right)^{\sigma}$, $\vartheta=a,b_i,i=1,\dots,k$.

If

$$|\theta|\|a\|\Pi_1+\sum_{i=1}^{k}\|b_i\||K_i|\Pi_{i+1}+\Delta_2<1, \tag{8}$$

where

$$\Pi_1:=\frac{T^{\alpha+\beta-\sigma}}{\Gamma(\alpha+\beta)}\left(\frac{1-\sigma}{\alpha+\beta-\sigma}\right)^{1-\sigma}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p-\sigma}+|A|T^{\alpha}\eta^{\alpha+\beta-p-\sigma}\right)}{\Gamma(\alpha+\beta-p)}\left(\frac{1-\sigma}{\alpha+\beta-p-\sigma}\right)^{1-\sigma}, \tag{9}$$

$$\Pi_{i+1}:=\frac{T^{\alpha+\beta+\delta_i-\sigma}}{\Gamma(\alpha+\beta+\delta_i)}\left(\frac{1-\sigma}{\alpha+\beta+\delta_i-\sigma}\right)^{1-\sigma}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i-\sigma}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i-\sigma}\right)}{\Gamma(\alpha+\beta-p+\delta_i)}\left(\frac{1-\sigma}{\alpha+\beta-p+\delta_i-\sigma}\right)^{1-\sigma},\ i=1,\dots,k,$$

and Δ_2 *is given by (7). Then the problem (1) admits a unique solution.*

Proof: For $z,w\in Z$ and $t\in\Theta$, by Hölder inequality and using (H_2), we have:

$$|\phi z(\tau)-\phi w(\tau)|$$

$$\leq\sup_{\tau\in\Theta}\Big\{|\theta|\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}a(s)|z(s)-w(s)|ds$$

$$+\sum_{i=1}^{k}|K_i|\int_0^{\tau}\frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)}b_i(s)|z(s)-w(s)|ds$$

$$+|\lambda|\int_0^{\tau}\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}|z(s)-w(s)|ds$$

$$+|\Lambda\theta|\tau^{\alpha}\int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}a(s)|z(s)-w(s)|ds$$
$$+|\Lambda|\tau^{\alpha}\sum_{i=1}^{k}|K_i|\int_0^T \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}b_i(s)|z(s)-w(s)|ds$$
$$+|\lambda\Lambda|\tau^{\alpha}\int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)-w(s)|ds$$
$$+|\Lambda A\theta|\tau^{\alpha}\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}a(s)|z(s)-w(s)|ds$$
$$+|\Lambda A|\tau^{\alpha}\sum_{i=1}^{k}|K_i|\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}b_i(s)|z(s)-w(s)|ds$$
$$+|\lambda\Lambda A|\tau^{\alpha}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)-w(s)|ds+\int_0^T|\varphi(s)||z(s)-w(s)|ds\Big\}.$$
$$\leq \sup_{\tau\in\Theta}\left\{\left[\frac{|\theta|}{\Gamma(\alpha+\beta)}\int_0^\tau\left((\tau-s)^{\frac{\alpha+\beta-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 a(s)^{\frac{1}{\sigma}}ds\right)^{\sigma}\right.\right.$$
$$+\sum_{i=1}^{k}\frac{|K_i|}{\Gamma(\alpha+\beta+\delta_i)}\int_0^\tau\left((\tau-s)^{\frac{\alpha+\beta+\delta_i-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 b_i(s)^{\frac{1}{\sigma}}ds\right)^{\sigma}$$
$$+\frac{|\Lambda\theta|\tau^{\alpha}}{\Gamma(\alpha+\beta-p)}\int_0^T\left((T-s)^{\frac{\alpha+\beta-p-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 a(s)^{\frac{1}{\sigma}}ds\right)^{\sigma}$$
$$+|\Lambda|\tau^{\alpha}\sum_{i=1}^{k}\frac{|K_i|}{\Gamma(\alpha+\beta-p+\delta_i)}\int_0^T\left((T-s)^{\frac{\alpha+\beta-p+\delta_i-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 b_i(s)^{\frac{1}{\sigma}}ds\right)^{\sigma}$$
$$+\frac{|\Lambda A\theta|\tau^{\alpha}}{\Gamma(\alpha+\beta-p)}\int_0^\eta\left((\eta-s)^{\frac{\alpha+\beta-p-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 a(s)^{\frac{1}{\sigma}}\right)^{\sigma}ds$$
$$+|\Lambda A|\tau^{\alpha}\sum_{i=1}^{k}\frac{|K_i|}{\Gamma(\alpha+\beta-p+\delta_i)}\int_0^\eta\left((\eta-s)^{\frac{\alpha+\beta-p+\delta_i-1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^1 b_i(s)^{\frac{1}{\sigma}}ds\right)^{\sigma}$$
$$\left.\left.+\frac{|\lambda|\tau^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|\tau^{2\alpha-p-1}}{\Gamma(\alpha-p+1)}+\frac{|\lambda\Lambda A|\tau^{\alpha}\eta^{\alpha-p}}{\Gamma(\alpha-p+1)}+MT\right]\right\}\|z-w\|$$
$$\leq\left(|\theta|\|a\|\left[\frac{T^{\alpha+\beta-\sigma}}{\Gamma(\alpha+\beta)}\left(\frac{1-\sigma}{\alpha+\beta-\sigma}\right)^{1-\sigma}\right.\right.$$
$$\left.+\frac{|\Lambda|(T^{2\alpha+\beta-p-\sigma}+|A|T^{\alpha}\eta^{\alpha+\beta-p-\sigma})}{\Gamma(\alpha+\beta-p)}\left(\frac{1-\sigma}{\alpha+\beta-p-\sigma}\right)^{1-\sigma}\right]$$
$$+\sum_{i=1}^{k}\|b_i\||K_i|\left[\frac{T^{\alpha+\beta+\delta_i-\sigma}}{\Gamma(\alpha+\beta+\delta_i)}\left(\frac{1-\sigma}{\alpha+\beta+\delta_i-\sigma}\right)^{1-\sigma}\right.$$
$$\left.+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i-\sigma}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i-\sigma}\right)}{\Gamma(\alpha+\beta-p+\delta_i)}\left(\frac{1-\sigma}{\alpha+\beta-p+\delta_i-\sigma}\right)^{1-\sigma}\right]$$
$$\left.+\left[\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|(T^{2\alpha-p-1}+|A|T^{\alpha}\eta^{\alpha-p})}{\Gamma(\alpha-p+1)}+MT\right]\right)\|z-w\|$$
$$=\left(|\theta|\|a\|\Pi_1+\sum_{i=1}^{k}\|b_i\||K_i|\Pi_{i+1}+\Delta_2\right)\|z-w\|.$$

Therefore,

$$\|\phi z-\phi w\|\leq\left(\|a\|\Pi_1+\sum_{i=1}^{k}\|b_i\||K_i|\Pi_{i+1}+\Delta_2\right)\|z-w\|.$$

Hence, by the Banach fixed point theorem.ϕ admits a unique fixed point which is a solution of the problem (1).

2.2 *Existence results*

Now, we show the existence result [25].

Theorem 6: *Let* $f,\ h_i: \Theta \times \mathbb{R} \to \mathbb{R},\ i \in \{1,\dots,k\}$, *are continuous functions. Assume that:*

(H_3) Let $\psi, \psi_i: [0,\infty) \to [0,\infty)$ *and* $q, q_i \in C(\Theta, \mathbb{R}^+),\quad i \in \{1,\dots,k\}$ *such that*

$$|f(\tau,z)| \le q(t)\psi(\|z\|), |h_i(\tau,z)| \le q_i(\tau)\psi_i(\|z\|), for\ all\ (\tau,z) \in \Theta \times \mathbb{R}.$$

(H_4) If $L > 0$, *then*

$$\frac{L}{|\theta|\|q\|\psi(L)\nabla_1 + \sum_{i=1}^{k} |K_i|\|q_i\|\psi_i(L)\nabla_{i+1} + \Delta_2 L} > 1,$$

where

$$\begin{aligned} \nabla_1 &:= \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p} + |A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}, \\ \nabla_{i+1} &:= \frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i} + |A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)}, i = 1,\dots,k, \end{aligned} \tag{0}$$

and Δ_2 *is given by (7). Then (1) admits one solution at least defined on* Θ.

Proof: Firstly, we will prove that ϕ is bounded. Let $B_r = \{z \in Z: \|z\| \le r, r > 0\}$ be bounded. So, for $\tau \in \Theta$, we have

$$\begin{aligned} &|\phi z(\tau)| \\ &\le |\theta| \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q(s)\psi(\|z\|)ds + \sum_{i=1}^{k} K_i \int_0^{\tau} \frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)} q_i(t)\psi_i(\|z\|)ds \\ &+|\lambda| \int_0^{\tau} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |z(s)|ds + |\Lambda\theta|\tau^{\alpha} \int_0^{T} \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} q(s)\psi(\|z\|)ds \\ &+|\Lambda|\tau^{\alpha} \sum_{i=1}^{k} |K_i| \int_0^{T} \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} q_i(t)\psi_i(\|z\|)ds \\ &+|\lambda\Lambda|\tau^{\alpha} \int_0^{T} \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |z(s)|ds + |\Lambda A\theta|\tau^{\alpha} \int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} q(s)\psi(\|z\|)ds \\ &+|\Lambda A|\tau^{\alpha} \sum_{i=1}^{k} |K_i| \int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} q_i(t)\psi_i(\|z\|)ds \\ &+|\lambda\Lambda A|\tau^{\alpha} \int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |z(s)|ds + \int_0^{T} |\varphi(s)||z(s)|ds. \\ &\le |\theta|\|q\|\psi(r)\left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda|T^{2\alpha+\beta-p}}{\Gamma(\alpha+\beta-p+1)} + \frac{|\Lambda A|T^{\alpha}\eta^{\alpha+\beta-p}}{\Gamma(\alpha+\beta-p+1)}\right] \\ &+\sum_{i=1}^{k} |K_i|\|q_i\|\psi_i(r)\left[\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)} + \frac{|\Lambda|T^{2\alpha+\beta-p+\delta_i}}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right. \\ &\left.+\frac{|\Lambda A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right] + \left[\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda\Lambda|T^{2\alpha-p}}{\Gamma(\alpha-p+1)} + \frac{|\lambda A\Lambda|T^{\alpha}\eta^{\alpha-p}}{\Gamma(\alpha-p+1)} + TM\right] r. \end{aligned}$$

Consequently,

$$\begin{aligned} &\|\phi z\| \\ &\le |\theta|\|q\|\psi(r)\left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p} + |A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}\right] \end{aligned}$$

$$+\sum_{i=1}^{k}|K_i|\|q_i\|\psi_i(r)\left[\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right]$$
$$+\left[\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|(T^{2\alpha-p}+|A|T^{\alpha}\eta^{\alpha-p})}{\Gamma(\alpha-p+1)}+TM\right]r$$
$$=|\theta|\|q\|\psi(r)\nabla_1+\sum_{i=1}^{k}|K_i|\|q_i\|\psi_i(r)\nabla_i+\Delta_2 r.$$

Next, we show that ϕ maps bounded sets into equicontinuous sets of Z. Let $\tau_1,\tau_2\in\Theta$ with $\tau_1<\tau_2$ and $z\in B_r$. Then, we have

$$|\phi z(\tau_2)-\phi z(\tau_1)|$$
$$\leq\frac{|\theta|}{\Gamma(\alpha+\beta)}\int_0^{\tau_1}\left((\tau_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right)\left|f\left(s,z(s)\right)\right|ds$$
$$+\frac{|\theta|}{\Gamma(\alpha+\beta)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha+\beta-1}\left|f\left(s,z(s)\right)\right|ds$$
$$+\sum_{i=1}^{k}\frac{|K_i|}{\Gamma(\alpha+\beta+\delta_i)}\int_0^{\tau_1}\left((\tau_2-s)^{\alpha+\beta+\delta_i-1}-(\tau_1-s)^{\alpha+\beta+\delta_i-1}\right)\left|h_i\left(s,z(s)\right)\right|ds$$
$$+\sum_{i=1}^{k}\frac{|K_i|}{\Gamma(\alpha+\beta+\delta_i)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha+\beta+\delta_i-1}\left|h_i\left(s,z(s)\right)\right|ds$$
$$+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)|z(s)|ds$$
$$+\frac{|\lambda|}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}|z(s)|ds$$
$$+(\tau_2^{\alpha}-\tau_1^{\alpha})|\Lambda\theta|\int_0^{T}\frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\left|f\left(s,z(s)\right)\right|ds$$
$$+(t_2^{\alpha}-t_1^{\alpha})|\Lambda|\sum_{i=1}^{k}|K_i|\int_0^{T}\frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\left|h_i\left(s,z(x)\right)\right|ds$$
$$+(\tau_2^{\alpha}-\tau_1^{\alpha})|\lambda\Lambda|\int_0^{T}\frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds$$
$$+(\tau_2^{\alpha}-\tau_1^{\alpha})|\Lambda A\theta|\int_0^{\eta}\frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)}\left|f\left(s,z(s)\right)\right|ds$$
$$+(\tau_2^{\alpha}-\tau_1^{\alpha})|\Lambda A|\sum_{i=1}^{k}|K_i|\int_0^{\eta}\frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)}\left|h_i\left(s,z(s)\right)\right|ds$$
$$+(\tau_2^{\alpha}-\tau_1^{\alpha})|\lambda\Lambda A|\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|z(s)|ds.$$
$$\leq\frac{|\theta|\|q\|\psi(r)}{\Gamma(\alpha+\beta+1)}\left({\tau_2}^{\alpha+\beta}-{\tau_1}^{\alpha+\beta}\right)$$
$$+\sum_{i=1}^{k}\frac{|K_i|\|q_i\|\psi_i(r)}{\Gamma(\alpha+\beta+\delta_i+1)}\left({\tau_2}^{\alpha+\beta+\delta_i}-{\tau_1}^{\alpha+\beta+\delta_i}\right)$$
$$+\left[\frac{|\Lambda\theta|\|q\|\psi(r)T^{\alpha+\beta-p}}{\Gamma(\alpha+\beta-p+1)}+\sum_{i=1}^{k}\frac{|K_i|\|q_i\|\psi_i(r)|\Lambda|T^{\alpha+\beta-p+\delta_i}}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right.$$
$$\left.+\frac{\|q\|\psi(r)|\Lambda A\theta|\eta^{\alpha+\beta-p}}{\Gamma(\alpha+\beta-p+1)}+\sum_{i=1}^{k}\frac{|K_i|\|q_i\|\psi_i(r)|\Lambda A|\eta^{\alpha+\beta-p+\delta_i}}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right.$$
$$\left.+\frac{|\lambda|r}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|T^{\alpha-p}r}{\Gamma(\alpha-p+1)}+\frac{|\lambda\Lambda A|\eta^{\alpha-p}r}{\Gamma(\alpha-p+1)}\right](\tau_2^{\alpha}-\tau_1^{\alpha}),$$

which imply that, $\|\phi z(\tau_2)-\phi z(\tau_1)\|\to 0$ independent of z as $\tau_2\to\tau_1$. Consequently, by the Arzela-Ascoli theorem, ϕ is completely continuous.

Finally, consider the equation $z = \rho\phi z$ for $0 < \rho < 1$ and suppose that z be a solution. Then, we have

$$\begin{aligned}\|z\| &= \|\rho\phi z\| \\ &\leq |\theta|\|q\|\psi(r)\left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p}+|A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}\right] \\ &+\sum_{i=1}^{k}|K_i|\|q_i\|\psi_i(r)\left[\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right] \\ &+\left[\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|T^{2\alpha-p}}{\Gamma(\alpha-p+1)}+\frac{|\lambda\Lambda A|T^{\alpha}\eta^{\alpha-p}}{\Gamma(\alpha-p+1)}+TM\right]\|z\|.\end{aligned}$$

Therefore,

$$\frac{\|z\|}{|\theta|\|q\|\psi(\|z\|)\nabla_1+\sum_{i=1}^{k}|K_i|\|q_i\|\psi_i(\|z\|)\nabla_{i+1}+\Delta_2\|z\|}\leq 1.$$

By (H_4) there existe L such that $L \neq \|z\|$. Let us set

$$\Omega = \{z \in Z: \|z\| < L\},$$

so, the $\phi: \overline{\Omega} \to Z$ is continuous and completely continuous. From the selection of $\overline{\Omega}$, we have $z \notin \partial\Omega$ with $z = \rho\phi z$, $0 < \rho < 1$. Hence, the ϕ has a fixed point $z \in \overline{\Omega}$.

Theorem 7: *Let* f, $h_i: \Theta \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, k\}$, *are continuous functions. Suppose that*

(H_5) *If* $0 \leq m < \frac{1-\Delta_2}{\left[|\theta|\nabla_1+\sum_{i=1}^{k}|K_i|\nabla_{i+1}\right]}$ *and* $L_i > 0$, $i = 1, \ldots, k+1$, *then*

$$|f(\tau,z)| \leq m_1(|z|)+L_1, |h_i(t,x)| \leq m_{i+1}(|z|)+L_{i+1}, i = 1, \ldots, k, (t,x) \in [0,T]\times\mathbb{R},$$

where $m = max\{m_i: i = 1, \ldots, k+1\}$, $L = max\{L_i: i = 1, \ldots, k+1\}$. *Then (1) admits one solution at least defined on* Θ.

Proof: Let

$$z = \phi z,$$

we shall demonstrate that there exists a fixed point $z \in Z$ satisfying (1). It is sufficient to show that $\phi: \overline{B}_r \to Z$ satisfies

$$z \neq \mu\phi z, \forall(z,\mu) \in \partial B_r \times [0,1], \tag{11}$$

where

$$B_r = \left\{z \in Z: \max_{\tau\in\Theta}|z(\tau)| < r, r > 0\right\}.$$

We define

$$V(\mu,z) = \mu\phi x, (z,\mu) \in Z \times \Theta.$$

As shown in Theorem 9, the operator ϕ is continuous, uniformly bounded, and equicontinuous. Then, v_μ, $v_\mu = z - V(\mu,z) = z - \mu\phi z$ is completely continuous. If (11) is holds, then we can write

$$\deg(v_\mu, B_r, 0) = \deg(I - \mu\phi, B_r, 0) = \deg(v_1, B_r, 0) = \deg(v_0, B_r, 0)$$
$$= \deg(I, B_r, 0) = 1 \neq 0, 0 \in B_r.$$

By the nonzero property of Leray-Schauder's degree, $v_1(z) = z - \phi z = 0$ for at least one $z \in B_r$. In order to demonstrate (11), we suppose that $z = \mu\phi z$ for $\mu \in \Theta$ and for all $\tau \in \Theta$. Then

$$\begin{aligned}
&\phi z(\tau) = |\mu\phi z(\tau)| \\
&\leq |\theta| \int_0^\tau \frac{(\tau-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left|f\left(s, z(s)\right)\right| ds + \sum_{i=1}^k |K_i| \int_0^\tau \frac{(\tau-s)^{\alpha+\beta+\delta_i-1}}{\Gamma(\alpha+\beta+\delta_i)} \left|h_i\left(s, z(s)\right)\right| ds \\
&+|\lambda| \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} |z(s)| ds + |\Lambda\theta|\tau^\alpha \int_0^T \frac{(T-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} \left|f\left(s, z(s)\right)\right| ds \\
&+|\Lambda|\tau^\alpha \sum_{i=1}^k |K_i| \int_0^T \frac{(T-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} \left|h_i\left(s, z(s)\right)\right| ds \\
&+|\lambda\Lambda|\tau^\alpha \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |x(s)| ds + |\Lambda A\theta|\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p-1}}{\Gamma(\alpha+\beta-p)} \left|f\left(s, z(s)\right)\right| ds \\
&+|\Lambda A|\tau^\alpha \sum_{i=1}^k |K_i| \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-p+\delta_i-1}}{\Gamma(\alpha+\beta-p+\delta_i)} \left|h_i\left(s, z(s)\right)\right| ds \\
&+|\lambda\Lambda A|\tau^\alpha \int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |z(s)| ds + \int_0^T |\varphi(s)||z(s)| ds \\
&\leq (m_1|z(\tau)| + L_1)|\theta| \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p-1} + |A|T^\alpha \eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}\right] \\
&+ \sum_{i=1}^k |K_i|(m_{i+1}|z(\tau)| + L_{i+1}) \left[\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i} + |A|T^\alpha \eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)}\right] \\
&+|z(\tau)| \left[\frac{|\lambda|T^\alpha}{\Gamma(\alpha+1)} + \frac{|\lambda\Lambda|T^{2\alpha-p}}{\Gamma(\alpha-p)} + \frac{|\lambda\Lambda A|T^\alpha \eta^{\alpha-p}}{\Gamma(\alpha-p+1)} + TM\right] \\
&\leq (m|z(\tau)| + L)\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right] + |z(\tau)|\Delta_2.
\end{aligned}$$

Taking norm $\sup_{\tau\in\Theta}|z(\tau)| = \|z\|$, we get

$$\|z\| \leq (m\|z\| + L)\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right] + \|z\|\Delta_2,$$

which implies that

$$\|z\| \leq \frac{L\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right]}{1 - m\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right] - \Delta_2}.$$

If $r = \frac{L\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right]}{1 - m\left[|\theta|\nabla_1 + \sum_{i=1}^k |K_i|\nabla_{i+1}\right] - \Delta_2} + 1$, then inequality (11) holds.

2.3 *Stability results*

Now, we prove the Ulam-Hyers stable (UH-stable) results for problem (1).

Definition 8: The fractional equation (1) is UH-stable if there exists a real number $C_{f,h_i} > 0, i \in \{1, \ldots, k\}$ such that for any $\varepsilon > 0$ and for all solution $z \in Z$ of

$$\left|D^\beta(D^\alpha + \lambda)w(t) - \theta f\left(t, w(t)\right) - \sum_{i=1}^k K_i I^{\delta_i} h_i\left(t, w(t)\right)\right| \leq \varepsilon, \qquad (12)$$

then ther exists $w \in Z$ of (1), with

$$\|w - z\| \leq C_{f,h_i}\varepsilon, \varepsilon > 0, i = 1, \ldots, k.$$

Definition 9: The fractional equation (1) is generalized UH-stable if there exists $F_{f,h_i} \in C(\mathbb{R}^+, \mathbb{R}^+), F_{f,h_i}(0) = 0, i \in \{1, \dots, k\}$ such that for any $\varepsilon > 0$ and for all solution $w \in Z$ of (12), there exists $z \in Z$ of (1) with

$$\|w - z\| \leq F_{f,h_i}(\varepsilon), i = 1, \dots, k.$$

Remark 10: Let $w \in Z$ is a solution of (12) if and only if there exists a function $g: \Theta \to \mathbb{R}$ with

1: $|\psi(\tau)| \leq \varepsilon, \tau \in \Theta$.

2: $D^{\beta}(D^{\alpha} + \lambda)w(\tau) = \theta f\big(\tau, w(\tau)\big) + \sum_{i=1}^{k} K_i I^{\delta_i} h_i\big(\tau, z(\tau)\big) + \psi(t), \tau \in \Theta$.

Theorem 11: *Let* f, $h_i: \Theta \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, k\}$ *are continuous functions satisfying the hypothesis* (H_1). *If*

$$\omega\Delta_1 < 1 - \Delta_2, \tag{13}$$

then the problem (1) is UH-stable and consequently, generalized UH-stable.

Proof: Let us denote by $z \in Z$ the unique solution of the problem

$$\begin{cases} D^{\beta}(D^{\alpha} + \lambda)z(\tau) = \theta f\big(\tau, z(\tau)\big) + \sum_{i=1}^{k} K_i I^{\delta_i} h_i\big(\tau, z(\tau)\big), \tau \in \Theta, \\ z(0) = w(0), D^{p} z(T) = D^{p} w(T), \end{cases}$$

such that $w \in Z$ is a solution of the inequality (12) Applying Remark 13, we can write

$$w(\tau) = I^{\alpha+\beta}[g_w(\tau)] - \lambda I^{\alpha}[z(\tau)] - \frac{c_0}{\Gamma(\alpha+1)}\tau^{\alpha} - c_1 + I^{\alpha+\beta}[\psi(\tau)], c_0, c_1 \in \mathbb{R},$$

where $g_w(\tau) = \theta f\big(\tau, w(\tau)\big) + \sum_{i=1}^{k} K_i I^{\delta_i} h_i\big(\tau, w(\tau)\big)$ and $|\psi(\tau)| \leq \varepsilon, \tau \in \Theta$.

Thanks to Lemma 5, we get

$$|w(\tau) - \phi w(\tau)| = \left|I^{\alpha+\beta}\psi(\tau)\right| \leq \varepsilon I^{\alpha+\beta}[1].$$

which implies that,

$$|w(\tau) - \phi w(\tau)| \leq \frac{\varepsilon}{\Gamma(\alpha+\beta+1)} T^{\alpha+\beta}.$$

Also, we have

$$\begin{aligned} & |w(\tau) - z(\tau)| \\ = & \left|w(t) - I^{\alpha+\beta} g_w(t) - \lambda I^{\alpha} z(t) - \frac{c_0}{\Gamma(\alpha+1)}\tau^{\alpha} - c_1\right| \\ = & |w(\tau) - \phi w(\tau) + \phi w(\tau) - \phi z(\tau)| \\ \leq & |w(\tau) - \phi w(\tau)| + |\phi w(\tau) - \phi z(\tau)|. \end{aligned}$$

Thanks to (H_1), we get

$$\begin{aligned} \|w - z\| & \leq \|w - \phi(w)\| + \|\phi(w) - \phi(z)\| \\ & \leq \frac{\varepsilon}{\Gamma(\alpha+\beta+1)} T^{\alpha+\beta} + (\omega\Delta_1 + \Delta_2)\|w - z\|, \end{aligned}$$

where $\Delta_i, i = 1,2$, are given by (6) and (7), respectively. Then

$$\|w - z\| \leq \frac{\varepsilon T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)\left(1-(\omega\Delta_1+\Delta_2)\right)},$$

If we put

$$C_{f,h_i} := \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)\left(1-(\omega\Delta_1+\Delta_2)\right)},$$

we obtain

$$\|w - z\| \leq C_{f,h_i}\varepsilon.$$

Hence, the E problem (1) is stable in Ulam-Hyers sense. If we take $F_{f,h_i}(\varepsilon) = C_{f,h_i}\varepsilon$. we see that, problem (1) is generalized Ulam-Uyers satable.

3. Examples

Example 12: Consider the fractional difference nonlocal boundary value problem:

$$\begin{cases} D^{\frac{1}{2}}\left(D^{\frac{2}{3}} + \frac{1}{15}\right)z(\tau) = \theta f\big(\tau, w(\tau)\big) + \sum_{i=1}^{k} K_i I^{\delta_i} h_i\big(\tau, w(\tau)\big), \tau \in \Theta, \\ z(0) = \int_0^1 \frac{\ln(1+s)}{17(1+s^2)} z(s)ds, D^{\frac{1}{7}}z(1) = 2D^{\frac{1}{7}}z\left(\frac{3}{5}\right). \end{cases} \tag{14}$$

For this example, we have $\beta = \frac{1}{2}$, $\alpha = \frac{2}{3}$, $\lambda = \frac{1}{15}$, $\theta = K_i = 1$, $(i = 1,2)$, $\delta_1 = \frac{1}{3}$, $\delta_2 = \frac{3}{4}$, $\varphi(s) = \frac{\ln(1+s)}{17(1+s^2)}$, $p = \frac{1}{7}$, $A = 2$ and $f(\tau, z) = \frac{e^{-\pi t^2} z(\tau)}{(32\sqrt{\pi}+e^{-\pi\tau})(1+z(\tau))}$, $h_1(\tau, z) = \frac{z(\tau)}{(15+\sqrt{\pi}e^{\tau})(2+z(\tau))}$, $h_2(\tau, z) = \frac{\sin(2\pi z(\tau))}{16\pi(\tau+2)^2}$. Also for $z, w \in \mathbb{R}$ and $\tau \in \Theta$, we have

$$|f(\tau, z) - f(\tau, w)| \leq \frac{1}{32\sqrt{\pi}}|z - w|,$$

$$|h_1(\tau, z) - h_1(\tau, w)| \leq \frac{1}{15 + \sqrt{\pi}}|z - w|,$$

$$|h_2(\tau, z) - h_2(\tau, w)| \leq \frac{1}{64\pi}|z - w|.$$

Hence,

$$\omega_1 = \frac{1}{32\sqrt{\pi}}, \omega_2 = \frac{1}{15+\sqrt{\pi}}, \omega_3 = \frac{1}{64\pi},$$

$$\Delta_1 := \frac{|\theta|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda\theta|\left(T^{2\alpha+\beta-p}+|A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)} + \sum_{i=1}^{k}|K_i|\frac{T^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}$$
$$+ \sum_{i=1}^{k}|K_i|\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_i}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_i}\right)}{\Gamma(\alpha+\beta-p+\delta_i+1)} = 7.982\,3,$$

$$\Delta_2 := \frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda\Lambda|\left(T^{2\alpha-p+1}+|A|T^{\alpha}\eta^{\alpha-p}\right)}{\Gamma(\alpha-p+1)} + MT = 0.222\,71,$$

and

$$\omega = \max\{\omega_i, i = 1, \ldots, 3\} = \frac{1}{15+\sqrt{\pi}}.$$

Therefore, we have

$$\omega\Delta_1 + \Delta_2 = 0.698\,63 < 1.$$

So, all the hypotheses of Theorem 6 are satisfied. Thus, the problem (14) has a unique solution.and by Theorem 14, (14) is stable in Ulam-Hyers sense with

$$\|y - x\| \leq 3.065\,7\varepsilon, \varepsilon > 0.$$

Example 13: Consider the example:

$$\begin{cases} D^{\frac{2}{5}}\left(D^{\frac{1}{4}} + \frac{1}{10}\right)z(\tau) = \theta f\big(\tau, z(\tau)\big) + \sum_{i=1}^{k} K_i I^{\delta_i} h_i\big(\tau, z(\tau)\big), \tau \in \Theta, \\ z(0) = \int_0^1 \frac{e^{-\pi s}}{20\sqrt{1+\pi s^2}} z(s)ds, D^{\frac{2}{9}}z(1) = \sqrt{3}D^{\frac{2}{9}}z\left(\frac{2}{7}\right). \end{cases} \tag{15}$$

Here, $\beta = \frac{2}{5}$, $\alpha = \frac{1}{4}$, $\lambda = \frac{1}{10}$, $\theta = 2$, $K_1 = \frac{1}{2}$, $K_2 = \frac{3}{2}$, $\delta_1 = \frac{1}{5}$, $\delta_2 = \frac{5}{6}$, $\varphi(s) = \frac{e^{-\pi s}}{20\sqrt{1+\pi s^2}}$, $p = \frac{2}{9}$, $A = \sqrt{3}$, $\eta = \frac{2}{7}$ and $f(\tau, z) = \frac{\sin z(\tau)}{16\pi + \cos^2 z(\tau)} + \frac{3+\sinh e^{\tau^2}}{2\pi}$, $h_1(\tau, z) = \frac{2\sin\left(\frac{z(\tau)}{2}\right)}{20\sqrt{\pi}+\cos^2 z(\tau)} + \frac{2+\cosh(\pi\tau+1)}{\sqrt{\pi}+3}$, $h_2(\tau, z) = \frac{\tau \sin z(\tau)}{19\pi} + \frac{2(t+1)|z|}{5|z|+1}$. Then we can find that

$$\Delta_2 := \frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\lambda\Lambda|\left(T^{2\alpha-p+1} + |A|T^{\alpha}\eta^{\alpha-p}\right)}{\Gamma(\alpha-p+1)} + MT = 0.219\,91,$$

$$\nabla_1 := \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p} + |A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)} = 2.028\,3,$$

$$\nabla_2 := \frac{T^{\alpha+\beta+\delta_1}}{\Gamma(\alpha+\beta+\delta_1+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_1} + |A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_1}\right)}{\Gamma(\alpha+\beta-p+\delta_1+1)} = 1.862\,6,$$

$$\nabla_3 := \frac{T^{\alpha+\beta+\delta_2}}{\Gamma(\alpha+\beta+\delta_2+1)} + \frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_2} + |A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_2}\right)}{\Gamma(\alpha+\beta-p+\delta_2+1)} = 1.695\,7.$$

Cleary,

$$|f(\tau, z)| = \left|\frac{\sin z(\tau)}{16\pi + \cos^2 z(\tau)} + \frac{3+\sinh e^{\tau^2}}{2\pi}\right| \leq \left(\frac{3+\sinh e^{\tau^2}}{16\pi}\right)(|z| + 8),$$

$$h_1(\tau, z) = \left|\frac{2\sin\left(\frac{z(\tau)}{2}\right)}{15\sqrt{\pi}+\cos^2 z(\tau)} + \frac{2+\cosh\left(\sqrt{\pi}\tau+1\right)}{3\sqrt{\pi}+3}\right| \leq \left(\frac{2+\cosh\left(\sqrt{\pi}\tau+1\right)}{15\sqrt{\pi}}\right)(|z| + 5),$$

$$h_2(\tau, z) = \left|\frac{\tau \sin z(\tau)}{19} + \frac{2(t+1)|z|}{5|z|+1}\right| \leq \frac{(t+1)}{5}\left(\frac{|z|}{19} + 2\right).$$

Choosing $q(\tau) = \frac{3+\sinh e^{\tau^2}}{16\pi}$, $q_1(\tau) = \frac{2+\cosh(\pi\tau+1)}{15\sqrt{\pi}}$, $q_2(\tau) = \frac{(\tau+1)}{5}$ and $\psi(|z|) = \frac{|z|+2}{16\pi}$, $\psi_1(|z|) = |z| + 5$, $\psi_2(|z|) = \frac{|z|}{19} + 2$, we can show that

$$\frac{L}{|\theta|\|q\|\left(\frac{L+2}{16\pi}\right)\nabla_1 + |K_1|\|q_1\|(L+5)\nabla_2 + |K_2|\|q_2\|\left(\frac{L}{19}+2\right)\nabla_3 + \Delta_2 L} > 1,$$

which implies $L > 10.677$. So, by Theorem 8, (15) admits one solution at least on Θ.

Example 14: We give the following fractional problem:

$$\begin{cases} D^{\frac{4}{5}}\left(D^{\frac{3}{4}}+\frac{1}{19}\right)z(\tau)=\theta f\left(\tau,z(\tau)\right)+\sum_{i=1}^{k}K_iI^{\delta_i}h_i\left(\tau,z(\tau)\right),\tau\in\Theta, \\ z(0)=\int_0^1\frac{\sqrt{\pi}\sinh s}{12\pi+s^2}z(s)ds, D^{\frac{3}{11}}z(1)=3D^{\frac{3}{11}}z\left(\frac{1}{3}\right). \end{cases} \tag{16}$$

Here, $\beta=\frac{4}{5}$, $\alpha=\frac{3}{4}$, $\lambda=\frac{1}{19}$, $\theta=K_i=1$, $(i=1,2),\delta_1=\frac{1}{5}$, $\delta_2=\frac{5}{6}$, $\varphi(s)=\frac{\sqrt{\pi}\sinh s}{12\pi+s^2}$, $p=\frac{3}{11}$, $A=3,\eta=\frac{1}{3}$ and $f(t,x)=\frac{3}{8}\sin\left(\frac{|z|}{3}\right)+\frac{2|z|}{1+|z|}$, $h_1(\tau,z)=\frac{3}{16\pi}\sin\left(\frac{2\pi}{3}|z|\right)+\frac{|z|}{1+2|x|}+\frac{1}{2}$, $h_2(\tau,z)=\frac{3}{10\pi}\sin(2|z|)+\frac{|z|}{1+|z|}+\frac{3}{2}$.

$$\begin{aligned} f(\tau,z) &= \left|\frac{3}{8}\sin\left(\frac{|z|}{3}\right)+\frac{2|z|}{1+|z|}\right|\leq\frac{1}{8}|z|+2, \\ h_1(\tau,z) &= \left|\frac{3}{16\pi^2}\sin\left(\frac{2\pi}{3}|z|\right)+\frac{|z|}{1+2|z|}+\frac{1}{2}\right|\leq\frac{1}{8\pi}|z|+1, \\ h_2(\tau,z) &= \left|\frac{3}{10\pi}\sin(2|z|)+\frac{|z|}{1+|x|}+\frac{3}{2}\right|\leq\frac{3}{5\pi}|z|+\frac{5}{2}. \end{aligned}$$

$$\Delta_2:=\frac{|\lambda|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\lambda\Lambda|\left(T^{2\alpha-p+1}+|A|T^{\alpha}\eta^{\alpha-p}\right)}{\Gamma(\alpha-p+1)}+MT=0.219\,91,$$

$$\nabla_1:=\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p}+|A|T^{\alpha}\eta^{\alpha+\beta-p}\right)}{\Gamma(\alpha+\beta-p+1)}=2.131\,9,$$

$$\nabla_2:=\frac{T^{\alpha+\beta+\delta_1}}{\Gamma(\alpha+\beta+\delta_1+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_1}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_1}\right)}{\Gamma(\alpha+\beta-p+\delta_1+1)}=1.755\,2,$$

$$\nabla_3:=\frac{T^{\alpha+\beta+\delta_2}}{\Gamma(\alpha+\beta+\delta_2+1)}+\frac{|\Lambda|\left(T^{2\alpha+\beta-p+\delta_2}+|A|T^{\alpha}\eta^{\alpha+\beta-p+\delta_2}\right)}{\Gamma(\alpha+\beta-p+\delta_2+1)}=0.884\,88.$$

Clearly $L=\max\{L_i,i=1,\ldots,3\}=\frac{5}{2}$ and $m=\max\{m_i,i=1,\ldots,3\}=\frac{1}{8}<\frac{1-\Delta_2}{\left[|\theta|\nabla_1+\sum_{i=1}^{3}|K_i|\nabla_{i+1}\right]}=0.154\,08.$

Thus, all the conditions of Theorem 10 are satisfied and accordingly the system (16) admits one solution at least.

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