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Smoothness via Geometry of orthogonality functional using 2-norm derivatives

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Abstract

In this paper, we introduce the notion of ρ_λ-orthogonality via 2-normed derivatives and discuss some of its essential geometric properties in the context of 2-$\mathcal{NS}$. Moreover, we also discuss the relation of ρ_λ-orthogonality with other renowned orthogonalities, and some examples are illustrated. In addition, we provide characterizations of the smooth 2-$\mathcal{NS}$ based on the ρ_λ-orthogonality. Also, we characterize 2-$\mathcal{JNPS}$s as an application in terms of ρ_λ.

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1. Introduction and Preliminaries

The idea of orthogonality is one of the most interesting ideas in the study of the geometry of spaces. Orthogonality relations which are taken from norm derivatives, is very useful for constructing investigations of the geometric structure of space. Suppose that $(\mathfrak{X}, \|.\|)$ be a normed space. In 1935, Birkhoff introduced one of the most significant orthogonality in normed space as follows: $\mathfrak{Q}_1$ is said to be Brikhoff orthogonal to $\mathfrak{Q}_2$ $(\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2)$ if $\|\mathfrak{Q}_1 + t\mathfrak{Q}_2\| \geq \|\mathfrak{Q}_1\|$ for all $t \in \mathbb{R}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \mathfrak{X}$. Amir [2] proposed the functional via norm derivative, which is helpful in the examination of the geometry of normed spaces. Furthermore, Milićić [16] combined norm derivatives to provide a new functional and its orthogonality. The norm orthogonality was then generalized by Zamani

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and Moslehian [19] as a convex combination of norm derivatives. Later, Abed [1] introduced a new form of orthogonality in real normed spaces based on the concept of norm derivative, explored its key properties, and used it to define the angular property.

Gähler [7] proposed the idea of a 2-normed space, which is an extension of a normed space. Throughout the paper, $\mathfrak{X}$ is a real linear space with $\dim(\mathfrak{X}) > 1$ and 2-normed space is denoted by $2\text{-}\mathcal{NS}$.

Definition 1.1: Consider a real-valued function $\|.,.\|$ defined on $\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}$ that fulfills the subsequent condition for every $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$:

(a) $\|\mathfrak{Q}_1,\mathfrak{Q}_2\| = 0$ if and only if $\mathfrak{Q}_1,\mathfrak{Q}_2$ are linearly dependent;

(b) $\|\mathfrak{Q}_1,\mathfrak{Q}_2\| = \|\mathfrak{Q}_2,\mathfrak{Q}_1\|$;

(c) $\|\varpi\mathfrak{Q}_1,\mathfrak{Q}_2\| = |\varpi|\|\mathfrak{Q}_1,\mathfrak{Q}_2\|$, for $\varpi \in \mathbb{R}$;

(d) $\|\mathfrak{Q}_1,\mathfrak{Q}_2+\mathfrak{Q}_3\| \le \|\mathfrak{Q}_1,\mathfrak{Q}_2\| + \|\mathfrak{Q}_1,\mathfrak{Q}_3\|$.

$\|.,.\|$ is called a 2-norm and $(\mathfrak{X},\|.,.\|)$ is called $2\text{-}\mathcal{NS}$.

A 2-norm function geometrically generalizes the idea of area function parallelogram because it usually reflects the area of the parallelogram formed by the two related vectors. Also, Gähler proved that $2\text{-}\mathcal{NS}$ is not metrizable. Specifically, it is not normable. Many authors gave partial answers to this (see [12]).

The idea of 2-inner product space was first proposed by Milicic [6] in 1969. Throughout the paper, 2-inner product and 2-inner product spaces are denoted by $2\text{-}\mathcal{IP}$ and $2\text{-}\mathcal{INPS}$ respectively.

Definition 1.2: Suppose that $\langle.,.|.\rangle$ is a real valued function on $\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}$ which satisfied the following condition for every $\mathfrak{Q}_1,\mathfrak{Q}_{1\prime\prime},\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$:

(a) $\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle \ge 0$, $\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle = 0$ if and only if $\mathfrak{Q}_1,\mathfrak{Q}_3$ are linearly dependent;

(b) $\langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \langle\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle$;

(c) $\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle = \langle\mathfrak{Q}_3,\mathfrak{Q}_3|\mathfrak{Q}_1\rangle$;

(d) $\langle\varpi\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \varpi\langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle$ for $\varpi \in \mathbb{R}$;

(e) $\langle\mathfrak{Q}_1+\mathfrak{Q}_{1\prime\prime},\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \langle\mathfrak{Q}_{1\prime\prime},\mathfrak{Q}_2|\mathfrak{Q}_3\rangle + \langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle$.

$\langle.,.|.\rangle$ is known as $2\text{-}\mathcal{IP}$ and $(\mathfrak{X},\langle.,.|.\rangle)$ is $2\text{-}\mathcal{INPS}$.

Ehret established that, $\|\mathfrak{Q}_1,\mathfrak{Q}_2\| = \langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_2\rangle^{\frac{1}{2}}$ defines a 2-norm if $(\mathfrak{X},\langle.,.|.\rangle)$ is a $2\text{-}\mathcal{IP}$.

It is clear that in $2\text{-}\mathcal{INPS}$, the equality

$$\begin{aligned}\|\mathfrak{Q}_1+\mathfrak{Q}_2,\mathfrak{Q}_3\|^4 - \|\mathfrak{Q}_1-\mathfrak{Q}_2,\mathfrak{Q}_3\|^4 &= 8(\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\langle\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3\rangle \\ &\quad +\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\langle\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3\rangle)\end{aligned} \tag{1.1}$$

holds, which is equivalent (see [3]) to

$$\|\mathfrak{Q}_1 - \mathfrak{Q}_2, \mathfrak{Q}_3\|^2 + \|\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_3\|^2 = 2(\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \|\mathfrak{Q}_2, \mathfrak{Q}_3\|^2).$$

Rizvi and Siddiqui [17] introduced the notion of a 2-semi inner product space. Throughout the paper, 2-semi inner product and 2-semi inner product space are denoted by $2\text{-}\mathcal{SIP}$ and $2\text{-}\mathcal{SIPS}$ respectively.

Definition 1.3: [11] A mapping $[.,.|.]: \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ is said to be $2\text{-}\mathcal{SIP}$ if it satisfies the following properties for all $\mathfrak{Q}_1, \mathfrak{Q}'_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$:

(i) $[\mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_3] \geq 0$, $[\mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_3] = 0$ if and only if $\mathfrak{Q}_1$ and $\mathfrak{Q}_3$ are linearly dependent;

(ii) $[\varpi\mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3] = \varpi[\mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3]$, for $\varpi \in \mathbb{R}$;

(iii) $[\mathfrak{Q}_1 + \mathfrak{Q}'_1, \mathfrak{Q}_2|\mathfrak{Q}_3] = [\mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3] + [\mathfrak{Q}'_1, \mathfrak{Q}_2|\mathfrak{Q}_3]$;

(iv) $|[\mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3]| \leq [\mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_3]^{\frac{1}{2}}[\mathfrak{Q}_2, \mathfrak{Q}_2|\mathfrak{Q}_3]^{\frac{1}{2}}$.

Then $(\mathfrak{X}, [.,.|.])$ is said to be a 2-semi-inner product space.

Also, every linear $2\text{-}\mathcal{NS}$ can be transformed into a $2\text{-}\mathcal{SIPS}$ and a $2\text{-}\mathcal{SIPS}$ is a linear 2-normed space with the 2-norm $\|\mathfrak{Q}_1, \mathfrak{Q}_3\| = [\mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_3]^{\frac{1}{2}}$ provided $[\mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_3] = [\mathfrak{Q}_3, \mathfrak{Q}_3|\mathfrak{Q}_1]$.

Numerous scholars have investigated the concept of orthogonality in $2\text{-}\mathcal{NS}$. In 1982, Khan and Siddiqui [13] introduced the notion of Birkhoff orthogonality in $2\text{-}\mathcal{NS}$ as follows:

$$\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2 \text{ if and only if } \|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| \geq \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \text{ for } \mathfrak{Q}_3 \neq 0,$$

where $t \in \mathbb{R}$.

In 1985, Godini [8] remarked that the phrase "for every $\mathfrak{Q}_3 \neq 0$" in the above definition should be replaced by "for every $\mathfrak{Q}_3 \notin span\{\mathfrak{Q}_1, \mathfrak{Q}_2\}$" for otherwise there do not exist two non-zero vectors $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ such that $\mathfrak{Q}_1$ is Birkhoff orthogonal to $\mathfrak{Q}_2$. Also, he provided the definition of orthogonality in $2\text{-}\mathcal{NS}$ as follows:

$$\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2 \text{ if and only if } \|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| \geq \|\mathfrak{Q}_1, \mathfrak{Q}_3\|$$
$$\text{for all } \mathfrak{Q}_3 \notin span\{\mathfrak{Q}_1, \mathfrak{Q}_2\}, \quad \text{and } t \in \mathbb{R}.$$

Later, Gunawan et al. [9] showed that even with the above correction, the definition of Birkhoff orthogonality are still "void" in the sense that there do not exist two non-zero vectors $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ such that $\mathfrak{Q}_1$ is Birkhoff orthogonal to $\mathfrak{Q}_2$ even in the standard case. The authors revised the concept of orthogonality in 2-normed space by imposing restrictions on the underlying set for $\mathfrak{Q}_3$.

$$\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2 \Leftrightarrow \text{ there exists a one co} - \text{dimensional subspace } G \text{ of } \mathfrak{X}$$
$$\text{for which } \|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| \geq \|\mathfrak{Q}_1, \mathfrak{Q}_3\|, \text{for all } t \in \mathbb{R}, \mathfrak{Q}_3 \in G.$$

Mazaheri and Nezhad [15] defined Birkhoff orthogonality in 2-normed space and presented some results on this topic.

Definition 1.4: Let $(\mathfrak{X}, \|.,.\|)$ be a 2-$\mathcal{NS}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \mathfrak{X}$. $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$ if there exists $\mathfrak{Q}_3 \in \mathfrak{X}$ such that $\|\mathfrak{Q}_1, \mathfrak{Q}_3\| \neq 0$ and $\|\mathfrak{Q}_1, \mathfrak{Q}_3\| \geq \|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\|$ for all scalar $t \in \mathbb{R}$.

In 2010, Gunawan et al. [10] proved that the conditions of the above Birkhoff orthogonality are equivalent to the conditions for linear independence of vectors.

Recently, Nur and Batkunde [18] defined the Dg-orthogonality through a 2-norm in normed spaces and established its connection with other existing orthogonality.

One method for describing the geometry of normed linear spaces is through the use of norm derivatives. They are utilized in the examination of smoothness and rigorous convexity. Orthogonality relations, computed from norm derivatives, allow for the geometric development of normed linear spaces. To investigate orthogonality in 2-$\mathcal{NS}$, one needs to generalize the norm derivatives to 2-norm derivatives (see [14]).

Using the partial derivatives of bifunctionals, Diminnie and White [5] characterized the 2-inner product space. They demonstrated that if $(\mathfrak{X}, \langle .,.|.\rangle)$ is a 2-$\mathcal{IP}$, where $||\mathfrak{Q}_1, \mathfrak{Q}_2|| = \langle \mathfrak{Q}_1, \mathfrak{Q}_1|\mathfrak{Q}_2\rangle^{\frac{1}{2}}$, then

$$\langle \mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \lim_{t\to 0} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2t}.$$

If the limit $\lim_{t\to 0} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2t}$ exists, then 2 -norm $\|.,.\|$ satisfies the Gateaux differentiablity conditions in the direction of $\mathfrak{Q}_2$ at the point $(\mathfrak{Q}_1, \mathfrak{Q}_3)$.

Definition 1.5: Let $(\mathfrak{X}, \|.,.\|)$ be a 2-$\mathcal{NS}$. Define mappings $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$, $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)\colon \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to \mathbb{R}$ are defined by

$$\begin{aligned}\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \lim_{t\to 0^+} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2t}\\ &= \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \lim_{t\to 0^+} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|}{t},\\ \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \lim_{t\to 0^-} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2t}\\ &= \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \lim_{t\to 0^-} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|}{t}.\end{aligned}$$

Also, define a mapping $\rho(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)\colon \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to \mathbb{R}$ is defined by

$$2\rho(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = (\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)).$$

The mappings $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ and $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$, are called 2-normed derivatives (2ND) and $\rho(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ is called average 2-normed derivative (A2ND).

To study the well-known properties of 2ND in a 2-$\mathcal{NS}$ see [14].

Proposition 1.6: [14] If $(\mathfrak{X}, \langle .,.|.\rangle)$ is a 2-$\mathcal{IP}$ with 2-norm, then $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \langle \mathfrak{Q}_1, \mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$.

For arbitrary non-zero elements $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \mathfrak{X}$, $V(\mathfrak{Q}_1, \mathfrak{Q}_2)$ denotes the subspace of $\mathfrak{X}$ generated by $\{\mathfrak{Q}_1, \mathfrak{Q}_2\}$.

Definition 1.7: [14] Suppose that $(\mathfrak{X}, \|.,.\|)$ is a 2-$\mathcal{NS}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$. An element $\mathfrak{Q}_1$ is $\mathcal{B}$-orthogonal to $\mathfrak{Q}_2$ if $\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| \geq \|\mathfrak{Q}_1, \mathfrak{Q}_3\|$ for each element $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$ and every $t \in \mathbb{R}$. Further it is denoted by $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$.

Theorem 1.8: [14*] Let* $(\mathfrak{X}, \|.,.\|)$ *be a* 2-$\mathcal{NS}$ *and* $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ *with* $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$. *Then the subsequent assertions are identical.:*

(i) $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$;

(ii) $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \leq 0 \leq \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$.

Definition 1.9: [14] Suppose that $(\mathfrak{X}, \|.,.\|)$ is a 2-$\mathcal{NS}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ where $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$. Then

(a) ρ_+ -orthogonality defined by $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$ and denoted by $\mathfrak{Q}_1 \perp_{\rho_+} \mathfrak{Q}_2$;

(b) ρ_- -orthogonality defined by $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$ and denoted by $\mathfrak{Q}_1 \perp_{\rho_-} \mathfrak{Q}_2$;

(c) ρ-orthogonality defined by $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$ and denoted by $\mathfrak{Q}_1 \perp_{\rho} \mathfrak{Q}_2$.

Definition 1.10: [4] $(\mathfrak{X}, \|.,.\|)$ is called smooth for $\mathfrak{Q}_1 \neq 0$ in $\mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$ the 2-norm $\|.,.\|$ is Gateaux differentiable at $(\mathfrak{Q}_1, \mathfrak{Q}_3)$ in any direction of $\mathfrak{Q}_2$.

Also, using the definition of smoothness in 2-$\mathcal{NS}$ we can say that $(\mathfrak{X}, \|.,.\|)$ is smooth if $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$.

The objective of this article is to present a more comprehensive mapping referred to as ρ_λ that encompasses A2ND. Additionally, it aims to explore the intriguing aspects of ρ_λ via a perspective of significant geometric notions, as facilitated by 2ND. In addition, we explore the correlation between ρ_λ-orthogonality and other well-known orthogonalities. Furthermore, we offer many characterizations of 2-normed smooth spaces that are founded on the concept of ρ_λ-orthogonality. In addition to its practical applications, we also provide a characterization of 2-inner product spaces using the concept of ρ_λ.

2. Main Results

In this section, we define ρ_λ-orthogonality and give some of its geometrical properties.

Throughout this section, $(\mathfrak{X}, \|.,.\|)$ is a 2-$\mathcal{NS}$.

First, we introduce the notion of a ρ_λ-orthogonality:

Definition 2.1: For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ with $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$, we have

$\mathfrak{Q}_1 \perp_{\rho_\lambda} \mathfrak{Q}_2$ if and only if
$\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$,
where $\lambda \in (0,1)\backslash\{\frac{1}{2}\}$.

From the above definition, we have $\perp_{\rho_\lambda} = \begin{cases} \perp_{\rho_+}, & \text{if } \lambda = 0 \\ \perp_{\rho_-}, & \text{if } \lambda = 1 \\ \perp_{\rho}, & \text{if } \lambda = \frac{1}{2}. \end{cases}$

The geometry of ρ_λ-orthogonality with $\lambda = 0, 1, \frac{1}{2}$ has been already defined in the literature (see [14] and references cited therein). So throughout the manuscript, we assume that $\lambda \in (0,1)\backslash\left\{\frac{1}{2}\right\}$.

To start with, we give some geometrical properties of ρ_λ.

Proposition 2.2: For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$

(i) $\rho_\lambda(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = t\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1, t\mathfrak{Q}_2; \mathfrak{Q}_3)$, where $t \geq 0$.

(ii) $\rho_\lambda(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = t\rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1, t\mathfrak{Q}_2; \mathfrak{Q}_3)$, where $t < 0$.

(iii) $\rho_\lambda(\mathfrak{Q}_1, t\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_3) = t\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3),\ t \in \mathbb{R}$.

(iv) $|\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)| \leq \|\mathfrak{Q}_1, \mathfrak{Q}_3\|\,\|\mathfrak{Q}_2, \mathfrak{Q}_3\|$.

Proof:

(i) For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$, we have

$$\begin{aligned} \rho_\lambda(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \lambda\rho_-(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= t\lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + t(1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= t(\lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)) \\ &= t\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3). \end{aligned}$$

(ii) For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$, we get

$$\begin{aligned} \rho_\lambda(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \lambda\rho_-(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(t\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= t(1-\lambda)\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + t\lambda\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= t((1-\lambda)\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \lambda\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)) \\ &= t\rho_{(1-\lambda)}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3). \end{aligned}$$

(iii) For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$, we have

$$\begin{aligned} &\rho_\lambda\,(p, t\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_3) \\ &= \lambda\rho_-(\mathfrak{Q}_1, t\mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, t\mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= \lambda(t\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)) \\ &+(1-\lambda)(t\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)) \\ &= t\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= t\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \rho_\pm(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3). \end{aligned}$$

(iv) For all $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_3 \notin V(\mathfrak{Q}_1, \mathfrak{Q}_2)$, we get

$$\begin{aligned} &|\rho_\lambda\,(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)| \\ &= |\lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)| \end{aligned}$$

$$\begin{aligned}
&\leq |\lambda\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)| + |(1-\lambda)\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)| \\
&= \lambda|\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)| + (1-\lambda)|\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)| \\
&\leq \lambda\|\mathfrak{Q}_1,\mathfrak{Q}_3\|\|\mathfrak{Q}_2,\mathfrak{Q}_3\| + (1-\lambda)\|\mathfrak{Q}_1,\mathfrak{Q}_3\|\|\mathfrak{Q}_2,\mathfrak{Q}_3\| \\
&= \|\mathfrak{Q}_1,\mathfrak{Q}_3\|\|\mathfrak{Q}_2,\mathfrak{Q}_3\|.
\end{aligned}$$

The following lemma is required for the next result.

Lemma 2.3: *For every* $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$, $\mathfrak{Q}_1 \perp_{\rho_\lambda} (a\mathfrak{Q}_1 + \mathfrak{Q}_2)$ *where* $a \in \mathbb{R}$ *and for any* $\lambda \in (0,1)\setminus\left\{\frac{1}{2}\right\}$.

Proof: Assume that $\mathfrak{Q}_1 \neq 0$. Put $a = -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}$. Using Proposition 2.2(iii), we have

$$\rho_\lambda(\mathfrak{Q}_1, a\mathfrak{Q}_1 + \mathfrak{Q}_2;\mathfrak{Q}_3) = -\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) + \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = 0.$$

Thus $\mathfrak{Q}_1 \perp_{\rho_\lambda} (a\mathfrak{Q}_1 + \mathfrak{Q}_2)$ where $a \in \mathbb{R}$.

A semi-orthogonality in a $2\text{-}\mathcal{SIP}$ $[.,.|.]$ is defined as

$$\mathfrak{Q}_1 \perp_s \mathfrak{Q}_2 \text{ if and only if } [\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3] = 0.$$

Theorem 2.4: $\perp_{\rho_\lambda} = \perp_s$ *if only if* $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = [\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3]$, *where* $[.,.|.]$ *is a* $2\text{-}\mathcal{SIP}$ *in a* $2\text{-}\mathcal{NS}$ $\mathfrak{X}$ *where* $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$.

Proof: First assume that $\perp_{\rho_\lambda} = \perp_s$. Let $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$. Taking $w = -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\mathrm{P}\mathfrak{Q}_1,\mathfrak{Q}_3\mathrm{P}^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. Using Lemma 2.3, we have $\mathfrak{Q}_1 \perp_{\rho_\lambda} w$. So, $\mathfrak{Q}_1 \perp_s w$. This implies that

$$0 = [w,\mathfrak{Q}_1|\mathfrak{Q}_3] = -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2} \times \|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2 + [\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3].$$

Therefore $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = [\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3]$.

Conversely, let $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = [\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3]$, for all

$\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$. If $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = 0$ then we have $[\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3] = 0$. So $\perp_{\rho_\lambda} \subseteq \perp_s$. Also if $[\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3] = 0$, then we get $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = 0$. Therefore $\perp_s \subseteq \perp_{\rho_\lambda}$. Hence $\perp_{\rho_\lambda} = \perp_s$.

Theorem 2.5: $\perp_{\rho_\lambda} \subseteq \perp_{\mathcal{B}}$, *for any* $\lambda \in (0,1)\setminus\left\{\frac{1}{2}\right\}$.

Proof: Let $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$ and $\mathfrak{Q}_1 \perp_{\rho_\lambda} \mathfrak{Q}_2$. Thus $\lambda\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = -(1-\lambda)\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. Since $\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) \leq \rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$, we get $\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) \leq 0 \leq \rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. Therefore, by the Theorem 1.8 we get $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$. Hence $\perp_{\rho_\lambda} \subseteq \perp_{\mathcal{B}}$, where $\lambda \in (0,1)\setminus\{\frac{1}{2}\}$.

The following results give some characterizations of smooth $2\text{-}\mathcal{NS}$ through ρ_λ-orthogonality.

Theorem 2.6: *For any* $\lambda \in (0,1)\backslash\left\{\frac{1}{2}\right\}$*, the following statements are equivalent:*

(a) $\perp_{\mathcal{B}} \subseteq \perp_{\rho_\lambda}$,

(b) $\perp_{\mathcal{B}} = \perp_{\rho_\lambda}$,

(c) $\mathfrak{X}$ is smooth.

Proof: Using Theorem 2.5, we have (a) implies (b).

Next, we prove that (c) implies (a). Assume that $\mathfrak{X}$ is smooth and $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$. Then by using Theorem 1.8, we have $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) \leq 0 \leq \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$. Since $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ so, $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$. It implies that $\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ $+(1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$. So $\mathfrak{Q}_1 \perp_{\rho_\lambda} \mathfrak{Q}_2$ and $\perp_{\mathcal{B}} \subseteq \perp_{\rho_\lambda}$.

Now, we show that (b) implies (c). Suppose that $\perp_{\mathcal{B}} = \perp_{\rho\lambda}$ holds. We shall prove that $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$. Assume that $\mathfrak{Q}_1 \neq 0$, otherwise $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ trivially holds. Then there is a number $a \in \mathbb{R}$ so that $\mathfrak{Q}_1 \perp_{\mathcal{B}} (a\mathfrak{Q}_1 + \mathfrak{Q}_2)$. Therefore $\mathfrak{Q}_1 \perp_{\rho_\lambda} (a\mathfrak{Q}_1 + \mathfrak{Q}_2)$ and we have $\rho_\lambda(\mathfrak{Q}_1, a\mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$. Using Proposition 2.2, we get

$$a\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 + \lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0. \quad (2.1)$$

We also have $(-\mathfrak{Q}_1) \perp_{\mathcal{B}} (a\mathfrak{Q}_1 + \mathfrak{Q}_2)$ and so $\rho_\lambda(-\mathfrak{Q}_1, a\mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3) = 0$. Then

$$-a\|\mathfrak{Q}_1, \mathfrak{Q}_3\|^2 - (1-\lambda)\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) - \lambda\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0. \quad (2.2)$$

Adding (2.1) and (2.2) we have

$$(2\lambda - 1)\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-2\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = 0.$$

This implies that $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$, that is $\mathfrak{X}$ is smooth.

The example below demonstrates that in general $\perp_{\mathcal{B}} \not\subset \perp_{\rho_\lambda}$.

Example 2.7: Consider $\mathfrak{X} = \mathbb{R}^3$ equipped with the 2-norm defined by

$$\|\mathfrak{Q}_1, \mathfrak{Q}_2\| = \sum_{i=1}^{3} |\mathfrak{Q}_{1_i}\mathfrak{Q}_{2_{i+1}} - \mathfrak{Q}_{1_{i+1}}\mathfrak{Q}_{2_i}|,$$

where $\mathfrak{Q}_{1_4} = \mathfrak{Q}_{1_1}, \mathfrak{Q}_{2_4} = \mathfrak{Q}_{2_1}$.

Let $\mathfrak{Q}_1 = (0,-1,1)$, $\mathfrak{Q}_2 = (1,0,0)$, and $\mathfrak{Q}_3 = (0,1,2)$. Now, $\|\mathfrak{Q}_1, \mathfrak{Q}_3\| = 3$, $\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| = |t| + 3 + |2t|$. Clearly, $\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| \geq 3 = \|\mathfrak{Q}_1, \mathfrak{Q}_3\|$. Therefore $\mathfrak{Q}_1 \perp_{\mathcal{B}} \mathfrak{Q}_2$.

$$\begin{aligned}
\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \lim_{t\to 0^+} \frac{\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| - \|\mathfrak{Q}_1, \mathfrak{Q}_3\|}{t} \\
&= 8 \lim_{t\to 0^+} \frac{(t+3+2t)-3}{t} \\
&= 8 \lim_{t\to 0^+} \frac{3t+3-3}{t} \\
&= 24.
\end{aligned}$$

$$\begin{aligned}\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) &= \|\mathfrak{Q}_1,\mathfrak{Q}_3\| \lim_{t\to 0^-} \frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|}{t} \\ &= 8\lim_{t\to 0^+} \frac{(-t+3-2t)-3}{t} \\ &= 8\lim_{t\to 0^+} \frac{-3t+3-3}{t} \\ &= -24.\end{aligned}$$

Now, $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \lambda\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = -24\lambda + 24(1-\lambda) = 24(1-2\lambda) \neq 0$. Therefore, $\perp_{\mathcal{B}} \not\subseteq \perp_{\rho_\lambda}$.

Theorem 2.8: *For any* $\lambda \in (0,1)\backslash\left\{\frac{1}{2}\right\}$, *the subsequent statements are equivalent:*

(a) $\perp_\rho \subseteq \perp_{\rho_\lambda}$,

(b) $\perp_{\rho_\lambda} \subseteq \perp_\rho$,

(c) $\perp_{\rho_\lambda} = \perp_\rho$,

(d) $\mathfrak{X}$ *is smooth.*

Proof: At first, we show that (a) implies (b). We have $\mathfrak{Q}_1 \perp_\rho -\frac{\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. From (a) it follows that $\mathfrak{Q}_1 \perp_{\rho_\lambda} -\frac{\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. Using Proposition 2.2(iii), we have

$$\begin{aligned}&\rho_\lambda\left(\mathfrak{Q}_1, -\frac{\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2} \times \mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3\right) \\ &\quad = -\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) + \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = 0.\end{aligned}$$

Thus $\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. This implies that $\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$ and $\mathfrak{X}$ is smooth.

Now, we prove that (b) implies (c). Also, we have $\mathfrak{Q}_1 \perp_{\rho_\lambda} -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. From (b) it follows that $\mathfrak{Q}_1 \perp_\rho -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. Using property of ρ, we have

$$\begin{aligned}0 &= \rho\left(\mathfrak{Q}_1, -\frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2} \times \mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3\right) \\ &= -\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) + \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3).\end{aligned}$$

Thus $\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. This implies that $\perp_\rho = \perp_{\rho_\lambda}$.

Similarly, the other implications can be proved.

The following example shows that, if $\mathfrak{X}$ is not smooth, then $\perp_{\rho_\lambda} \neq \perp_\rho$.

Example 2.9: Consider $\mathfrak{X} = \mathbb{R}^3$ equipped with the 2-norm defined by

$$\|\mathfrak{Q}_1,\mathfrak{Q}_2\| = \sum_{i=1}^{3} |\mathfrak{Q}_{1_i}\mathfrak{Q}_{2_{i+1}} - \mathfrak{Q}_{1_{i+1}}\mathfrak{Q}_{2_i}|,$$

where $\mathfrak{Q}_{1_4} = \mathfrak{Q}_{1_1}, \mathfrak{Q}_{2_4} = \mathfrak{Q}_{2_1}$.

Let $\mathfrak{Q}_1 = (0,1,-1)$, $\mathfrak{Q}_2 = (2,0,0)$, and $\mathfrak{Q}_3 = (0,1,3)$. Now, $\|\mathfrak{Q}_1,\mathfrak{Q}_3\| = 3$, $\|\mathfrak{Q}_1 + t\mathfrak{Q}_2,\mathfrak{Q}_3\| = |2t| + 4 + |6t|$.

$$\begin{aligned}\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \lim_{t\to 0^+} \frac{\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| - \|\mathfrak{Q}_1, \mathfrak{Q}_3\|}{t} \\ &= 4 \lim_{t\to 0^+} \frac{(2t+4+6t)-4}{t} \\ &= 4 \lim_{t\to 0^+} \frac{8t}{t} \\ &= 32. \\ \rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) &= \|\mathfrak{Q}_1, \mathfrak{Q}_3\| \lim_{t\to 0^-} \frac{\|\mathfrak{Q}_1 + t\mathfrak{Q}_2, \mathfrak{Q}_3\| - \|\mathfrak{Q}_1, \mathfrak{Q}_3\|}{t} \\ &= 4 \lim_{t\to 0^+} \frac{(-2t+4-6t)-4}{t} \\ &= 4 \lim_{t\to 0^+} \frac{-8t}{t} \\ &= -32.\end{aligned}$$

Here $\rho(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \frac{-32+32}{2} = 0$

Now, $\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \lambda\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + (1-\lambda)\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = -32\lambda + 3(1-\lambda) = 32(1-2\lambda) \neq 0$. Therefore $\perp_\rho \not\subseteq \perp_{\rho_\lambda}$.

In the next result, we establish another way of describing smooth space.

Theorem 2.10: *For any* $\lambda \in (0,1)\setminus\left\{\frac{1}{2}\right\}$, *the subsequent assertions are equivalent.:*

(a) $\perp_{\rho_+} \subseteq \perp_{\rho_\lambda}$,

(b) $\perp_{\rho_\lambda} \subseteq \perp_{\rho_+}$,

(c) $\perp_{\rho_\lambda} = \perp_{\rho_+}$,

(d) $\mathfrak{X}$ is smooth.

Proof: First, we establish that (a) implies (d). We have $\mathfrak{Q}_1 \perp_{\rho_+} -\frac{\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. From (i) it follows that

$\mathfrak{Q}_1 \perp_{\rho_\lambda} -\frac{\rho(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}\mathfrak{Q}_1 + \mathfrak{Q}_2$. Using Proposition 2.2(iii), we have

$$\begin{aligned}0 &= \rho_\lambda(\mathfrak{Q}_1, -\frac{\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2} \times \mathfrak{Q}_1 + \mathfrak{Q}_2; \mathfrak{Q}_3) \\ &= -\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3).\end{aligned}$$

Thus $\rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$. This implies that $\rho_-(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) = \rho_+(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)$ and hence $\mathfrak{X}$ is smooth.

In the same way, the other implications can be shown to be true.

Theorem 2.11: *For any* $\lambda \in (0,1)\setminus\left\{\frac{1}{2}\right\}$, *the statement given bellow are equivalent:*

(a) $\perp_{\rho_-} \subseteq \perp_{\rho_\lambda}$,

(b) $\perp_{\rho_\lambda} \subseteq \perp_{\rho_-}$,

(c) $\perp_{\rho_\lambda} = \perp_{\rho_-}$,

(d) $\mathfrak{X}$ is smooth.

In a $2\text{-}\mathcal{NS}$, the equality

$$\|\mathfrak{Q}_1+\mathfrak{Q}_2,\mathfrak{Q}_3\|^4-\|\mathfrak{Q}_1-\mathfrak{Q}_2,\mathfrak{Q}_3\|^4$$
$$=8(\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)+\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3))$$

is a generalization of (1.1).

Next, we find the sufficient condition for the smoothness of a $2\text{-}\mathcal{NS}$.

Theorem 2.12: *Let* $(\mathfrak{X}.\|.,.\|)$ *be a* $2\text{-}\mathcal{NS}$ *and for all* $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3\in\mathfrak{X}$, $\lambda\in(0,1)\backslash\left\{\frac{1}{2}\right\}$, *we have*

$$\begin{aligned}&\|\mathfrak{Q}_1+\mathfrak{Q}_2,\mathfrak{Q}_3\|^4-\|\mathfrak{Q}_1-\mathfrak{Q}_2,\mathfrak{Q}_3\|^4\\&=8(\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)+\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3).\end{aligned}\tag{2.3}$$

Then $\mathfrak{X}$ is smooth.

Proof. Suppose that (2.3) holds. It follows that

$$\begin{aligned}&8\,(\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)+\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3))\\
&=\|\mathfrak{Q}_1+\mathfrak{Q}_2,\mathfrak{Q}_3\|^4-\|\mathfrak{Q}_1-\mathfrak{Q}_2\|^4\\
&=\lim_{t\to0^+}\left(\left\|(\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2)+\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^4-\left\|(\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2)-\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^4\right)\\
&=8\lim_{t\to0^+}\left(\left\|\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2\rho_\lambda\left(\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\right.\\
&\qquad\left.\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\rho_\lambda\left(\mathfrak{Q}_2,\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2;\mathfrak{Q}_3\right)\right)\\
&=8\lim_{t\to0^+}\left(\left\|\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2\rho_\lambda\left(\mathfrak{Q}_1+\tfrac{t}{2}q,\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\right.\\
&\qquad\left.\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\left(\tfrac{t}{2}\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2+\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)\right)\right)\\
&=8(\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\rho_\lambda\left(\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\\
&\qquad\|\mathfrak{Q}_2,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)).\end{aligned}$$

Therefore,

$$\lim_{t\to0^+}\rho_\lambda\left(\mathfrak{Q}_1+\tfrac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_2;\mathfrak{Q}_3\right)=\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3).\tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}&\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)\\
&=\lim_{t\to0^+}\frac{\|\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2-\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2t}\\
&=\lim_{t\to0^+}\frac{\left\|(\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2)+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2-\left\|(\mathfrak{Q}_1-\frac{t}{2}\mathfrak{Q}_2)+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2}{2t}\\
&=\lim_{t\to0^+}\frac{\left\|(\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2)+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^4-\left\|(\mathfrak{Q}_1-\frac{t}{2}\mathfrak{Q}_2)+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^4}{2t(\|(\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2+\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2)}\\
&=\lim_{t\to0^+}\frac{8\left(\left\|\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2\rho_\lambda\left(\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\frac{t}{2}\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\left\|\frac{t}{2}\mathfrak{Q}_2\right\|^2\rho_\lambda\left(\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2;\mathfrak{Q}_3\right)\right)}{2t(\|(\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2+\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2)}\end{aligned}$$

$$= \lim_{t\to 0^+} \frac{4t\left(\left\|\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2 \rho_\lambda\left(\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\left\|\frac{t}{2}\mathfrak{Q}_2\right\|^2 \rho_\lambda\left(\mathfrak{Q}_2,\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2;\mathfrak{Q}_3\right)\right)}{2t(\|(\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2+\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2)}$$

$$= 2\lim_{t\to 0^+} \frac{\left\|\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_3\right\|^2 \rho_\lambda\left(\mathfrak{Q}_1+\frac{t}{2}\mathfrak{Q}_2,\mathfrak{Q}_2;\mathfrak{Q}_3\right)+\frac{t^3}{8}\|\mathfrak{Q}_2\|^4+\frac{t^2}{4}\|\mathfrak{Q}_2\|\rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)}{(\|(\mathfrak{Q}_1+t\mathfrak{Q}_2,\mathfrak{Q}_3\|^2+\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2)}$$

$$= 2\times \frac{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)+0}{2\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2} = \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$$

Since $\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$, this implies that $\rho_+(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. Hence $\mathfrak{X}$ is smooth.

Theorem 2.13: $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$ *if and only if the norm is designed in* 3*-Dimensional subspace of* $\mathfrak{X}$ *comes from* 2-$\mathcal{INPS}$ *for all* $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3 \in \mathfrak{X}$ *and* $\lambda \in (0,1)\setminus\left\{\frac{1}{2}\right\}$.

Proof: Assume that the norm in $\mathfrak{X}$ is derived from a 2-$\mathcal{INPS}$. Then $\rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \rho_-(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)$. Therefore using the definition we have $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle$. By the property of 2-$\mathcal{IP}$ $\langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \langle\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle$ we have $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$.

Conversely, let $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$. This condition implies that $\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_{1-\lambda}(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$ for all $\mathfrak{Q}_1,\mathfrak{Q}_2 \in \mathfrak{X}$. Using Theorem 2.2 (ii), we have $\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = -\rho_\lambda(-\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_2,-\mathfrak{Q}_1;\mathfrak{Q}_3) = \rho_{1-\lambda}(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$.

Assume that, P be a three dimensional subpace of $\mathfrak{X}$. We define a mapping $\langle.,.|.\rangle: P\times P\times P \to \mathbb{R}$ by

$$\langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)+\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3)}{2}.$$

We prove that $\langle.,.|.\rangle$ is a 2-$\mathcal{IP}$ in P.

(i) Based on Proposition 2.2(iii), we obtain $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3) = \mathrm{P}\mathfrak{Q}_1,\mathfrak{Q}_3\mathrm{P}^2 \geq 0$. Also $\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3) \geq 0$. Therefore $\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle \geq 0$. We Know that $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3) = \rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3) = 0$ if and only if $\mathfrak{Q}_1$ and $\mathfrak{Q}_3$ are linearly dependent. Therefore $\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle = 0$ if and only if $\mathfrak{Q}_1$ and $\mathfrak{Q}_3$ are linearly dependent.

(ii) By the condition $\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_\lambda(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$ and $\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_2;\mathfrak{Q}_3) = \rho_{1-\lambda}(\mathfrak{Q}_2,\mathfrak{Q}_1;\mathfrak{Q}_3)$. Therefore $\langle\mathfrak{Q}_1,\mathfrak{Q}_2|\mathfrak{Q}_3\rangle = \langle\mathfrak{Q}_2,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle$.

(iii) Consider

$$\begin{aligned}\langle\mathfrak{Q}_1,\mathfrak{Q}_1|\mathfrak{Q}_3\rangle &= \frac{\rho_\lambda(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3)+\rho_{1-\lambda}(\mathfrak{Q}_1,\mathfrak{Q}_1;\mathfrak{Q}_3)}{2}\\ &= \frac{\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2+\|\mathfrak{Q}_1,\mathfrak{Q}_3\|^2}{2}\\ &= \frac{\|\mathfrak{Q}_3,\mathfrak{Q}_1\|^2+\|\mathfrak{Q}_3,\mathfrak{Q}_1\|^2}{2}\\ &= \frac{\rho_\lambda(\mathfrak{Q}_3,\mathfrak{Q}_3;\mathfrak{Q}_1)+\rho_{1-\lambda}(\mathfrak{Q}_3,\mathfrak{Q}_3;\mathfrak{Q}_1)}{2}\\ &= \langle\mathfrak{Q}_3,\mathfrak{Q}_3;\mathfrak{Q}_1\rangle.\end{aligned}$$

(iv) For $\alpha \geq 0$ we have

$$\begin{aligned}\langle \alpha\mathfrak{Q}_1, \mathfrak{Q}_2 | \mathfrak{Q}_3\rangle &= \frac{\rho_\lambda(\alpha\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_{1-\lambda}(\alpha\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)}{2} \\ &= \frac{\alpha\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \alpha\rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)}{2} \\ &= \alpha\langle \mathfrak{Q}_1, \mathfrak{Q}_2 | \mathfrak{Q}_3\rangle.\end{aligned}$$

For $\alpha < 0$ we have

$$\begin{aligned}\langle \alpha\mathfrak{Q}_1, \mathfrak{Q}_2 | \mathfrak{Q}_3\rangle &= \frac{\rho_\lambda(\alpha\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_{1-\lambda}(\alpha\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)}{2} \\ &= \frac{\alpha\rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \alpha\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)}{2} \\ &= \alpha\langle \mathfrak{Q}_1, \mathfrak{Q}_2 | \mathfrak{Q}_3\rangle.\end{aligned}$$

(v) It is sufficient to illustrate the additivity concerning the second variable. Take $\mathfrak{Q}_1, \mathfrak{Q}_2, w, \mathfrak{Q}_3 \in \mathfrak{X}$. We have

$$\begin{aligned}&\langle \mathfrak{Q}_1, \mathfrak{Q}_2 + w | \mathfrak{Q}_3\rangle \\ &= \frac{\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2 + w; \mathfrak{Q}_3) + \rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2 + w; \mathfrak{Q}_3)}{2} \\ &= \frac{1}{2}(\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_\lambda(\mathfrak{Q}_1, w; \mathfrak{Q}_3) \\ &\qquad + \rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_{1-\lambda}(\mathfrak{Q}_1, w; \mathfrak{Q}_3)) \\ &= \frac{1}{2}(\rho_\lambda(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3) + \rho_{1-\lambda}(\mathfrak{Q}_1, \mathfrak{Q}_2; \mathfrak{Q}_3)) + \\ &\qquad \frac{1}{2}(\rho_\lambda(\mathfrak{Q}_1, w; \mathfrak{Q}_3) + \rho_{1-\lambda}(\mathfrak{Q}_1, w; \mathfrak{Q}_3)) \\ &= \langle \mathfrak{Q}_1, \mathfrak{Q}_2 | \mathfrak{Q}_3\rangle + \langle \mathfrak{Q}_1, w | \mathfrak{Q}_3\rangle.\end{aligned}$$

Thus $\langle .,. | . \rangle$ is an 2-$\mathcal{IP}$ in P.

Conclusion

This article defines the ρ_λ functional in 2-$\mathcal{NS}$ via 2-norm derivatives and discusses some of its intriguing geometric properties. In addition, we give some characterizations of the smooth 2-$\mathcal{NS}$ which are based on ρ_λ-orthogonality.

In the future, is it possible to get a relation between the linear preserving mapping and ρ_λ-orthogonality in 2-$\mathcal{NS}$?

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