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Representation of convex sets

Stefan M. Stefanov
Department of Informatics
Neofit Rilski South-West University
Blagoevgrad 2700
Bulgaria

Abstract

Some results concerning representation of convex sets are stated and proved.

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1. Introduction

Definition 1: A vector $z \in \mathbb{R}^n$ is called a *recession direction* for the convex set X if $\forall\ x \in X$ and $\lambda \geq 0$, $x + \lambda z \in X$.

If X is a convex closed set, then in accordance with a known theorem, the following implication holds true

$$x_0 \in X, x_0 + \lambda z \in X \forall\ \lambda \geq 0 \text{ imply } x + \lambda z \in X \forall\ x \in X, \forall\ \lambda \geq 0.$$

Therefore z is a recession direction for the set X when X contains all rays whose origin is a point of X and whose direction is z.

Remark 1: Obviously $z = 0 \in \mathbb{R}^n$ is a recession direction because $x \in X$, $\lambda \geq 0$ imply $x + \lambda . 0 \equiv x \in X$.

Definition 2: The set of all recession directions of a convex closed set X is called the *recession cone* of the set X, denoted by $K(X)$:

$$K(X) = \{z \in \mathbb{R}^n : x + \lambda z \in X, x \in X, \lambda \geq 0\}.$$

Recall the definition of a cone ([19]).

The set $K \subseteq \mathbb{R}^n$ is called a *cone* (*with vertex at* $\mathbf{0}$) if $x \in K$, $t \geq 0$ imply $tx \in K$, i.e., if $x \in K$ then K also contains the ray $\Lambda = \{tx : t \geq 0\}$.

Each ray Λ with an origin at the vertex $\mathbf{0}$ of the cone K and which ray is contained in K is called *ray* of the cone K.

E-mail: **stefm@swu.bg**

Convex cone is a set which is convex and which is a cone.

The concepts and problems, discussed in the paper, and similar topics are considered in references [1]-[22], etc.

Some results concerning representation of convex sets are stated and proved in Section 2. In Section 3, an open problem for research is formulated.

2. Representation of convex sets

Theorem 1: *If $X \subseteq \mathbb{R}^n$ is a closed convex set then the following properties are satisfied.*

1) $K(X)$ *is a convex closed cone;*
2) X *is a unbounded set* $\Leftrightarrow$ $K(X) \neq \{0\}$;
3) $\hat{X} \neq \emptyset$ *if and only if* $K(X)$ *is a pointed cone.*

Proof:

1) If $z_1,\ z_2 \in K(X)$ and $\alpha \geq 0,\ \beta \geq 0$, then for arbitrary $x \in X,\ \lambda \geq 0$ we get $y = x + \lambda\alpha z_1 \in X$, $y + \lambda\beta z_2 \in X$. Then $x + \lambda(\alpha z_1 + \beta z_2) = y + \lambda\beta z_2 \in X$, that is, $\alpha z_1 + \beta z_2 \in K(X)$. Therefore $K(X)$ is a convex cone according to Theorem 1 ([19]).

   We must prove that $K(X)$ is a closed set. Let $\{z_k\}_{k=1}^{\infty} \to z^*,\ z_k \in K(X)$. Then for an arbitrary $x \in X$ and $\lambda \geq 0$ we have $x + \lambda z_k \in X,\ k = 1,2,\ldots$ Therefore $x + \lambda z^* \in X$ for every $\lambda \geq 0$ because X is closed by the assumption, that is, $z^* \in K(X)$, which means that $K(X)$ is a closed set by definition.

2) *Necessity.* Let X be a unbounded set. We will prove that there exists $z \neq 0, z \in K(X)$. Because the set X is unbounded by assumption, we can choose a sequence $\{x_k\}$ of points of X, such that $\|x_k\| \to \infty$ for $k \to \infty$. The sequence $\left\{\frac{x_k}{\|x_k\|}\right\}$ is bounded, therefore a convergent subsequence $\left\{\frac{x_{k_i}}{\|x_{k_i}\|}\right\} \to z, \|z\| = 1$ of this sequence can be selected in accordance with the theorem of Weierstrass-Bolzano. Then for arbitrary (and after that fixed) $x \in X,\ \lambda \geq 0$ we get

$$x + \lambda z = \lim_{i\to\infty}\left\{x + \frac{\lambda x_{k_i}}{\|x_{k_i}\|}\right\} \equiv \lim_{i\to\infty}\left\{x + \frac{\lambda}{\|x_{k_i}\|}\left(x_{k_i} - x\right) + \lambda\frac{x}{\|x_{k_i}\|}\right\}$$
$$= \lim_{i\to\infty}\left\{x + \frac{\lambda}{\|x_{k_i}\|}\left(x_{k_i} - x\right)\right\}.$$

   Since for sufficiently large values of k_i we have $0 < \frac{\lambda}{\|x_{k_i}\|} < 1$, then $x + \frac{\lambda}{\|x_{k_i}\|}(x_{k_i} - x) \in X$ because X is convex. Therefore $x + \lambda z \in X$ since X is closed, that is, $z \in K(X)$, hence $K(X) \neq \emptyset$.

   Sufficiency. The converse statement of 2) is obvious because if $K(X) \neq \{0\}$, then X is a unbounded set according to Definition 1 and Definition 2.

3) The set X contains the straight line $\{x + tb: t \in \mathbb{R}\}$ if and only if $K(X)$ contains the straight line $\{tb: t \in \mathbb{R}\}$. From this observation it immediately follows that $\hat{X} \neq \emptyset \Leftrightarrow \hat{K}(X) \neq \emptyset$, that is, if and only if $K(X)$ is a pointed cone by definition (Definition 4 [19]).

Remark 2: Statement 1) of Theorem 1 justifies the name of the concept of *recession cone* for $K(X)$ in Definition 2.

Lemma 1: *If X is a closed convex set in $\mathbb{R}^n$ and $\hat{X} \neq \emptyset$, then each relative interior point of X is represented as convex combination of two relative boundary points of X or in the form $\overline{x} + \overline{z}$, where $\overline{x} \in r\partial X$, $\overline{z} \in K(X)$.*

Proof: By the assumption, $x \in \mathrm{ri}\, X$. Let $z \in \mathbb{R}^n$, $z \neq 0$.

Construct the straight line $\{x + tz: t \in \mathbb{R}\} \subseteq \mathrm{a}ffX$ through the point x, where aff X denotes the affine hull of the set X (cf. Definition 8 [21]).

Denote

$$t_1 = \sup\{t: x + tz \in X\}\ (> 0),$$
$$t_2 = \sup\{t: x - tz \in X\}\ (> 0).$$

The following two cases are possible.

Case 1. $0 < t_1 < \infty$, $0 < t_2 < \infty$ (that is, the intersection of the straight line $\{x + tz: t \in \mathbb{R}\}$ and the set X is a closed line segment, which is a compact set). Then $x_1 \overset{\text{def}}{=} x + t_1 z$ and $x_2 \overset{\text{def}}{=} x - t_2 z$ are relative boundary points of X and

$$x = \frac{t_2}{t_1 + t_2} x_1 + \frac{t_1}{t_1 + t_2} x_2, \qquad x_1, x_2 \in r\,\partial X,$$

where

$$\frac{t_1}{t_1 + t_2} > 0, \quad \frac{t_2}{t_1 + t_2} > 0, \quad \frac{t_1}{t_1 + t_2} + \frac{t_2}{t_1 + t_2} = 1$$

according to Theorem 3 ([22]), that is, in Case 1, a relative interior point of X is represented as a convex combination of two relative boundary points.

Case 2: Only one of t_1, t_2 is finite. Let, for example, $t_2 < \infty$. Then $\overline{x} \overset{\text{def}}{=} x - t_2 z \in r\,\partial X$ according to definition of t_2 and using that $z \in K(X)$. Hence $x = \overline{x} + t_2 z \overset{\text{def}}{=} \overline{x} + \overline{z}$, where $\overline{z} \overset{\text{def}}{=} t_2 z \in K(X)$ because $t_2 \geq 0$, $z \in K(X)$, and $K(X)$ is a cone.

The case when $t_1 = t_2 = \infty$ is impossible because $\hat{X} \neq \emptyset$, that is, X does not contain a straight line according to Theorem 2 ([22]).

Theorem 2: *If $X \subseteq \mathbb{R}^n$ is closed and convex and $\hat{X} \neq \emptyset$, then $X = co\,\hat{X} + K(X)$.*

Proof: The equality of the sets X and $co\,\hat{X} + K(X)$ will be proved by the two-way inclusion.

1) Because X is a convex set then $co\,\hat{X} \subseteq X$, and if $x \in X$, $z \in K(X)$, then $x + z \in X$. Therefore $co\,\hat{X} + K(X) \subseteq X$.
2) The converse inclusion will be proved by induction on the dimension of X.

Let $\dim X = 1$. Under the assumptions made, the one-dimensional sets X are closed line segments and rays, because the set X cannot be a straight line according to Theorem 2 ([22]) since $\hat{X} \neq \emptyset$ by assumption. In this case, using an extreme point and a recession direction, we obtain the whole set X, i.e., $X \subseteq co\,\hat{X} + K(X)$.

Induction hypothesis. Assume that the statement of the converse inclusion of Theorem 2 is true for the sets with dimension $\dim X < k$, i.e., let $X \subseteq co\,\hat{X} + K(X)$.

We have to prove this statement for $\dim X = k$.

Taking into account the representation of the relative interior points, it suffices to prove that each relative boundary point of the set X is represented as a convex combination of extreme points and a recession direction.

a) Let $\overline{x} \in r\,\partial X$. Construct a supporting hyperplane H through the point $\overline{x}$, such that $X \not\subseteq H$, according to Theorem 7 ([21]). Consider the convex closed set $X_1 \equiv X \cap H$, $dim\, X_1 < k$, $\hat{X}_1 \neq \emptyset$. If we assume that X_1 does not have an extreme point, that is, if $\hat{X}_1 = \emptyset$, then X_1 would contain a straight line according to Theorem 2 ([22]), hence X would also contain a straight line and again according to Theorem 2 ([22]), we would have $\hat{X} = \emptyset$, a contradiction with the hypothesis of Theorem 2.

   Since $\overline{x} \in H$, and since $\overline{x} \in r\,\partial X$ implies $\overline{x} \in X$, then $\overline{x} \in X_1 \overset{\text{def}}{=} H \cap X$.

   From the Induction hypothesis, applied to $\overline{x}$ and the set X_1 with dimension $\dim X_1 < k$, we have

$$\overline{x} \in co\hat{X}_1 + K(X_1) \subseteq co\hat{X} + K(X),$$

   where for the second inclusion we have used that $\hat{X}_1 \subseteq \hat{X}$ according to Theorem 1 ([22]), and

$$K(X_1) \equiv K(H \cap X) = K(H) \cap K(X) \subseteq K(X).$$

   We have proved that $\overline{x} \in r\,\partial X$ implies $\overline{x} \in co\hat{X} + K(X)$, that is, $r\,\partial X \subseteq co\hat{X} + K(X)$.

   Using the Principle of mathematical induction, it follows that the statement, which has to be proved, is true for the set X with $\dim X = k$ for every positive integer k.

b) Let $x \in ri\,X$. According to Lemma 1, there are two possible subcases.

   b1) $x = \alpha x_1 + \beta x_2$, $x_1, x_2 \in r\,\partial X$, where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$. Therefore $x \in co\,\hat{X} + K(X)$ according to case a).

b2) $x = \overline{x} + \overline{z}$, $\overline{x} \in r\,\partial X$, $\overline{z} \in K(X)$. Since $\overline{x} \in r\,\partial X$ then by case a) we get $\overline{x} = \hat{x} + z_1$, $\hat{x} \in co\,\hat{X}$, $z_1 \in K(X)$. Therefore

$$x = \overline{x} + \overline{z} = (\hat{x} + z_1) + \overline{z} = \hat{x} + (z_1 + \overline{z}) \in co\,\hat{X} + K(X).$$

From subcases b1) and b2) we have that $\mathrm{r}i\,X \subseteq co\,\hat{X} + K(X)$, the case b).

Cases a) and b), which cover all possible cases, imply $X \subseteq co\,\hat{X} + K(X)$.

The two opposite inclusions, established in 1) and 2), imply $X = co\,\hat{X} + K(X)$.

Remark 3: When the set X is not only closed but also bounded (that is, when X is compact), then $K(X) = \{0\}$, and Theorem 2 implies $X = co\,\hat{X}$, that is, the Minkowski-Krein-Milman theorem (Theorem 4 [22]) is a corollary of Theorem 2.

Definition 3: Two subspaces S and T of $\mathbb{R}^n$ are said to be *complementary* if $S + T = \mathbb{R}^n$, where" $+$ " is notation of the Minkowski sum of sets, and $S \cap T = \{0\}$. Each of these subspaces is a *complement* to the other subspace.

Two subspaces S and T of $\mathbb{R}^n$ are complementary (complemented) if and only if $S \cap T = \{0\}$ and $\dim S + \dim T = n$.

Each point $x \in \mathbb{R}^n$ can be presented as follows

$$x = x_1 + x_2, \tag{1}$$

where $x_1 \in S$, $x_2 \in T$, and this presentation is unique.

The points x_1 and x_2 are called the *projections* of x onto the subspaces S and T, respectively, and this is denoted as follows

$$x_1 = \mathrm{p}r_S x, \qquad x_2 = \mathrm{p}r_T x.$$

Denote by

$$x_1 = \mathrm{p}r_S X, \qquad x_2 = \mathrm{p}r_T X$$

the sets of projections of the points of X onto S and T, respectively.

Definition 4: Let X be convex and closed set. The set

$$S(X) \overset{\text{def}}{=} K(X) \cap \{-K(X)\}$$

is called the *recession subspace* of the set X.

Remark 4: From Definition 4 and Remark 1 it follows that $s = 0 \in \mathbb{R}^n$ always belongs to $S(X)$ because $0, -0 \in K(X)$.

The following **properties** are an immediate consequence of Definition 4.

1) $S(X)$ is a subspace (this justifies the name of the concept of *recession subspace*);
2) $s \in S(X) \Leftrightarrow s \in K(X)$ and $-s \in K(X)$;
3) $\hat{X} \neq \emptyset$ if and only if $S(X) = \{0\}$ (also using 3) of Theorem 1);
4) $S(K(X)) = S(X)$.

Theorem 3: *If X is a convex closed set in $\mathbb{R}^n$ and T is a complement of the subspace $S(X)$, then the following statements hold true.*

1) $K(X) \cap T$ *is a pointed cone;*
2) $\widehat{X \cap T} \neq \emptyset$, i.e., $X \cap T$ *has at least* 1 *extreme point;*
3) $X = X \cap T + S(X)$;
4) a) $\mathrm{pr}_T X = X \cap T$; b) $\mathrm{pr}_S X = S(X)$.

Proof:

1) If we assume the contrary, that $K(X) \cap T$ is not a pointed cone under the hypothesis of Theorem 3, then there will exist $z \neq 0$ with $z \in K(X) \cap T$ and $-z \in K(X) \cap T$. Then we would have
   $0 \neq z \in [K(X) \cap T] \cap [-K(X) \cap T] = K(X) \cap (-K(X)) \cap T = S(X) \cap T.$
   However, $S(X) \cap T = \{0\}$ because T is a complementary subspace of $S(X)$, a contradiction with the above conclusion. Therefore the assumption is wrong, therefore $K(X) \cap T$ is a pointed cone.
2) Because X and T are closed convex sets, and $X \cap T \neq \emptyset$, then $K(X \cap T) = K(X) \cap T$, and $K(X) \cap T$ is a pointed cone according to statement 1) of Theorem 3. From this conclusion and from Theorem 1, part 3), it follows that $\widehat{X \cap T} \neq \emptyset$.
3) The statement will be proved by two-way inclusion of X and $X \cap T + S(X)$.
   a) Under the hypothesis of Theorem 3 we obtain $X \cap T + S(X) \subseteq X + S(X) \subseteq X$.
   b) Let $x \in X \subset \mathbb{R}^n$. Therefore $x = s + t$, $s \in S(X)$, $t \in T$, and this representation follows from the hypothesis of Theorem 3 that T is a complementary subspace of $S(X)$. Hence $t = x - s \equiv x + 1.(-s) \in X$, where we have used Definition 1 of recessive direction and the fact that $-s$ is a recession direction according to Definition 4 of $S(X)$. Since $t \in T$, $t \in X$, then $t \in X \cap T$. Therefore $x = t + s \in X \cap T + S(X)$ for each $x \in X$. Because $x \in X$ was arbitrarily chosen, then $X \subseteq X \cap T + S(X)$.

   The two opposite inclusions, established in a) and b), imply that $X = X \cap T + S(X)$.
4) a) The statement will be proved by two-way inclusion of the sets $\mathrm{pr}_T X$ and $X \cap T$.
   a1) Obviously $X \cap T \subseteq \mathrm{pr}_T X$.
   a2) Let $x_1 \in \mathrm{pr}_T X$, that is, $x_1 \in T$ and there exist points $x \in X$, $x_2 \in S(X)$ such that $x = x_1 + x_2$. Therefore $x_1 = x - x_2 \in X$. From $x_1 \in T$ and $x_1 \in X$ we have that $x_1 \in X \cap T$. Hence $\mathrm{pr}_T X \subseteq X \cap T$.

   The two opposite inclusions, established in a1) and a2), imply that $\mathrm{pr}_T X = X \cap T$.

b) The statement will be proved by two-way inclusion of the sets $\mathrm{p}r_S X$ and $S(X)$.

b1) Obviously $\mathrm{p}r_S X \subseteq S(X)$.

b2) Let $x_2 \in S(X)$ and $x_1 \in X \cap T$. Then $x \overset{\text{def}}{=} x_1 + x_2 \in X \cap T + S(X) \overset{3)}{=} X$, that is, $x_2 \in \mathrm{p}r_S X$. Hence $S(X) \subseteq \mathrm{p}r_S X$.

The two opposite inclusions, established in b1) and b2), imply that $\mathrm{p}r_S X = S(X)$.

An alternative proof of 3) b) by using statement 4) of Theorem 3.

From definition of projection of a set over a subspace and from representation (1) we get

$$X \subseteq \mathrm{p}r_T X + \mathrm{p}r_S X \overset{4)}{=} X \cap T + S(X).$$

Theorem 4: *If X is a convex closed set in $\mathbb{R}^n$, T is a complement of $S(X)$ and $X_1 = X \cap T$, then $X = co\, \hat{X}_1 + K(X_1) + S(X)$.*

Proof: According to Theorem 3, part 3), we have $X = X_1 + S(X)$, and according to Theorem 3, part 2), we have $\hat{X}_1 \equiv \widehat{X \cap T} \neq \emptyset$. Therefore $X_1 = \mathrm{c}o\, \hat{X}_1 + K(X_1)$ by Theorem 2. Then

$$X = X_1 + S(X) = co\, \hat{X}_1 + K(X_1) + S(X).$$

3. Conclusion

In this paper, some results concerning representation of convex sets are stated and proved.

Open problem for future research is to establish other results regarding the considered topic.

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