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Relation-theoretic fuzzy metric fixed point results through $\mathcal{R}$-$\mathcal{FZ}$-contractions

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Abstract

We explore $\mathcal{R}$-$\mathcal{FZ}$-contractions in fuzzy metric spaces endowed with a binary relation $\mathcal{R}$, deriving fixed point theorems via fuzzy simulation functions. Illustrative examples and corollaries confirm the validity and applicability of the approach, which extends and refines several existing fixed point results.

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1. Introduction

Fixed point (FP) theory remains a central branch of nonlinear functional analysis, providing fundamental techniques to address problems arising in diverse mathematical contexts. The classical Banach contraction principle is the cornerstone of metric fixed point results and has been refined through various extensions in generalized metric frameworks. Recently, Khojasteh, Shukla, and Radenović [16] introduced the concept of simulation functions, offering a new perspective for developing FP results.

Another growing direction is the relation-theoretic approach, initially proposed by Turinici [27] and later extended by Ran and Reurings [28], who formulated an order-theoretic version of the Banach contraction principle with applications to matrix equations. Alam and Imdad [11] further generalized these

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findings through banari relations, inspiring numerous results under different relational structures [20-22, 24-26].

The emergence of fuzzy set theory by Zadeh [1] provided a natural framework to model uncertainty and imprecision. Building on this idea, Kramosil and Michalek [6] introduced fuzzy metric spaces, later refined by George and Veeramani [3] to ensure a Hausdorff topology. In such spaces, the degree of proximity between two points depends on a parameter $\theta > 0$, reflecting real-life variability. For further examples of fuzzy metrics and their applications, we refer the reader to [5].

Considerable research has been devoted to FP results within fuzzy settings. Gregori and Sapena [4] studied fuzzy contractive mappings, while Mihet [14] and Wardowski [15] introduced ψ- and $\mathcal{H}$-contractive mappings, respectively. Melliani and Moussaoui later proposed the fuzzy simulation approach through $\mathcal{FZ}$-contractions, followed by several developments in various contraction forms [2, 10, 18, 19, 21, 23, 29-41].

In this paper, we extend these ideas by defining the notion of $\mathcal{R}$-$\mathcal{FZ}$-contractions on fuzzy metric spaces endowed with a binary elation. The main theorems, supported by illustrative examples and consequences, unify and broaden several existing results in the literature.

2. Preliminaries

Definition 1: [7] A continuous operation $*: [0,1] \times [0,1] \to [0,1]$ is termed a continuous t-norm (CTN) if it is commutative, associative, and

1. For every $\tau_1 \in [0,1]$, $\tau_1 * 1 = \tau_1$,
2. For all $\tau_1, \tau_2, \tau_3, \tau_4 \in [0,1]$, if $\tau_1 \leq \tau_3$ and $\tau_2 \leq \tau_4$, then $\tau_1 * \tau_2 \leq \tau_3 * \tau_4$.

Example 1: $\mu_1 *_m \mu_2 = \min\{\mu_1, \mu_2\}$; $\mu_1 *_L \mu_2 = \max\{\mu_1 + \mu_2 - 1, 0\}$; $\mu_1 *_p \mu_2 = \mu_1 \cdot \mu_2$, for all $\mu_1, \mu_2 \in [0,1]$.

Definition 2: [2] Assume that F is a non-empty set, $*$ a CTN, and $\mathcal{J}: \mathcal{F}^2 \times (0,\infty) \to [0,1]$ a fuzzy set. The triple $(\mathrm{F}, \mathcal{J}, *)$ is termed a fuzzy metric space (FMS) if for all $s, t, r \in \mathrm{F}$, $\theta, \kappa > 0$, we have:

($\mathcal{F}1$) $\mathcal{J}(s,t,\theta) > 0$,

($\mathcal{F}2$) $\mathcal{J}(s,t,\theta) = 1$ iff $s = t$,

($\mathcal{F}3$) $\mathcal{J}(s,t,\theta) = \mathcal{J}(t,s,\theta)$,

($\mathcal{F}4$) $\mathcal{J}(s,t,\theta) * \mathcal{J}(t,r,\kappa) \leq \mathcal{J}(s,r,\theta+\kappa)$,

($\mathcal{F}5$) $\theta \mapsto \mathcal{J}(s,t,\theta)$ is continuous from $(0,\infty)$ into $[0,1]$.

Lemma 1: [2] *$\mathcal{J}(u,v,.)$ is nondecreasing for all u, v in F.*

Definition 3: [3] Consider $(F, \mathcal{J}, *)$ be a FMS and a sequence $\{u_q\} \subseteq F$.

1. $\{u_q\} \subseteq F$ is said to converge to $u \in F$ iff

$$\lim_{q\to\infty} \mathcal{J}(u_q, u, \theta) = 1 \quad \forall \theta > 0.$$

2. $\{u_q\} \subseteq F$ is termed a Cauchy sequence if $\forall \varepsilon \in (0,1)$ and $\theta > 0$, $\exists q_0 \in \mathbb{N}$ such that
$$\mathcal{J}(u_q, u_p, \theta) > 1 - \varepsilon \quad \forall q, p \geq q_0.$$
3. A FMS is termed complete if every Cauchy sequence in F converges.

Definition 4: ([17-19]) A function $\Im: (0,1] \times (0,1] \to \mathbb{R}$ is termed an $\mathcal{FZ}$-simulation function if it satisfies the following conditions:

($\Im$1) : $\Im(1,1) = 0$;

($\Im$2) : $\Im(u, v) < \frac{1}{v} - \frac{1}{u}$ for every $u, v \in (0,1)$;

($\Im$3) : For sequences $\{u_q\}$ and $\{v_q\}$ in $(0,1]$ such that $\lim_{q\to\infty} u_q = \lim_{q\to\infty} v_q < 1$, it holds that
$$\limsup_{q\to\infty} \Im(u_q, v_q) < 0.$$

We denote the family of all such functions by $\mathcal{FZ}$.

Definition 5: ([17-19]) Consider a FMS $(\mathrm{F}, \mathcal{J}, *)$, $\hbar: \mathrm{F} \to \mathrm{F}$ a mapping and $\Im \in \mathcal{FZ}$. Then $\hbar$ is termed an $\mathcal{FZ}$-contraction w.r.t $\Im$ if
$$\Im(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{J}(u, v, \theta)) \geq 0, \forall u, v \in \hat{o}, \theta > 0.$$

Example 2: ([17-19]). Let $\Im_i: (0,1] \times (0,1] \to \mathbb{R}$ for $i = 1,2,3$ be the following functions:

Each of the above functions $\Im_i$ $(i = 1,2,3)$ satisfies the required conditions to be regarded as an $\mathcal{FZ}$-simulation function.

Definition 6: ([8]). Let F be a nonempty set. A subset $\mathcal{R} \subseteq F \times F$ is called a binary relation on F. Moreover, if either $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$, we denote this by $[u, v] \in \mathcal{R}$.

In the study of FP theory and related structures, the concept of a binary relation plays an essential role in establishing various generalized contraction frameworks. Let F be a nonempty set and $\mathcal{R}$ a binary relation on F. The relation $\mathcal{R}$ is said to be reflexive if $u \,\mathcal{R}\, u$ for all $u \in F$; transitive if $u \,\mathcal{R}\, v$ and $v \,\mathcal{R}\, w$ together imply $u \,\mathcal{R}\, w$, for all $u, v, w \in F$; antisymmetric if $u \,\mathcal{R}\, v$ and $v \,\mathcal{R}\, u$ imply $u = v$, for all $u, v \in F$; complete if $[u, v] \in \mathcal{R}$ for all $u, v \in F$; and $\hbar$-closed if $(u, v) \in \mathcal{R}$ implies $(\hbar u, \hbar v) \in \mathcal{R}$ for all $u, v \in F$, where $\hbar$ is a self-mapping on F, see [8, 9].

Definition 7: ([11]). Let F be a nonempty set and let $\mathcal{R}$ be a binary relation on F. A sequence $\{u_q\} \subseteq F$ is called $\mathcal{R}$-preserving (denoted by $\mathcal{R}$P) if $(u_q, u_{q+1}) \in \mathcal{R}$ for all $q \in \mathbb{N}$.

Definition 8: ([12]). A binary relation $\mathcal{R}$ on F is said to be $\mathcal{J}$-self-closed (abbreviated as $\mathcal{J}$-SC) if for every $\mathcal{R}$P sequence $\{u_q\} \subseteq F$ satisfying

$$u_q \xrightarrow[q \to \infty]{\mathcal{J}} u,$$

there exists a subsequence $\{u_{q_k}\}$ of $\{u_q\}$ such that $(u_{q_k}, u) \in \mathcal{R}$.

Definition 9: ([24]). Let $(F, \mathcal{J}, *)$ be a fuzzy metric space (FMS) and $\mathcal{R}$ a binary relation on F. A sequence $\{u_q\} \subseteq F$ is said to be $\mathcal{R}$-Cauchy if $(u_q, u_{q+1}) \in \mathcal{R}$ for all $q \in \mathbb{N}$, and for every $\varepsilon \in (0,1)$ and $\theta > 0$, there exists $q_0 \in \mathbb{N}$ such that

$$\mathcal{J}(u_{q+p}, u_q, \theta) > 1 - \varepsilon \quad \text{for all } q \geq q_0 \text{ and } p \in \mathbb{N}.$$

Remark 1: ([24]). For any arbitrary binary relation $\mathcal{R}$ on F, every Cauchy sequence is also an $\mathcal{R}$-Cauchy sequence. Moreover, the notions of $\mathcal{R}$-Cauchyness and ordinary Cauchyness coincide whenever $\mathcal{R}$ is taken to be the universal relation.

Definition 10: ([24]). A fuzzy metric space $(F, \mathcal{J}, *)$ endowed with a binary relation $\mathcal{R}$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-Cauchy sequence in ô is convergent.

Remark 2: ([24]). For any arbitrary binary relation $\mathcal{R}$ on F, every complete FMS is also $\mathcal{R}$-complete.

Definition 11: [13]. Let F be a nonempty set and $\mathcal{R}$ a binary relation on F. For any $a, b \in F$, a sequence $\{\delta_0, \delta_1, \ldots, \delta_h\} \subset \mathrm{F}$ is called a path of length h in $\mathcal{R}$ from a to b if the following conditions are satisfied:

$(\mathcal{W}1)$ $\delta_0 = a$ and $\delta_h = b$;

$(\mathcal{W}2)$ $(\delta_j, \delta_{j+1}) \in \mathcal{R}$ for all j such that $0 \leq j \leq h - 1$.

Let F be a nonempty set, $\mathcal{S} \subseteq F$, and let $\hbar: F \to F$ be a self-mapping. The following notations will be used throughout this paper:

$$F(\hbar, \mathcal{R}) := \{\, u \in F : (u, \hbar u) \in \mathcal{R} \,\},$$

$$\gamma(a, b, \mathcal{R}) := \text{the set of all paths in } \mathcal{R} \text{ connecting } a \text{ and } b,$$

and

$$\mathcal{R}|_{\mathcal{S}} := \mathcal{R} \cap \mathcal{S}^2,$$

which denotes the restriction of $\mathcal{R}$ to $\mathcal{S}$. In other words, $\mathcal{R}|_{\mathcal{S}}$ represents the relation on $\mathcal{S}$ induced by $\mathcal{R}$.

Definition 12: Let $(F, \mathcal{J}, *)$ be an FMS and let $\mathcal{R}$ be a binary relation on F. A mapping $\hbar: F \to F$ is said to be $\mathcal{R}$-continuous (abbreviated as $\mathcal{R}$-C) at a point $u \in F$ if for every $\mathcal{R}$P sequence $\{u_q\} \subseteq \mathrm{F}$ satisfying

$$u_q \xrightarrow[q \to \infty]{\mathcal{J}} u,$$

it follows that

$$\hbar u_q \xrightarrow[q \to \infty]{\mathcal{J}} \hbar u.$$

The mapping $\hbar$ is called $\mathcal{R}$-continuous on F if it is $\mathcal{R}$-continuous at every point of F.

Remark 3: Every continuous mapping is $\mathcal{R}$-C for any binary relation $\mathcal{R}$. In particular, when $\mathcal{R}$ is taken as the universal relation, $\mathcal{R}$-C coincides with the usual notion of continuity.

To develop our main results, we shall make use of the following lemma.

Lemma 2: ([24]). Let $\hbar: F \to F$ be a self-mapping, and let $\mathcal{R}$ be a transitive and $\hbar$-closed binary relation on F. Assume that there exists an element $u_0 \in F$ such that $u_0 \,\mathcal{R}\, \hbar u_0$. Define a sequence $\{u_q\}$ in F recursively by $u_q = \hbar u_{q-1}$ for all $q \in \mathbb{N}$. Then the following holds:

$$u_p \,\mathcal{R}\, u_q \quad \text{for all } p, q \in \mathbb{N} \text{ with } q < p.$$

3. Main results

Definition 13: Let $(F, \mathcal{J}, *)$ be a fuzzy metric space (FMS), $\mathcal{R}$ a binary relation on F, and $\hbar: F \to F$ a self-mapping. The mapping $\hbar$ is said to be an $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\mathfrak{I} \in \mathcal{FZ}$ if

$$\mathfrak{I}(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{O}(u, v, \theta)) \geq 0, \quad \forall\, u, v \in F, \theta > 0, \text{with } (u, v) \in \mathcal{R}, \qquad (1)$$

where $\mathcal{O}(u, v, \theta) = \min\{\, \mathcal{J}(u, v, \theta), \mathcal{J}(u, \hbar u, \theta), \mathcal{J}(v, \hbar v, \theta) \,\}$.

Remark 1 Note that if $(F, \mathcal{J}, *)$ is an FMS, $\mathcal{R}$ a binary relation on F, and $\hbar: F \to F$ an $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\mathfrak{I} \in \mathcal{FZ}$, then the following two statements complement each other; $(\mathcal{E}_1)$: $\mathfrak{I}(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{O}(u, v, \theta)) \geq 0$ for all $u, v \in$ F such that $(u, v) \in \mathcal{R}$; and $(\mathcal{E}_2)$: $\mathfrak{I}(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{O}(u, v, \theta)) \geq 0$ for all $u, v \in$ F such that $[u, v] \in \mathcal{R}$.

The implication $(\mathcal{E}_2) \Rightarrow (\mathcal{E}_1)$ is immediate. Conversely, if $(\mathcal{E}_1)$ holds, then $(\mathcal{E}_2)$ follows because whenever $(v, u) \in \mathcal{R}$, the symmetry of $\mathcal{J}$ ensures that

$$\mathfrak{I}(\mathcal{J}(\hbar v, \hbar u, \theta), \mathcal{O}(v, u, \theta)) = \mathfrak{I}(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{O}(u, v, \theta)) \geq 0,$$

and we also have $\mathcal{O}(u, v, \theta) = \mathcal{O}(v, u, \theta)$, since

$$\begin{aligned}\mathcal{O}(u, v, \theta) &= \min\{\mathcal{J}(u, v, \theta), \mathcal{J}(u, \hbar u, \theta), \mathcal{J}(v, \hbar v, \theta)\} \\ &= \min\{\mathcal{J}(v, u, \theta), \mathcal{J}(v, \hbar v, \theta), \mathcal{J}(u, \hbar u, \theta)\}.\end{aligned}$$

Therefore, $(\mathcal{E}_1)$ also implies $(\mathcal{E}_2)$.

Theorem 1: *Let* $(\hat{o}, \mathcal{J}, *)$ *be an FMS equipped with a binary relation* $\mathcal{R}$, *and let* $\hbar: F \to F$ *be a self-mapping. Assume that the following conditions hold:*

(i) There exists a subset $\mathcal{S} \subseteq F$ such that $\hbar(F) \subseteq \mathcal{S} \subseteq F$, and $(\mathcal{S}, \mathcal{J}, *)$ is $\mathcal{R}$-complete.

(2) $F(\hbar, \mathcal{R}) \neq \emptyset$.

(iii) $\mathcal{R}$ is both $\hbar$-closed and transitive.
(iv) $\hbar$ is an $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t some $\Im \in \mathcal{FZ}$.
(v) Either $\mathcal{R}|_{\mathcal{S}}$ is $\mathcal{J}$-SC or $\hbar$ is *RC*.
Then $\hbar$ has a *FP* in *F*.

Proof: Since $F(\hbar, \mathcal{R}) \neq \emptyset$, choose an element $u_0 \in F(\hbar, \mathcal{R})$. Construct as equence $\{u_q\}$ in *F* by the recursive rule $u_{q+1} = \hbar(u_q)$ for every $q \in \mathbb{N}$. Because $\mathcal{R}$ is $\hbar$-closed and $(u_0, \hbar u_0) \in \mathcal{R}$, it follows that

$$(\hbar u_0, \hbar^2 u_0), (\hbar^2 u_0, \hbar^3 u_0), \ldots, (\hbar^q u_0, \hbar^{q+1} u_0) \in \mathcal{R}.$$

Hence,

$$\left(u_q, u_{q+1}\right) \in \mathcal{R} \quad \text{for all } q \in \mathbb{N}, \tag{3.2}$$

which means that $\{u_q\}$ is an $\mathcal{R}$P sequence.

Since $\hbar$ satisfies the $\mathcal{R}$-$\mathcal{FZ}$-contraction condition associated with $\Im \in \mathcal{FZ}$, we obtain

$$\Im\big(\mathcal{J}(\hbar u_{q-1}, \hbar u_q, \theta), \mathcal{O}(u_{q-1}, u_q, \theta)\big) \geq 0, \tag{3.3}$$

where

$$\begin{aligned}\mathcal{O}(u_{q-1}, u_q, \theta) &= \min\{\, \mathcal{J}(u_{q-1}, u_q, \theta), \mathcal{J}(u_{q-1}, \hbar u_{q-1}, \theta), \mathcal{J}(u_q, \hbar u_q, \theta) \,\} \\ &= \min\{\, \mathcal{J}(u_{q-1}, u_q, \theta), \mathcal{J}(u_q, u_{q+1}, \theta) \,\}.\end{aligned}$$

Suppose that, for some $q \in \mathbb{N}$, we have $\mathcal{O}(u_{q-1}, u_q, \theta) = \mathcal{J}(u_q, u_{q+1}, \theta)$. Then, using (3.3) together with property $(\Im 2)$, we obtain

$$\begin{aligned}0 &\leq \Im\big(\mathcal{J}(\hbar u_{q-1}, \hbar u_q, \theta), \mathcal{O}(u_{q-1}, u_q, \theta)\big) \\ &= \Im\big(\mathcal{J}(u_q, u_{q+1}, \theta), \mathcal{J}(u_q, u_{q+1}, \theta)\big) \\ &< \frac{1}{\mathcal{J}(u_q, u_{q+1}, \theta)} - \frac{1}{\mathcal{J}(u_q, u_{q+1}, \theta)}.\end{aligned}$$

This is clearly a contradiction. Hence, for every $q \in \mathbb{N}$, it must hold that $\mathcal{O}(u_{q-1}, u_q, \theta) = \mathcal{J}(u_{q-1}, u_q, \theta)$. Consequently,

$$\begin{aligned}0 &\leq \Im\big(\mathcal{J}(\hbar u_{q-1}, \hbar u_q, \theta), \mathcal{O}(u_{q-1}, u_q, \theta)\big) \\ &= \Im\big(\mathcal{J}(u_q, u_{q+1}, \theta), \mathcal{J}(u_{q-1}, u_q, \theta)\big) \\ &< \frac{1}{\mathcal{J}(u_{q-1}, u_q, \theta)} - \frac{1}{\mathcal{J}(u_q, u_{q+1}, \theta)}.\end{aligned}$$

From the preceding inequality, we have

$$\mathcal{J}(u_{q-1}, u_q, \theta) < \mathcal{J}(u_q, u_{q+1}, \theta),$$

which shows that the sequence $\{\mathcal{J}(u_{q-1}, u_q, \theta)\}$ is nondecreasing in $(0,1]$. Hence, there exists $l(\theta) \leq 1$ such that

$$\lim_{q \to \infty} \mathcal{J}(u_{q-1}, u_q, \theta) = l(\theta), \quad \forall\, \theta > 0.$$

We now demonstrate that $l(\theta) = 1$. Assume, for contradiction, that $l(\theta_0) < 1$ for some $\theta_0 > 0$. Using (3.2) and the property $(\Im 3)$, we obtain

$$0 \leq \limsup_{q\to\infty} \Im\big(\mathcal{J}(u_q, u_{q+1}, \theta_0), \mathcal{J}(u_{q-1}, u_q, \theta_0)\big) < 0,$$

which is a contradiction. Therefore, $l(\theta) = 1$, and consequently,

$$\lim_{q\to\infty} \mathcal{J}(u_q, u_{q+1}, \theta) = 1, \quad \forall\, \theta > 0. \tag{3.4}$$

Next, we show that the sequence $\{u_q\}$ is Cauchy. Assume the contrary: there exist $\tau \in (0,1)$, $\theta_0 > 0$, and two subsequences $\{u_{q_i}\}$ and $\{u_{p_i}\}$ of $\{u_q\}$, with $q_i > p_i \geq i$ for all $i \in \mathbb{N}$, such that

$$\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0) \leq 1 - \tau. \tag{3.5}$$

By Lemma 1, it follows that

$$\mathcal{J}(u_{p_i}, u_{q_i}, \tfrac{\theta_0}{2}) \leq 1 - \tau. \tag{3.6}$$

Choosing p_i to be the smallest index for which (3.6) holds, we obtain

$$\mathcal{J}(u_{p_i}, u_{q_i-1}, \tfrac{\theta_0}{2}) > 1 - \tau. \tag{3.7}$$

By Lemma 2 together with (3.1), we obtain

$$0 \leq \Im\big(\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0), \mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta)\big), \tag{3.8}$$

where

$$\begin{aligned}\mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta) &= \min\{\mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta), \mathcal{J}(u_{p_i-1}, \hbar u_{p_i-1}, \theta),\\ &\qquad \mathcal{J}(u_{q_i-1}, \hbar u_{q_i-1}, \theta)\}\\ &= \min\{\mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta), \mathcal{J}(u_{p_i-1}, u_{p_i}, \theta), \mathcal{J}(u_{q_i-1}, u_{q_i}, \theta)\}.\end{aligned}$$

If $\mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0) = \mathcal{J}(u_{p_i-1}, u_{p_i}, \theta_0)$, then from (3.8) we have

$$\begin{aligned}0 &\leq \Im\big(\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0), \mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0)\big)\\ &= \Im\big(\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0), \mathcal{J}(u_{p_i-1}, u_{p_i}, \theta_0)\big)\\ &< \frac{1}{\mathcal{J}(u_{p_i-1}, u_{p_i}, \theta_0)} - \frac{1}{\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0)}.\end{aligned}$$

Hence,

$$\mathcal{J}(u_{p_i-1}, u_{p_i}, \theta_0) < \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0). \tag{3.9}$$

Taking the limit as $i \to \infty$ and using (3.4), we obtain

$$\lim_{i\to\infty} \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0) = 1,$$

which contradicts (3.5). By applying a similar argument, if we assume that $\mathcal{O}\big(u_{p_i-1}, u_{q_i-1}, \theta_0\big) = \mathcal{J}\big(u_{q_i-1}, u_{q_i}, \theta_0\big)$, we again reach a contradiction. Consequently, the only possibility is

$$\mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0) = \mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0).$$

Then, by (3.8) and property $(\Im 2)$, we have

$$\mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0) < \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0). \tag{3.10}$$

Combining (3.5), (3.7), and the triangular inequality, we derive

$$\begin{aligned}1-\varepsilon &\geq \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0)\\ &> \mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0)\\ &\geq \mathcal{J}\left(u_{p_i-1}, u_{p_i}, \frac{\theta_0}{2}\right) * \mathcal{J}\left(u_{p_i}, u_{q_i-1}, \frac{\theta_0}{2}\right)\\ &> \mathcal{J}\left(u_{q_i-1}, u_{q_i}, \frac{\theta_0}{2}\right) * (1-\tau).\end{aligned}$$

Taking the limit as $i \to \infty$ and using (3.4), we conclude that

$$\lim_{i\to\infty} \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0) = \lim_{i\to\infty} \mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0) = 1-\tau, \tag{3.11}$$

which is again a contradiction.

From (3.4) and (3.11), we obtain

$$\begin{aligned}\lim_{i\to\infty} \mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0) &= \lim_{i\to\infty} \min\{\mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0), \mathcal{J}(u_{p_i-1}, \hbar u_{p_i-1}, \theta_0),\\ &\qquad \mathcal{J}(u_{q_i-1}, \hbar u_{q_i-1}, \theta_0)\}\\ &= \lim_{i\to\infty} \min\{\mathcal{J}(u_{p_i-1}, u_{q_i-1}, \theta_0), \mathcal{J}(u_{p_i-1}, u_{p_i}, \theta_0),\\ &\qquad \mathcal{J}(u_{q_i-1}, u_{q_i}, \theta_0)\}\\ &= \min\{1-\tau, 1, 1\} = 1-\tau.\end{aligned}$$

Thus, the sequences $x_i = \mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0)$ and $y_i = \mathcal{J}(u_{p_i}, u_{q_i}, \theta_0)$ share the same limit $1-\tau < 1$. Since $\hbar$ is an $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\mathfrak{I} \in \mathcal{FZ}$, property $(\mathfrak{I}3)$ implies that

$$0 \leq \limsup_{i\to\infty} \mathfrak{I}\big(\mathcal{J}(u_{p_i}, u_{q_i}, \theta_0), \mathcal{O}(u_{p_i-1}, u_{q_i-1}, \theta_0)\big) < 0,$$

which is a contradiction. Hence, $\{u_q\}$ must be a Cauchy sequence. Because $\{u_q\} \subseteq \hbar(\hat{o}) \subseteq \mathcal{S}$, the sequence $\{u_q\}$ is an $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{S}$. Since $(\mathcal{S}, \mathcal{J}, *)$ is $\mathcal{R}$-complete, there exists an element $z \in \mathcal{S}$ such that $u_q \to z$.

If $\hbar$ is $\mathcal{R}$-C, then

$$z = \lim_{q\to\infty} u_{q+1} = \lim_{q\to\infty} \hbar u_q = \hbar z,$$

so z is a FP of $\hbar$.

If, instead, $\mathcal{R}$ is $\mathcal{J}$-self-closed, note that $\{u_q\}$ is $\mathcal{R}$-preserving and $u_q \to u$. Hence, there exists a subsequence $\{u_{q_i}\}$ of $\{u_q\}$ such that $[u_{q_i}, u] \in \mathcal{R}|_{\mathcal{S}}$ for all $i \in \mathbb{N}$. From Remark 1, we also have $[u_{q_i}, u] \in \mathcal{R}$ and $u_{q_i} \to u$. Therefore,

$$\begin{aligned}0 &\leq \mathfrak{I}\big(\mathcal{J}(u_{q_i+1}, \hbar u, \theta), \mathcal{O}(u_{q_i}, u, \theta)\big)\\ &= \mathfrak{I}\big(\mathcal{J}(\hbar u_{q_i}, \hbar u, \theta), \mathcal{O}(u_{q_i}, u, \theta)\big)\\ &< \frac{1}{\mathcal{O}(u_{q_i}, u, \theta)} - \frac{1}{\mathcal{J}(\hbar u_{q_i}, \hbar u, \theta)}.\end{aligned}$$

This inequality implies that $\mathcal{O}(u_{q_i}, u, \theta) < \mathcal{J}(\hbar u_{q_i}, \hbar u, \theta)$, where

$$\mathcal{O}(u_{q_i}, u, \theta) = \min\{\mathcal{J}(u_{q_i}, u, \theta), \mathcal{J}(u_{q_i-1}, \hbar u_{q_i+1}, \theta), \mathcal{J}(u, \hbar u, \theta)\}.$$

By (3.4) and taking the limit as $i \to \infty$, we have

$$\lim_{i\to\infty} \mathcal{J}(u_{q_i+1}, \hbar u, \theta) = \lim_{i\to\infty} \mathcal{O}(u_{q_i}, u, \theta) = \mathcal{J}(u, \hbar u, \theta) < 1.$$

Applying property $(\Im 3)$ once more yields

$$0 \leq \limsup_{i\to\infty} \Im\big(\mathcal{J}(u_{q_i+1}, \hbar u, \theta), \mathcal{O}(u_{q_i}, u, \theta)\big) < 0,$$

a contradiction. Therefore, $\mathcal{J}(u, \hbar u, \theta) = 1$, which implies $\hbar u = u$. Hence, u is a FP of $\hbar$.

Theorem 2: *In addition to the assumptions of Theorem 1, if $\gamma(a, b, \mathcal{R}) \neq \emptyset$, then $\hbar$ has a unique FP.*

Proof: by contradiction, suppose that u and u^* are two distinct FPs of $\hbar$. As $\gamma(u, u^*, \mathcal{R}) \neq \emptyset$, $\exists\{\delta_0, \delta_1, \delta_2, \dots, \delta_q\}$ of some finite length q in $\mathcal{R}$ from u to u^* such that

$$\delta_0 = u, \delta_k = u^*, (\delta_j, \delta_{j+1}) \in \mathcal{R}, j = 0,1,2, \dots, q-1.$$

As $\mathcal{R}$ is transitive, then $(\delta_0, \delta_q) \in \mathcal{R}$. Since $\hbar$ is an $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\Im \in \mathcal{FZ}$, we get

$$\begin{aligned} 0 \leq\ & \Im(\mathcal{J}(\hbar\delta_0, \hbar\delta_q, \theta), \mathcal{O}(\delta_0, \delta_q, \theta)) \\ <\ & \frac{1}{\mathcal{O}(\delta_0, \delta_q, \theta)} - \frac{1}{\mathcal{J}(\hbar\delta_0, \hbar\delta_q, \theta)} \\ =\ & \frac{1}{\mathcal{O}(u, u^*, \theta)} - \frac{1}{\mathcal{J}(u, u^*, \theta)}. \end{aligned}$$

It is a contradiction. Thus, the FP of $\hbar$ is unique.

Corollary 1: *If the requirement of $\mathcal{S}$-completeness in Theorem 1 is replaced by the usual completeness of $(F, \mathcal{J}, *)$, and the assumption of $\mathcal{R}$-C is substituted by the standard notion of continuity, then the conclusion of Theorem 1 remains valid.*

Corollary 2: *Let $(F, \mathcal{J}, *)$ be an FMS equipped with a binary relation $\mathcal{R}$, and let $\hbar: F \to F$ be a self-mapping satisfying the following conditions:*

(i) $\mathcal{J}(\hbar u, \hbar v, \theta) \geq \psi(\mathcal{O}(u, v, \theta))$, for all $u, v \in F$ and $\theta > 0$ with $(u, v) \in \mathcal{R}$;
(ii) There exists a subset $\mathcal{S} \subseteq F$ such that $\hbar(F) \subseteq \mathcal{S} \subseteq F$, and $(\mathcal{S}, \mathcal{J}, *)$ is $\mathcal{R}$-complete;
(iii) $F(\hbar, \mathcal{R}) \neq \emptyset$;
(iv) $\mathcal{R}$ is both $\hbar$-closed and transitive;
(v) Either $\mathcal{R}|_{\mathcal{S}}$ is $\mathcal{J}$-SC or $\hbar$ is $\mathcal{R}$-C.

Then $\hbar$ has a FP in F.

Proof: The result follows from Theorem 1 by taking $\Im: (0,1] \times (0,1] \to \mathbb{R}$ defined as $\Im(y, x) = \frac{1}{\psi(x)} - \frac{1}{y}, \quad \forall\, y, x \in (0,1]$.

Example 3: Consider $F = [0,4)$ endowed with the standard fuzzy metric $\mathcal{J}(u, v, \theta) = \frac{\theta}{\theta + d(u,v)}$, where $d(u, v) = |u - v|$ and a binary relation $\mathcal{R} = \{(0,0), (0,1), (0,2),$

$(0,3),(1,1),(1,3),(2,2)\}$. Consider the mapping $\hbar: F \to F$ defined by

$$\hbar u = \begin{cases} 0, & 0 \leq u \leq 1, \\ 1, & 1 < u < 4. \end{cases}$$

Define the function $\Im: (0,1] \times (0,1] \to \mathbb{R}$ by

$$\Im(y,x) = \frac{7}{8}\left(\frac{1}{x} - 1\right) - \left(\frac{1}{y} - 1\right).$$

Clearly, $\mathrm{F}(\hbar,\mathcal{R}) \neq \emptyset$ since $(0,\hbar 0) \in \mathcal{R}$. For $\mathcal{S} = [0,1]$, we have $\{\hbar u, u \in F\} = \{0,1\} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{R}$-complete. It is easy to check that $\mathcal{R}$ is transitive, $\hbar$-closed and $\mathcal{R}|_{\mathcal{S}}$ is $\mathcal{J}$-SC. In order to show that $\hbar$ is $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\Im$. Consider the cases $(0,2)$,$(0,3)$ and $(1,3)$, as $\mathcal{J}(\hbar u, \hbar v, \theta) = 1$ in the other cases.

Case (1): if $(u,v) = (0,2)$, then

$$\begin{aligned}
&\Im(\mathcal{J}(\hbar 0, \hbar 2, \theta), \min\{\mathcal{J}(0,2,\theta), \mathcal{J}(0,\hbar 0,\theta), \mathcal{J}(2,\hbar 2,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \min\{\mathcal{J}(0,2,\theta), 1, \mathcal{J}(2,1,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \mathcal{J}(0,2,\theta)) \\
&= \frac{7}{8}\left(\frac{1}{\mathcal{J}(0,2,\theta)} - 1\right) - \left(\frac{1}{\mathcal{J}(0,1,\theta)} - 1\right) \\
&= \frac{7}{8}\left(\frac{d(0,2)}{\theta}\right) - \frac{d(0,1)}{\theta} = \frac{3}{4\theta} \geq 0.
\end{aligned}$$

Case (2): if $(u,v) = (0,3)$, then

$$\begin{aligned}
&\Im(\mathcal{J}(\hbar 0, \hbar 3, \theta), \min\{\mathcal{J}(0,3,\theta), \mathcal{J}(0,\hbar 0,\theta), \mathcal{J}(3,\hbar 3,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \min\{\mathcal{J}(0,3,\theta), 1, \mathcal{J}(3,1,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \mathcal{J}(0,3,\theta)) \\
&= \frac{7}{8}\left(\frac{1}{\mathcal{J}(0,3,\theta)} - 1\right) - \left(\frac{1}{\mathcal{J}(0,1,\theta)} - 1\right) \\
&= \frac{7}{8}\left(\frac{d(0,3)}{\theta}\right) - \frac{d(0,1)}{\theta} = \frac{13}{8\theta} \geq 0.
\end{aligned}$$

Case 3: If $(u,v) = (1,3)$, then

$$\begin{aligned}
&\Im(\mathcal{J}(\hbar 1, \hbar 3, \theta), \min\{\mathcal{J}(1,3,\theta), \mathcal{J}(1,\hbar 1,\theta), \mathcal{J}(3,\hbar 3,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \min\{\mathcal{J}(1,3,\theta), \mathcal{J}(1,0,\theta), \mathcal{J}(3,1,\theta)\}) \\
&= \Im(\mathcal{J}(0,1,\theta), \mathcal{J}(1,3,\theta)) \\
&= \frac{7}{8}\left(\frac{1}{\mathcal{J}(1,3,\theta)} - 1\right) - \left(\frac{1}{\mathcal{J}(0,1,\theta)} - 1\right) \\
&= \frac{7}{8}\left(\frac{d(1,3)}{\theta}\right) - \frac{d(0,1)}{\theta} = \frac{3}{4\theta} \geq 0.
\end{aligned}$$

Hence, $\Im(\mathcal{J}(\hbar u, \hbar v, \theta), \mathcal{O}(u,v,\theta)) \geq 0$, $\forall u,v \in \mathrm{F}$ such that $(u,v) \in \mathcal{R}$. Thus, $\hbar$ is $\mathcal{R}$-$\mathcal{FZ}$-contraction w.r.t $\Im$. the hypotheses of Theorem 1 are verified, and $\hbar$ admits the FP $u = 0$.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control,* vol. 8, pp. 338-353 (1965).

[2] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385-389 (1988).

[3] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395-399 (1994).

[4] V. Gregori and A. Sapena, "On fixed-point theorems in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 245-252 (2002).

[5] V. Gregori, S. Morillas, and A. Sapena, "Examples of fuzzy metrics and applications," *Fuzzy Sets and Systems*, vol. 170, pp. 95-111 (2011).

[6] I. Kramosil and J. Michalek, "Fuzzy metrics and statistical metric spaces," *Kybernetika,* vol. 11, no. 5, pp. 336-344 (1975).

[7] B. Schweizer and A. Sklar, "Statistical metric spaces," *Pacific Journal of Mathematics,* vol. 10, no. 1, pp. 313-334 (1960).

[8] S. Lipschutz, Schaum's Outline of Theory and Problems of Set Theory and Related Topics. New York, NY, USA: McGraw-Hill (1976).

[9] R. Maddux, "Boolean algebras," Studies in Logic and the Foundations of Mathematics, vol. 150, Elsevier, Amsterdam, Netherlands (2006).

[10] Abdelhamid Moussaoui, "Exploring fixed points via admissibility criteria for fuzzy θ_f-weak contractions," *Nonlinear Analysis: Modelling and Control,* vol. 30, no. 6, pp. 1067-1080 (2025).

[11] A. Alam and M. Imdad, "Relation-theoretic contraction principle," *Journal of Fixed Point Theory and Applications,* vol. 17, pp. 693-702 (2015).

[12] W. M. Alfaqih, B. Ali, M. Imdad, and S. Sessa, "Fuzzy relation-theoretic contraction principle," *Journal of Intelligent and Fuzzy Systems,* vol. 40, pp. 4491-4501 (2021).

[13] B. Kolman, R. C. Busby, and S. Ross, Discrete Mathematical Structures, 3rd ed. New Delhi, India: PHI Pvt. Ltd. (2000).

[14] D. Mihet, "Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces," *Fuzzy Sets and Systems,* vol. 159, no. 6, pp. 739-744 (2008).

[15] D. Wardowski, "Fuzzy contractive mappings and fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 222, pp. 108-114 (2013).

[16] F. Khojasteh, S. Shukla, and S. Radenović, "A new approach to the study of fixed point theory for simulation functions," *Filomat,* vol. 29, no. 6, pp. 1189-1194 (2015).

[17] S. Melliani and A. Moussaoui, "Fixed point theorem using a new class of fuzzy contractive mappings," *Journal of Universal Mathematics,* vol. 1, no. 2, pp. 148-154 (2018).

[18] A. Moussaoui, N. Hussain, S. Melliani, N. Hayel, and M. Imdad, "Fixed point results via extended FZ-simulation functions in fuzzy metric spaces," *Journal of Inequalities and Applications*, vol. 2022, Article 69 (2022).

[19] A. Moussaoui, N. Hussain, and S. Melliani, "Global optimal solutions for proximal fuzzy contractions involving control functions," *Journal of Mathematics*, vol. 2021.

[20] M. Hasanuzzaman and M. Imdad, "Relation-theoretic metrical fixed point results for Suzuki type Z_R-contraction with an application," AIMS Mathematics, vol. 5, no. 3, pp. 2071-2087 (2020).

[21] W. M. Alfaqih, M. Imdad, R. Gubran, and I. A. Khan, "Relation-theoretic coincidence and common fixed point results under $(\mathcal{F},\mathcal{R})_g$-contractions with an application," *Fixed Point Theory and Applications,* vol. 2019, Article 12.

[22] M. Imdad, Q. Khan, W. M. Alfaqih, and R. Gubran, "A relation-theoretic $(\mathcal{F},\mathcal{R})$-contraction principle with applications to matrix equations," *Bulletin of Mathematical Analysis and Applications*, vol. 10, pp. 1-12 (2018).

[23] A. Moussaoui, N. Saleem, S. Melliani, and M. Zhou, "Fixed point results for new types of fuzzy contractions via admissible functions and FZ-simulation functions," *Axioms*, vol. 11, Article 87 (2022).

[24] S. M. Saleh, W. M. Alfaqih, S. Sessa, and F. Di Martino, "New relation-theoretic fixed point theorems in fuzzy metric spaces with an application to fractional differential equations," *Axioms*, vol. 11, Article 117 (2022).

[25] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy b-metric spaces with an

application," *Journal of Mathematics,* vol. 2021, Article ID 6663707, 7 pages (2021).

[26] S. Antal, A. Tomar, and U. Gairola, "Relation theoretic results via simulation function with applications," *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 1, pp. 1769-1783 (2022).

[27] M. Turinici, "Abstract comparison principles and multivariable Gronwall-Bellman inequalities," *Journal of Mathematical Analysis and Applications*, vol. 117, pp. 100-127 (1986).

[28] A. C. Ran and M. C. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society,* vol. 132, pp. 1435-1443 (2004).

[29] D. Gopal and C. Vetro, "Some new fixed point theorems in fuzzy metric spaces," *Iranian Journal of Fuzzy Systems*, vol. 11, no. 3, pp. 95-107 (2014).

[30] A. Moussaoui, S. Radenović, and S. Melliani, "Fixed point theorems involving $\mathcal{FZ}$-ϑ_f-contractions in GV-fuzzy metrics," *Filomat,* vol. 38, no. 6, pp. 1973-1985 (2024).

[31] M. Sangurlu, H. Isik, and D. Turkoglu, "A new method for some fixed point theorems in fuzzy metric spaces," *Bangmod International Journal of Mathematical and Computer Sciences*, vol. 3, pp. 10-15 (2017).

[32] A. Moussaoui, C. Park, and S. Melliani, "New best proximity point results via simulation functions in fuzzy metric spaces," *Boletim da Sociedade Paranaense de Matemática*, vol. 43 (2025).

[33] N. S. Hayel, I. A. Khan, M. Imdad, and W. M. Alfaqih, "New fuzzy φ-fixed point results employing a new class of fuzzy contractive mappings," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 4, pp. 5391-5402 (2019).

[34] A. Moussaoui, S. Radenović, O. T. Birgani, and S. Melliani, "A new contraction principle via fuzzy L-simulation functions," *Boletim da Sociedade Paranaense de Matemática*, vol. 43 (2025).

[35] S. Ur Rehman and H. Aydi, "Rational fuzzy cone contractions on fuzzy cone metric spaces with an application to Fredholm integral equations," *Journal of Function Spaces*, vol. 2021, Article ID 5527864, 13 pages (2021).

[36] A. Moussaoui, S. Melliani, and S. Radenović, "A nonlinear fuzzy contraction principle via control functions," *Filomat,* vol. 38, no. 6, pp. 1963-1972 (2024).

[37] A. Moussaoui, F. A. I. Amir, S. Radenović, S. Melliani, and M. Elomari, "Fixed point theorems for ξ-α-η-ΓF-fuzzy contraction with an application to neutral fractional integro-differential equations with nonlocal conditions," *Nonlinear Analysis: Modelling and Control,* vol. 29, no. 5, pp. 939–957 (2024).

[38] A. Moussaoui and S. Melliani, "Fixed point results under admissible α-η-F simulation fuzzy contraction with application," *International Journal of System Assurance Engineering and Management*, vol. 15, pp. 3807–3816 (2024).

[39] R. Batra, R. Gupta, and P. Sahni, "Some results related to fixed points of F-G-contractions on metric spaces equipped with w-distance," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 2, pp. 413–423 (2020).

[40] S. H. Hadi and Z. S. Madhi, "Fixed-point theorems of FR-Ćirić-Pata type in Fuzzy-Riesz I-Convex Metric Space (FRCMS)," *Journal of Interdisciplinary Mathematics,* vol. 24, no. 8, pp. 2181–2193 (2021).

[41] A. Moussaoui and S. Radenovi/c, "Fuzzy metric spaces: a survey on fixed point results, contraction principles and simulation functions," *Fixed Point Theory Algorithms Sci. Eng.*, vol. 2025, Article 24 (2025).

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