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q-numerical radius inequalities for two by two operator matrices

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Abstract

This work explores new upper bounds for the q-numerical radius of $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$, extending and strengthening existing advancements in the field. Specific instances of q-numerical radius inequalities for operator sums are derived using these findings. Additionally, the paper presents new inequalities for the q-numerical radius of two by two matrices, offering valuable contributions to the study of operator theory.

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1. Introduction and Preliminaries

Let $\mathcal{L}(\mathcal{K})$ be the space of all bounded operators on $\mathcal{K}$, and let $\mathcal{K}$ be a Hilbert space. An operator $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$ on a Hilbert space $\mathcal{K}$ has a classical numerical range and numerical radius defined by $W(\mathcal{Y}) = \{< \mathcal{Y}\xi, \xi >: \|\xi\| = 1\}$ and $\omega(\mathcal{Y}) = \sup\{|\lambda|: \lambda \in W(\mathcal{Y})\}$, respectively. For further information, see [1, 8, 10, 15]. The spectral norm of $\mathcal{L}(\mathcal{K})$ is known to be identical to the numerical radius. For the numerical radius, several mathematicians came up with upper (and lower) boundaries. In [16], Kittaneh specifically showed that for each operator $\mathcal{Y}$ in $\mathcal{L}(\mathcal{K})$, the disparity listed below is true.

$$\omega^2(\mathcal{Y}) \leq \frac{1}{2}\left(\|\mathcal{Y}\|^2 + \|\mathcal{Y}^2\|^{\frac{1}{2}}\right)^2. \tag{1.1}$$

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Also in [16], he showed that for any $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$,

$$\frac{1}{4}\|\mathcal{Y}^*\mathcal{Y}+\mathcal{Y}\mathcal{Y}^*\| \leq \omega^2(\mathcal{Y}) \leq \frac{1}{2}\|\mathcal{Y}^*\mathcal{Y}+\mathcal{Y}\mathcal{Y}^*\|. \tag{1.2}$$

Dragomir established the following inequalities for every $\mathcal{Y}, \mathcal{S} \in \mathcal{L}(\mathcal{K})$ in [8, Theorem 4, 5]

$$\left\|\frac{\mathcal{Y}^*\mathcal{Y}+\mathcal{S}^*\mathcal{S}}{2}\right\| \leq \omega(\mathcal{Y}^*\mathcal{S}) + \frac{1}{2}\|\mathcal{Y}-\mathcal{S}\|^2. \tag{1.3}$$

In [11], the author introduces new inequalities for the numerical radius of normal operators on a Hilbert space. One of these inequalities states that for a normal operator $\mathcal{T} \in \mathcal{L}(\mathcal{K})$, the following is true:

$$\frac{1}{\sqrt{2}}\omega(\mathcal{T}_1+\mathcal{T}_2) \leq \omega(\mathcal{T}) \leq \omega(|\mathcal{T}_1|+|\mathcal{T}_2|), \tag{1.4}$$

where $\mathcal{T} = \mathcal{T}_1 + i\mathcal{T}_2$ is the Cartesian decomposition of $\mathcal{T}$, and $|\mathcal{T}_j| = \sqrt{\mathcal{T}_j^*\mathcal{T}_j}, j = 1,2$.

On the flip side, Marcus and Andresen [19] introduced a broader concept known as the $\mathfrak{q}$-numerical range, denoted by $W_{\mathfrak{q}}(\cdot)$, in 1977. This concept applies to a space with an inner product and unitary properties, specifically in n dimensions. They showed that an annulus is formed when the "$\mathfrak{q}$-numerical range" is rotated about the origin for $\mathfrak{q} \in \mathbb{C}$ with $q \in (0,1]$. For more details, see [4], [5] and [22].

Li, Metha, and Rodman [17] investigated the basic characteristics of $W_{\mathfrak{q}}(T)$ in 1994. Li and Nakazato [18], Rajic [21], and Chien [6] conducted follow-up research on the $\mathfrak{q}$-numerical range, radius, and associated shift operators.

Let $0 < \mathfrak{q} \leq 1$. The $\mathfrak{q}$-numerical range of $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$ is established by

$$W_{\mathfrak{q}}(\mathcal{Y}) = \{\langle \mathcal{Y}\xi, \zeta\rangle : \xi, \zeta \in \mathcal{K}, \|\xi\| = \|\zeta\| = 1, <\xi, \zeta> = \mathfrak{q}\}.$$

Consequently, the $\mathfrak{q}$-numerical radius of $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$ has the following form:

$$\omega_{\mathfrak{q}}(\mathcal{Y}) = \sup\left\{|\mu| : \mu \in W_{\mathfrak{q}}(\mathcal{Y})\right\}.$$

The $\mathfrak{q}$-numerical range's characteristics show that for every $\mathcal{Y}, \mathfrak{P} \in \mathcal{L}(\mathcal{K})$, and $\alpha \in \mathbb{C}$,

(a) $\omega_{\mathfrak{q}}(\alpha\mathcal{Y}) = |\alpha|\omega_{\mathfrak{q}}(\mathcal{Y})$;

(b) $\omega_{\mathfrak{q}}(\mathcal{Y}+\mathfrak{P}) \leq \omega_{\mathfrak{q}}(\mathcal{Y}) + \omega_{\mathfrak{q}}(\mathfrak{P})$;

(c) $\omega_{\mathfrak{q}}(\mathfrak{U}^*\mathcal{Y}\mathfrak{U}) = \omega_{\mathfrak{q}}(\mathcal{Y})$ for every unitary operator $\mathfrak{U} \in \mathcal{L}(\mathcal{K})$;

(d) $\omega_{\mathfrak{q}}(\mathcal{Y}^*) = \omega_{\mathfrak{q}}(\mathcal{Y})$.

The authors in [20] established the equivalence between the operator norm and the $\mathfrak{q}$-numerical radius as a result of these features. In particular, they showed that for $0 < \mathfrak{q} < 1$ and $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$ (see [20, Theorem 2.1]),

$$\frac{\mathfrak{q}}{2(2-\mathfrak{q}^2)}\|\mathcal{Y}\| \leq w_{\mathfrak{q}}(\mathcal{Y}) \leq \|\mathcal{Y}\| \tag{1.5}$$

Additionally, they offered proof that, for each normal operator $\mathcal{Y}$,

$$\frac{\mathfrak{q}}{2-\mathfrak{q}^2}\|\mathcal{Y}\| \leq w_{\mathfrak{q}}(\mathcal{Y}) \leq \|\mathcal{Y}\|. \tag{1.6}$$

For the "q-numerical radius $\omega_{\mathfrak{q}}(\mathcal{Y})$", we provide upper and lower bounds in this work. The requirement is that $0 < \mathfrak{q} < 1$. By providing insights into a broader range of circumstances, this builds upon the already established information in the literature.

2. Main Result

We use a number of well-known lemmas to illustrate our expanded numerical radius inequalities.

Lemma 2.1: (Hölder Mc-Carthy inequality)**:** *Let $\xi \in \mathcal{K}$ be any unit vector, and let $F \in \mathcal{L}(\mathcal{K})$, $F \geq 0$, be allowed. Next, we have*

(i) $\langle F\xi, \xi\rangle^{\kappa} \leq \langle F^{\kappa}\xi, \xi\rangle$ for $\kappa \geq 1$.

(ii) $\langle F^{\kappa}\xi, \xi\rangle \leq \langle F\xi, \xi\rangle^{\kappa}$ for $0 < \kappa \leq 1$.

Lemma 2.2: [13] *Let $\varrho, \sigma > 0$ and $0 \leq \gamma \leq 1$ and $\mu, \nu > 1$ such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Then*

(i) $\varrho^{\gamma}\sigma^{1-\gamma} \leq \gamma\varrho + (1-\gamma)\sigma \leq (\gamma\varrho^{\kappa} + (1-\gamma)\sigma^{\kappa})^{\frac{1}{\kappa}}$ for $\kappa \geq 1$.

(ii) $\sqrt{\varrho\sigma} \leq \frac{\rho+\sigma}{2}$.

(iii) $\varrho\sigma \leq \frac{\varrho^{\mu}}{\mu} + \frac{\sigma^{\nu}}{\nu} \leq \left(\frac{\varrho^{\mu\kappa}}{\mu} + \frac{\sigma^{\nu\kappa}}{\nu}\right)^{\frac{1}{\kappa}}$ for $\kappa \geq 1$.

Lemma 2.3: [2] *Let $\varrho_i \in \mathbb{R}^+$, $i \in \{1, \cdots, m\}$. Then*

$$\left(\sum_{i=1}^{m} \varrho_i\right)^{\kappa} \leq m^{\kappa-1}\sum_{i=1}^{m} \varrho_i^{\kappa} \; for \; \kappa \geq 1. \tag{2.1}$$

Lemma 2.4: [13] *Let $\mathcal{Y} \in \mathcal{L}(\mathcal{K})$ and $\xi, \zeta \in \mathcal{K}$ be any vectors.*

(i) If $\gamma, \delta \geq 0$ such that $\gamma + \delta = 1$, then $|\langle \mathcal{Y}\xi, \zeta\rangle|^2 \leq \langle |\mathcal{Y}|^{2\gamma}\xi, \xi\rangle \langle |\mathcal{Y}^*|^{2\delta}\zeta, \zeta\rangle$,

(ii) If ψ, ϕ are non-negative continuous functions on $[0, \infty)$ satisfying $\psi(\tau)\phi(\tau) = \tau$ $(\tau \geq 0)$, $| < \mathcal{Y}\xi, \zeta > | \leq \|\psi(|\mathcal{Y}|)\xi\| \, \|\phi(|\mathcal{Y}^*|)\zeta\|$.

Lemma 2.5: ([9]): *If the operators $\mathfrak{P}, \mathcal{Y}$ are positive, then*

$$\left\|\mathfrak{P}^{\frac{1}{2}}\mathcal{Y}^{\frac{1}{2}}\right\| \leq \|\mathfrak{P}\mathcal{Y}\|^{\frac{1}{2}}. \tag{2.2}$$

Lemma 2.6 ([14]): *If $\mathcal{Y}, \mathfrak{B} \in \mathcal{L}(\mathcal{K})$ are positive operators, then*

$$\|\mathcal{Y} + \mathfrak{B}\| \leq \frac{1}{2}\left(\|\mathcal{Y}\| + \|\mathfrak{B}\| + \sqrt{(\|\mathcal{Y}\| - \|\mathfrak{B}\|)^2 + 4\left\|\mathcal{Y}^{\frac{1}{2}}\mathfrak{B}^{\frac{1}{2}}\right\|^2}\right). \tag{2.3}$$

Lemma 2.7 ([12])**:** *Let $\mathcal{J}, \mathcal{G} \in \mathcal{L}(\mathcal{K})$. Then the following claims are true:*

(a) $w\left(\begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{G} \end{bmatrix}\right) = \max\{w(\mathcal{J}), w(\mathcal{G})\}$.

(b) $w\left(\begin{bmatrix} 0 & \mathcal{J} \\ \mathcal{G} & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & \mathcal{G} \\ \mathcal{J} & 0 \end{bmatrix}\right)$.

(c) $w\left(\begin{bmatrix} 0 & \mathcal{J} \\ \mathcal{G} & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & \mathcal{J} \\ e^{i\theta}\mathcal{G} & 0 \end{bmatrix}\right)$ for all $\theta \in \mathbb{R}$.

(d) $w\left(\begin{bmatrix} \mathcal{J} & \mathcal{G} \\ \mathcal{G} & \mathcal{J} \end{bmatrix}\right) = \max\{w(\mathcal{J}-\mathcal{G}), w(\mathcal{J}+\mathcal{G})\}$. In particular, $w\left(\begin{bmatrix} 0 & \mathcal{G} \\ \mathcal{G} & 0 \end{bmatrix}\right) = w(\mathcal{G})$.

(e) $\left\|\begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{G} \end{bmatrix}\right\| = \max\{\|\mathcal{J}\|, \|\mathcal{G}\|\}$.

Theorem 2.8: *If* $\Psi = \begin{bmatrix} 0 & \mathcal{J} \\ \mathcal{K} & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ *and* $\mathfrak{q} \in (0,1)$, *then*

$$\omega_{\mathfrak{q}}^2(\Psi) \le \frac{\mathfrak{q}^2}{4}\left(\max\{\|\mathcal{J}\|, \|\mathcal{G}\|\} + \sqrt{\max\{\|\mathcal{J}\mathcal{G}\|, \|\mathcal{G}\mathcal{J}\|\}}\right) + (1-\mathfrak{q}^2)\max\{\|\mathcal{J}\|^2, \|\mathcal{G}\|^2\} + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\max\{\|\mathcal{J}\|^2, \|\mathcal{G}\|^2\}.$$

Proof: Let $\hbar = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \wp = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ such that $\|\hbar\| = \|\wp\| = 1$ and $\langle \hbar, \wp \rangle = \mathfrak{q}$. Then $\hbar \ne \wp$ and $\hbar \ne -\wp$. Put $\wp = \mathfrak{q}\hbar + \sqrt{1-\mathfrak{q}^2}\aleph$. Hence $\langle \hbar, \aleph \rangle = 0$.

Now, by employing Lemma 2.4 (i) and Lemma 2.2, we get

$$\begin{aligned}
|\langle \Psi\hbar, \wp \rangle|^2 &\le \langle |\Psi|\hbar, \hbar \rangle \langle |\Psi^*|\wp, \wp \rangle \\
&= \langle |\Psi|\hbar, \hbar \rangle \left\langle |\Psi^*|\left(\mathfrak{q}\hbar + \sqrt{1-\mathfrak{q}^2}\aleph\right), \mathfrak{q}\hbar + \sqrt{1-\mathfrak{q}^2}\aleph \right\rangle \\
&= \mathfrak{q}^2 \langle |\Psi|\hbar, \hbar \rangle \langle |\Psi^*|\hbar, \hbar \rangle + (1-\mathfrak{q}^2)\langle |\Psi|\hbar, \hbar \rangle \langle |\Psi^*|\aleph, \aleph \rangle \\
&\quad + 2Re\,\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\langle |\Psi|\hbar, \hbar \rangle \langle |\Psi^*|\hbar, \aleph \rangle
\end{aligned}$$

$$\begin{aligned}
&\le \mathfrak{q}^2\left(\frac{\langle |\Psi|\hbar,\hbar\rangle + \langle |\Psi^*|\hbar,\hbar\rangle}{2}\right)^2 + (1-\mathfrak{q}^2)\left(\frac{\langle |\Psi|\hbar,\hbar\rangle^2 + \langle |\Psi^*|\aleph,\aleph\rangle^2}{2}\right) \\
&\quad + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\langle |\Psi|\hbar, \hbar \rangle |\langle |\Psi^*|\hbar, \aleph \rangle| \\
&\le \frac{\mathfrak{q}^2}{4}(\langle (|\Psi| + |\Psi^*|)\hbar, \hbar \rangle)^2 + \frac{1-\mathfrak{q}^2}{2}(\langle |\Psi|\hbar, \hbar \rangle^2 + \langle |\Psi^*|\aleph, \aleph \rangle^2) \\
&\quad + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\||\Psi|\|\||\Psi^*|\|\|\hbar\|^3\|\aleph\| \\
&\le \frac{\mathfrak{q}^2}{4}\omega(|\Psi| + |\Psi^*|) + \frac{1-\mathfrak{q}^2}{2}(\omega^2(|\Psi|) + \omega^2(|\Psi^*|)) + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\||\Psi|\|\,\||\Psi^*|\| \\
&\le \frac{\mathfrak{q}^2}{4}\||\Psi| + |\Psi^*|\| + \frac{1-\mathfrak{q}^2}{2}(\||\Psi|\|^2 + \||\Psi^*|\|^2) + 2\mathfrak{q}\sqrt{1-\mathfrak{q}^2}\||\Psi|\|\,\||\Psi^*|\|.
\end{aligned}$$

Using Lemma 2.6 for the positive operators $|\Psi|$ and $|\Psi^*|$, and the fact that $\||\Psi|\| = \||\Psi^*|\| = \|\Psi\|$, $\||\Psi||\Psi^*|\| = \|\Psi^2\|$, we obtain

$$\begin{aligned}
\||\Psi| + |\Psi^*|\| &\le \frac{1}{2}\left(\||\Psi|\| + \||\Psi^*|\| + \sqrt{(\||\Psi|\| - \||\Psi^*|\|) + 4\left\||\Psi|^{\frac{1}{2}}|\Psi^*|^{\frac{1}{2}}\right\|^2}\right) \\
&\le \frac{1}{2}\left(2\||\Psi|\| + 2\sqrt{\||\Psi||\Psi^*|\|}\right) = \|\Psi\| + \|\Psi^2\|^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$|< \Psi\hbar, \wp >|^2 \leq \frac{\mathrm{q}^2}{4}\left(\|\Psi\| + \|\Psi^2\|^{\frac{1}{2}}\right) + (1-\mathrm{q}^2)\||\Psi|\|^2 + 2\mathrm{q}\sqrt{1-\mathrm{q}^2}\||\Psi|\|^2.$$

The supremum over all vectors $\hbar, \wp \in \mathcal{K} \oplus \mathcal{K}$ is found by using $\|\hbar\| = \|\wp\| = 1$ and $< \hbar, \wp >= \mathrm{q}$

$$\omega_{\mathrm{q}}^2(\Psi) \leq \frac{\mathrm{q}^2}{4}\left(\|\Psi\| + \|\Psi^2\|^{\frac{1}{2}}\right) + (1-\mathrm{q}^2)\||\Psi|\|^2 + 2\mathrm{q}\sqrt{1-\mathrm{q}^2}\||\Psi|\|^2.$$

Lemma 2.7 for Ψ, Ψ^2, and $|\Psi|$ is now applied, and we obtain

$$\omega_{\mathrm{q}}^2(\Psi) \leq \frac{\mathrm{q}^2}{4}\left(\max\{\|\mathcal{J}\|, \|\mathcal{G}\|\} + \sqrt{\max\{\|\mathcal{J}\mathcal{G}\|, \|\mathcal{G}\mathcal{J}\|\}}\right)$$
$$+(1-\mathrm{q}^2)\max\{\|\mathcal{J}\|^2, \|\mathcal{G}\|^2\} + 2\mathrm{q}\sqrt{1-\mathrm{q}^2}\max\{\|\mathcal{J}\|^2, \|\mathcal{G}\|^2\}.$$

Theorem 2.9: *If* $\Psi = \begin{bmatrix} O & \mathcal{J} \\ \mathcal{G} & O \end{bmatrix} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ *and* $q \in (0,1)$*, then*

$$\frac{1}{4}\left(\frac{q}{2-q^2}\right)^2 \max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{G}^*|^2\|\} \leq \omega_{\mathrm{q}}^2\left(\begin{bmatrix} O & \mathcal{J} \\ \mathcal{G} & O \end{bmatrix}\right)$$
$$\leq \frac{q^2}{2\left(1-\sqrt{1-q^2}\right)^2}\max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{J}^*|^2\|\}. \quad (2.7)$$

Proof: Let $\hbar = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \wp = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ such that $\|\hbar\| = \|\wp\| = 1$ and $< \hbar, \wp >= q$. Then $\hbar \neq \wp$ and $\hbar \neq -\wp$. By using $\wp = q\hbar + \sqrt{1-q^2}\aleph$ which $\|\aleph\| = 1$, $< \hbar, \aleph >= 0$, $\lambda = \frac{q}{2+q}$ and $\chi = \frac{q}{2-q}$. Then

$$\begin{aligned} |\langle\Psi\hbar, \wp\rangle|^2 &= \left|\langle\Psi\hbar, q\hbar + \sqrt{1-q^2}\aleph\rangle\right|^2 \\ &\leq \left(q|\langle\Psi\hbar, \hbar\rangle| + \sqrt{1-q^2}|\langle\Psi\hbar, \aleph\rangle|\right)^2 \\ &\leq \left(q\omega(\Psi) + \sqrt{1-q^2}\omega_{\mathrm{q}}(\Psi)\right)^2. \end{aligned}$$

Utilizing $\langle\hbar, \wp\rangle = q$ and $\|\hbar\| = \|\wp\| = 1$ to determine the supremum over all vectors $\hbar, \wp \in \mathcal{K} \oplus \mathcal{K}$, we arrive at

$$\omega_{\mathrm{q}}(\Psi) \leq q\omega(\Psi) + \sqrt{1-q^2}\omega_{\mathrm{q}}(\Psi).$$

Consequently,

$$\omega_{\mathrm{q}}(\Psi) \leq \frac{q}{1-\sqrt{1-q^2}}\omega(\Psi). \quad (2.5)$$

By using the right-hand side of the inequality (1.2) and the inequality (2.5) and Lemma 2.7, we have

$$\begin{aligned} \omega_{\mathrm{q}}^2(\Psi) &\leq \frac{q^2}{2\left(1-\sqrt{1-q^2}\right)^2}\|\Psi^*\Psi + \Psi\Psi^*\| \\ &= \frac{q^2}{2\left(1-\sqrt{1-q^2}\right)^2}\left\|\begin{bmatrix} |\mathcal{G}|^2 + |\mathcal{J}^*|^2 & O \\ O & |\mathcal{J}|^2 + |\mathcal{G}^*|^2 \end{bmatrix}\right\| \end{aligned}$$

$$= \frac{q^2}{2\left(1-\sqrt{1-q^2}\right)^2} \max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{G}^*|^2\|\}.$$

This proves the right-hand side of (2.4).

To prove the other side of the inequality (2.4). Put $u = \frac{\hbar+\wp}{\|\hbar+\wp\|}$, $w = \frac{\hbar-\wp}{\|\hbar-\wp\|}$ and $v = qu + \sqrt{1-q^2}w$. Trivially $\langle u, w\rangle = 0$. Also $\langle u, v\rangle = q$ and $\|v\| = 1$. Letting $v' = qw + \sqrt{1-q^2}u$, we get

$$Re(\langle \Psi u, v\rangle + \langle \Psi w, v'\rangle) \leq |\langle \Psi u, v\rangle| + |\langle \Psi w, v'\rangle| \leq 2\omega_{\mathrm{q}}(\Psi). \qquad (2.6)$$

On can see that $\|\hbar + \wp\|^2 = 2 + 2q$ and $\|\hbar - \wp\|^2 = 2 - 2q$. On the other hand,

$$\begin{aligned}
&Re(\langle \Psi u, v\rangle + \langle \Psi w, v'\rangle) = \\
&Re\Big[\lambda\langle \Psi(\hbar+\wp), \hbar+\wp\rangle + \tfrac{\sqrt{1-q^2}}{\sqrt{2+2q}\sqrt{2-2q}}\langle \Psi(\hbar+\wp), \hbar-\wp\rangle \\
&\chi\langle \Psi(\hbar-\wp), \hbar-\wp\rangle + \tfrac{\sqrt{1-q^2}}{\sqrt{2+2q}\sqrt{2-2q}}\langle \Psi(\hbar-\wp), \hbar+\wp\rangle\Big] \\
&= Re[\lambda\langle \Psi\hbar, \hbar\rangle + \lambda\langle \Psi\hbar, \wp\rangle + \lambda\langle \Psi\wp, \hbar\rangle + \lambda\langle \Psi\wp, \wp\rangle \\
&+\tfrac{1}{2}\langle \Psi\hbar, \hbar\rangle - \tfrac{1}{2}\langle \Psi\hbar, \wp\rangle + \tfrac{1}{2}\langle \Psi\wp, \hbar\rangle - \tfrac{1}{2}\langle \Psi\wp, \wp\rangle \\
&+\chi\langle \Psi\hbar, \hbar\rangle - \chi\langle \Psi\hbar, \wp\rangle - \chi\langle \Psi\wp, \hbar\rangle + \chi\langle \Psi\wp, \wp\rangle \\
&+\tfrac{1}{2}\langle \Psi\hbar, \hbar\rangle + \tfrac{1}{2}\langle \Psi\hbar, \wp\rangle - \tfrac{1}{2}\langle \Psi\wp, \hbar\rangle - \tfrac{1}{2}\langle \Psi\wp, \wp\rangle\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\lambda+\chi+1)Re\langle \Psi\hbar, \hbar\rangle + (\lambda+\chi-1)Re\langle \Psi\wp, \wp\rangle \\
&\leq 2\omega_{\mathrm{q}}(\Psi) + (\chi-\lambda)Re[\langle \Psi\hbar, \wp\rangle + \langle \Psi\wp, \hbar\rangle] \\
&\leq 2\omega_{\mathrm{q}}(\Psi) + (\chi-\lambda)|\langle \Psi\hbar, \wp\rangle + \langle \Psi\wp, \hbar\rangle| \\
&\leq 2\omega_{\mathrm{q}}(\Psi) + \tfrac{q^2}{1-q^2}(|\langle \Psi\hbar, \wp\rangle| + |\langle \Psi\wp, \hbar\rangle|) \\
&\leq \left(2 + \tfrac{q^2}{1-q^2}\right)\omega_{\mathrm{q}}(\Psi) = \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi).
\end{aligned}$$

So,

$$\begin{aligned}
(\lambda+\chi+1)Re\langle \Psi\hbar, \hbar\rangle &\leq \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi) + (1-\lambda-\chi)|\langle \Psi\wp, \wp\rangle| \\
&\leq \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi) + |\langle \Psi\wp, \wp\rangle| \qquad (2.7) \\
&\leq \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi) + \omega(\Psi).
\end{aligned}$$

Replacing Ψ by $e^{i\theta}\Psi$ in (2.7), we have

$$(\lambda+\chi+1)Re\langle (e^{i\theta}\Psi)\hbar, \hbar\rangle \leq \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi) + \omega(\Psi),$$

The supremum over all $\theta \in \mathbb{R}$ is found by

$$(\lambda+\chi+1)|\langle \Psi\hbar, \hbar\rangle| \leq \tfrac{2-q^2}{1-q^2}\omega_{\mathrm{q}}(\Psi) + \omega(\Psi).$$

We now determine that by calculating the supremum of this inequality over all $\hbar \in \mathcal{K} \oplus \mathcal{K}$ that fulfill $\|\hbar\| = 1$,

$$(\lambda + \chi + 1)\omega(\Psi) \le \frac{2-q^2}{1-q^2}\omega_q(\Psi) + \omega(\Psi). \tag{2.8}$$

Consequently,

$$\frac{q}{1-q^2}\omega(\Psi) \le \frac{2-q^2}{1-q^2}\omega_q(\Psi). \tag{2.9}$$

Therefore,

$$\frac{q}{2-q^2}\omega(\Psi) \le \omega_q(\Psi). \tag{2.10}$$

The inequality (2.10) and the left side of (1.2) are now used to suggest that

$$\begin{aligned}\omega_q^2(\Psi) &\ge \frac{1}{4}\left(\frac{q}{2-q^2}\right)^2 \|\Psi^*\Psi + \Psi\Psi^*\| \\ &= \frac{1}{4}\left(\frac{q}{2-q^2}\right)^2 \max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{G}^*|^2\|\}.\end{aligned}$$

The evidence is now complete.

Remark 2.10: (i) In Theorem 2.9, if we tend q to 1, we get

$$\frac{1}{4}\max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{G}^*|^2\|\} \le \omega(\Psi) \le \frac{1}{2}\max\{\||\mathcal{G}|^2 + |\mathcal{J}^*|^2\|, \||\mathcal{J}|^2 + |\mathcal{G}^*|^2\|\}. \tag{2.11}$$

(ii) Using Lemma 2.7, allowing q tend to 1, and taking into account $\mathcal{J} = \mathcal{G}$ in Theorem 2.9, we obtain

$$\frac{1}{4}\|\mathcal{J}^*\mathcal{J} + \mathcal{J}\mathcal{J}^*\| \le \omega(\mathcal{J}) \le \frac{1}{2}\|\mathcal{J}^*\mathcal{J} + \mathcal{J}\mathcal{J}^*\|. \tag{2.12}$$

By using the inequality (2.5) and the inequality (2.10) and Lemma 2.7, we have

Corollary 2.11: *If* $\Psi = \begin{bmatrix} O & \mathcal{J} \\ \mathcal{G} & O \end{bmatrix} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ *and* $q \in (0,1)$, *then*

$$\frac{q}{2-q^2}\max\{\omega(\mathcal{J}), \omega(\mathcal{G})\} \le \omega_q(\Psi) \le \frac{q}{1-\sqrt{1-q^2}}\max\{\omega(\mathcal{J}), \omega(\mathcal{G})\}. \tag{2.13}$$

Therefore, as two norms, the q-numerical radius is equal to the numerical radius. For q-numerical radius, our final findings are extensions of (1.2) and (1.3).

Theorem 2.12: *If* $\Psi = \begin{bmatrix} O & \mathcal{J} \\ \mathcal{G} & O \end{bmatrix}, \Upsilon = \begin{bmatrix} O & \mathcal{D} \\ \mathcal{M} & O \end{bmatrix} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ *and* $q \in (0,1)$, *then*

$$\frac{q}{2-q^2}\max\{\||\mathcal{G}|^2 + |\mathcal{M}|^2\|, \||\mathcal{J}|^2 + |\mathcal{D}|^2\|\} \le \max\{\|\mathcal{J} - \mathcal{D}\|^2, \|\mathcal{G} - \mathcal{M}\|^2\} + 2\omega_q\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathcal{J} \end{bmatrix}\right) \tag{2.14}$$

and

$$\frac{q}{2-q^2}\max\{\|\mathcal{J} + \mathcal{D}\|^2, \|\mathcal{G} + \mathcal{M}\|^2\} \le 2\omega_q\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathcal{J} \end{bmatrix}\right) + \max\{\||\mathcal{G}|^2 + |\mathcal{M}|^2\|, \||\mathcal{J}|^2 + |\mathcal{D}|^2\|\}. \tag{2.15}$$

Proof: Let $\hbar = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \wp = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ such that $\|\hbar\| = \|\wp\| = 1$ and $\langle \hbar, \wp \rangle = q$. Then

$$\begin{aligned}
|\langle (\Psi^*\Psi + \Upsilon^*\Upsilon)\hbar, \wp\rangle| &= |\langle (\Psi^*\Psi + \Upsilon^*\Upsilon)\hbar, \wp\rangle \\
&\quad -(\langle \Psi^*\Upsilon\hbar, \wp\rangle + \langle \Upsilon^*\Psi\hbar, \wp\rangle) \\
&\quad +(\langle \Psi^*\Upsilon\hbar, \wp\rangle + \langle \Upsilon^*\Psi\hbar, \wp\rangle)| \\
&\le |\langle (\Psi^*\Psi + \Upsilon^*\Upsilon)\hbar, \wp\rangle - \langle \Psi^*\Upsilon\hbar, \wp\rangle - \langle \Upsilon^*\Psi\hbar, \wp\rangle| \\
&\quad +|(\langle \Psi^*\Upsilon\hbar, \wp\rangle + \langle \Upsilon^*\Psi\hbar, \wp\rangle)| \\
&\le |\langle (\Psi - \Upsilon)\hbar, (\Psi - \Upsilon)\wp\rangle| + |\langle \Psi^*\Upsilon\hbar, \wp\rangle| + |\langle \Upsilon^*\Psi\hbar, \wp\rangle| \\
&= |\langle (\Psi - \Upsilon)^*(\Psi - \Upsilon)\hbar, \wp\rangle| + |\langle \Psi^*\Upsilon\hbar, \wp\rangle| + |\langle \Upsilon^*\Psi\hbar, \wp\rangle| \\
&\le \omega_{\mathrm{q}}((\Psi - \Upsilon)^*(\Psi - \Upsilon)) + \omega_{\mathrm{q}}(\Psi^*\Upsilon) + \omega_{\mathrm{q}}(\Upsilon^*\Psi) \\
&\le \|(\Psi - \Upsilon)^*(\Psi - \Upsilon)\| + 2\omega_{\mathrm{q}}(\Upsilon^*\Psi) \\
&= \|\Psi - \Upsilon\|^2 + 2\omega_{\mathrm{q}}(\Upsilon^*\Psi) \\
&= \max\{\|\mathcal{J} - \mathcal{D}\|^2, \|\mathcal{G} - \mathcal{M}\|^2\} + 2\omega_{\mathrm{q}}\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathcal{J} \end{bmatrix}\right).
\end{aligned}$$

It is possible to calculate the supremum over all vectors $\hbar, \wp \in \mathcal{K} \oplus \mathcal{K}$ with $\|\hbar\| = \|\wp\| = 1$ and $< \hbar, \wp >= q$ to obtain

$$\omega_{\mathrm{q}}(\Psi^*\Psi + \Upsilon^*\Upsilon) \le \max\{\|\mathcal{J} - \mathcal{D}\|^2, \|\mathcal{G} - \mathcal{M}\|^2\} + 2\omega_{\mathrm{q}}\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathcal{J} \end{bmatrix}\right).$$

Now, using Lemma 2.7 and the fact that $\omega_{\mathrm{q}}(\Psi) \ge \frac{q}{2-q^2}\|\Psi\|$, we get that

$$\begin{aligned}
\omega_{\mathrm{q}}(\Psi^*\Psi + \Upsilon^*\Upsilon) &\ge \frac{q}{2-q^2}\|\Psi^*\Psi + \Upsilon^*\Upsilon\| \\
&= \frac{q}{2-q^2}\max\{\||\mathcal{G}|^2 + |\mathcal{M}|^2\|, \||\mathcal{J}|^2 + |\mathcal{D}|^2\|\}.
\end{aligned}$$

The proof of (2.14) is therefore finished.

Let $\hbar = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \wp = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ so that $\|\hbar\| = \|\wp\| = 1$ in order to demonstrate the inequality (2.15) as well as $< \hbar, \wp >= q$. Next,

$$\begin{aligned}
&|\langle (\Psi + \Upsilon)^*(\Psi + \Upsilon)\hbar, \wp\rangle| \\
&\quad = |\langle (\Psi^*\Psi + \Upsilon^*\Upsilon)\hbar, \wp\rangle + \langle \Upsilon^*\Psi\hbar, \wp\rangle + \langle \Psi^*\Upsilon\hbar, \wp\rangle| \\
&\quad \le |\langle (\Psi^*\Psi + \Upsilon^*\Upsilon)\hbar, \wp\rangle| + |\langle \Upsilon^*\Psi\hbar, \wp\rangle| + |\langle \Psi^*\Upsilon\hbar, \wp\rangle| \\
&\quad \le \|\Psi^*\Psi + \Upsilon^*\Upsilon\| + 2\omega_{\mathrm{q}}(\Upsilon^*\Psi).
\end{aligned}$$

Using $\|\hbar\| = \|\wp\| = 1$ and $< \hbar, \wp >= q$ to obtain the supremum over all vectors $\hbar, \wp \in \mathcal{K} \oplus \mathcal{K}$, we obtain

$$\begin{aligned}
\omega_{\mathrm{q}}((\Psi + \Upsilon)^*(\Psi + \Upsilon)) &\le \|\Psi^*\Psi + \Upsilon^*\Upsilon\| + 2\omega_{\mathrm{q}}(\Upsilon^*\Psi) \\
&= 2\omega_{\mathrm{q}}\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathcal{J} \end{bmatrix}\right) + \max\{\||\mathcal{G}|^2 + |\mathcal{M}|^2\|, \||\mathcal{J}|^2 + |\mathcal{D}|^2\|\}.
\end{aligned}$$

Note that $(\Psi + \Upsilon)^*(\Psi + \Upsilon)$ is self-adjoint. Using [20]'s Theorem 2.1(b), we obtain

$$\frac{q}{2-q^2}\|(\Psi + \Upsilon)^*(\Psi + \Upsilon)\| = \frac{q}{2-q^2}\|\Psi + \Upsilon\|^2$$

$$= \frac{q}{2-q^2} \max\{\|\mathcal{J}+\mathcal{D}\|^2, \|\mathcal{G}+\mathcal{M}\|^2\}$$
$$\leq \omega_q((\Psi+\Upsilon)^*(\Psi+\Upsilon)) = \omega_q\left(\begin{bmatrix} |\mathcal{G}+\mathcal{M}|^2 & O \\ O & |\mathcal{J}+\mathcal{D}|^2 \end{bmatrix}\right)$$
$$\leq 2\omega_q\left(\begin{bmatrix} \mathcal{M}^*\mathcal{G} & O \\ O & \mathcal{D}^*\mathfrak{J} \end{bmatrix}\right) + \max\{\||\mathcal{G}|^2+|\mathcal{M}|^2\|, \||\mathcal{J}|^2+|\mathcal{D}|^2\|\}.$$

The proof is complete.

The sequel makes extensive use of the next two lemmas.

Lemma 2.13: ([7]). *Let* $\gamma, \delta, \rho \in \mathcal{K}$. *Then*
$$|\langle\delta,\gamma\rangle|^2 + |\langle\gamma,\rho\rangle|^2 \leq \|\gamma\|^2(\max\{\|\delta\|^2, \|\rho\|^2\} + |\langle\delta,\rho\rangle|). \tag{2.16}$$

Lemma 2.14: *Let* $\gamma, \eta, \rho \in \mathcal{K}$ *and* $p \geq 1$. *Then*
$$|\langle\eta,\gamma\rangle|^{2p} + |\langle\gamma,\rho\rangle|^{2p} \leq \|\gamma\|^{2p}(\max\{\|\eta\|^{2p}, \|\rho\|^{2p}\} + |\langle\eta,\rho\rangle|^p). \tag{2.17}$$

Theorem 2.15: *Let* $P, Q \in \mathcal{L}(\mathcal{K})$, $q \in (0,1)$ *and* $p \geq 1$. *Then*
$$\omega_q^{2p}(P+Q) \leq 2^{2p-1}(\max\{\|P\|^{2p}, \|Q\|^{2p}\} + \omega^p(Q^*P)).$$

Proof: Let $\xi, \zeta \in \mathcal{K}$ with $\|\xi\| = \|\zeta\| = 1$ and $\langle\xi,\zeta\rangle = q$. Then
$$2^{1-2p}|\langle(P+Q)\xi,\zeta\rangle|^{2p} = 2^{1-2p}|\langle P\xi,\zeta\rangle + \langle Q\xi,\zeta\rangle|^{2p}$$
$$\leq 2^{1-2p}(|\langle P\xi,\zeta\rangle| + |\langle Q\xi,\zeta\rangle|)^{2p}$$
$$\leq |\langle P\xi,\zeta\rangle|^{2p} + |\langle Q\xi,\zeta\rangle|^{2p} \text{ (by Lemma 2.3)}$$
$$\leq \|\zeta\|^{2p}(\max\{\|P\xi\|^{2p}, \|Q\xi\|^{2p}\} + |\langle P\xi, Q\xi\rangle|^p) \text{ (by Lemma 2.14)}$$
$$\leq \max\{\|P\|^{2p}, \|Q\|^{2p}\} + |\langle Q^*P\xi,\xi\rangle|^p$$
$$\leq \max\{\|P\|^{2p}, \|Q\|^{2p}\} + \omega^p(Q^*P).$$

Using $\|\xi\| = \|\zeta\| = 1$ and $\langle\xi,\zeta\rangle = q$ to obtain the supremum over all vectors $\xi, \zeta \in \mathcal{K}$, we obtain
$$\omega_q^{2p}(P+Q) \leq 2^{2p-1}(\max\{\|P\|^{2p}, \|Q\|^{2p}\} + \omega^p(Q^*P)).$$

The evidence is now complete.

Theorem 2.16: *Let* $\Psi = \begin{bmatrix} \mathcal{J} & \mathcal{G} \\ \mathcal{D} & \mathcal{M} \end{bmatrix} \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ *and* $q \in (0,1)$ *and* $p \geq 1$. *Then*
$$\omega_q^{2p}\left(\begin{bmatrix} \mathcal{J} & \mathcal{G} \\ \mathcal{D} & \mathcal{M} \end{bmatrix}\right) \leq 2^{p-1}(\max\{\||\mathcal{J}|^2+|\mathcal{G}^*|^2\|, \||\mathcal{M}|^2+|\mathcal{D}^*|^2\|\}$$
$$+ \max\{\||\mathcal{J}|^2-|\mathcal{G}^*|^2\|, \||\mathcal{M}|^2-|\mathcal{D}^*|^2\|\})^p + 2^{2p-1}\omega^p\left(\begin{bmatrix} O & \mathcal{D}^*\mathcal{M} \\ \mathcal{G}^*\mathcal{J} & O \end{bmatrix}\right).$$

Proof: Let $\Theta = \begin{bmatrix} \mathcal{J} & O \\ O & \mathcal{M} \end{bmatrix}$ and $\Xi = \begin{bmatrix} O & \mathcal{G} \\ \mathcal{D} & O \end{bmatrix}$, and let $\hbar = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \wp = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ such that $\|\hbar\| = \|\wp\| = 1$ and $< \langle\hbar,\wp\rangle = q$. Then

$$\begin{aligned}2^{1-2p}|\langle\Psi\hbar,\wp\rangle|^{2p} &= 2^{1-2p}|\langle\Theta\hbar,\wp\rangle+\langle\Xi\hbar,\wp\rangle|^{2p}\\ &\le 2^{1-2p}(|\langle\Theta\hbar,\wp\rangle|+|\langle\Xi\hbar,\wp\rangle|)^{2p} \text{ (by the triangle inequality)}\\ &\le |\langle\Theta\hbar,\wp\rangle|^{2p}+|\langle\Xi\hbar,\wp\rangle|^{2p} \text{ (by Lemma 2.3)}\\ &\le (\max\{\|\Theta\hbar\|^2,\|\Xi\hbar\|^2\})^p+|\langle\Theta\hbar,\Xi\hbar\rangle|^p \text{ (by Lemma 2.14)}\\ &= \frac{1}{2^p}(\|\Theta\hbar\|^2+\|\Xi\hbar\|^2+|\|\Theta\hbar\|^2-\|\Xi\hbar\|^2|)^p+|\langle\Theta\hbar,\Xi\hbar\rangle|^p\\ &= \frac{1}{2^p}(\langle(|\Theta|^2+|\Xi|^2)\hbar,\hbar\rangle+|\langle(|\Theta|^2-|\Xi|^2)\hbar,\hbar\rangle|)^p+|\langle\Xi^*\Theta\hbar,\hbar\rangle|^p\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2^p}\left(\begin{bmatrix}|\mathcal{J}|^2+|\mathcal{G}^*|^2 & O\\ O & |\mathcal{M}|^2+|\mathcal{D}^*|^2\end{bmatrix}\hbar,\hbar\right.\\ &\quad\left.+\begin{bmatrix}|\mathcal{J}|^2-|\mathcal{G}^*|^2 & O\\ O & |\mathcal{M}|^2-|\mathcal{D}^*|^2\end{bmatrix}\hbar,\hbar\right)^p+\left|\left\langle\begin{bmatrix}O & \mathcal{D}^*\mathcal{M}\\ \mathcal{G}^*\mathcal{J} & O\end{bmatrix}\hbar,\hbar\right\rangle\right|\\ &\le \frac{1}{2^p}(\max\{\||\mathcal{J}|^2+|\mathcal{G}^*|^2\|,\||\mathcal{M}|^2+|\mathcal{D}^*|^2\|\}\\ &\quad+\max\{\||\mathcal{J}|^2-|\mathcal{G}^*|^2\|,\||\mathcal{M}|^2-|\mathcal{D}^*|^2\|\})^p+\omega^p\left(\begin{bmatrix}O & \mathcal{D}^*\mathcal{M}\\ \mathcal{G}^*\mathcal{J} & O\end{bmatrix}\right).\end{aligned}$$

Utilizing $\langle\hbar,\wp\rangle=q$ and $\|\hbar\|=\|\wp\|=1$ to determine the supremum over all vectors $\hbar,\wp\in\mathcal{K}$, we arrive at

$$\begin{aligned}\omega_q^{2p}\left(\begin{bmatrix}\mathcal{J} & \mathcal{G}\\ \mathcal{D} & \mathcal{M}\end{bmatrix}\right) &\le 2^{p-1}(\max\{\||\mathcal{J}|^2+|\mathcal{G}^*|^2\|,\||\mathcal{M}|^2+|\mathcal{D}^*|^2\|\}\\ &\quad+\max\{\||\mathcal{J}|^2-|\mathcal{G}^*|^2\|,\||\mathcal{M}|^2-|\mathcal{D}^*|^2\|\})^p+2^{2p-1}\omega^p\left(\begin{bmatrix}O & \mathcal{D}^*\mathcal{M}\\ \mathcal{G}^*\mathcal{J} & O\end{bmatrix}\right).\end{aligned}$$

Remark 2.17: The following series of refinements may be obtained by using $\mathcal{J}=\mathcal{G}=\mathcal{D}=\mathcal{M}$ from Theorem 2.16

$$\begin{aligned}\omega_q^{2p}(\mathcal{J}) &\le 2^{p-1}(\||\mathcal{J}|^2+|\mathcal{J}^*|^2\|+\||\mathcal{J}|^2-|\mathcal{J}^*|^2\|)^p+2^{2p-1}\omega^p(|\mathcal{J}|^2)\\ &= 2^{2p-1}\left(\frac{\||\mathcal{J}|^2+|\mathcal{J}^*|^2\|+\||\mathcal{J}|^2-|\mathcal{J}^*|^2\|}{2}\right)^p+2^{2p-1}\omega^p(|\mathcal{J}|^2)\\ &= 2^{2p-2}\left(\frac{\||\mathcal{J}|^2+|\mathcal{J}^*|^2\|^p+\||\mathcal{J}|^2-|\mathcal{J}^*|^2\|^p}{2}\right)+2^{2p-1}\omega^p(|\mathcal{J}|^2).\end{aligned}$$

Lemma 2.18 ([3]). *Assuming ρ is a nonzero complex integer and $\|\xi\|=1$, let $\xi,\zeta,\eta\in\mathcal{K}$. Then,*

$$|\langle\zeta,\xi\rangle\langle\xi,\eta\rangle|\le\frac{1}{|\rho|}(\max\{1,|\rho-1|\}\|\zeta\|\|\eta\|+|\langle\zeta,\eta\rangle|).$$

Theorem 2.19: *Let $\mathcal{P},\mathcal{Q}\in\mathcal{L}(\mathcal{K})$ and the complex number τ should not be zero. After that,*

$$\omega_q^2(\mathcal{P}+\mathcal{Q})\le\left(1+\frac{1}{|\tau|}\max\{1,|\tau-1|\}\right)\||\mathcal{P}|^2+|\mathcal{Q}|^2\|+\frac{2}{|\tau|}\omega(\mathcal{Q}^*\mathcal{P}). \tag{2.18}$$

Proof: Let $\xi,\zeta\in\mathcal{K}$ with $\|\xi\|=\|\zeta\|=1$ and $\langle\xi,\zeta\rangle=q$. Then

$|\langle(\mathcal{P}+\mathcal{Q})\xi,\zeta\rangle|^2 \leq (|\langle\mathcal{P}\xi,\zeta\rangle| + |\langle\mathcal{Q}\xi,\zeta\rangle|)^2$
$= |\langle\mathcal{P}\xi,\zeta\rangle|^2 + |\langle\mathcal{Q}\xi,\zeta\rangle|^2 + 2|\langle\mathcal{P}\xi,\zeta\rangle||\langle\mathcal{Q}\xi,\zeta\rangle|$
$\leq |\langle\mathcal{P}\xi,\zeta\rangle|^2 + |\langle\mathcal{Q}\xi,\zeta\rangle|^2 + \frac{2}{|\tau|}(\max\{1,|\tau-1|\}\|\mathcal{P}\xi\|\|\mathcal{Q}\xi\| + |\langle\mathcal{P}\xi,\mathcal{Q}\xi\rangle|)$
(by Lemma 2.18)
$\leq \|\mathcal{P}\xi\|^2 + \|\mathcal{Q}\xi\|^2 + \frac{2}{|\tau|}\left(\max\{1,|\tau-1|\}\sqrt{\langle\mathcal{P}\xi,\mathcal{P}\xi\rangle\langle\xi,\mathcal{Q}\xi\rangle\mathcal{Q}} + |\langle\mathcal{P}\xi,\mathcal{Q}\xi\rangle|\right)$
$= \langle|\mathcal{P}|^2\xi,\xi\rangle + \langle|\mathcal{Q}|^2\xi,\xi\rangle + \frac{2}{|\tau|}\left(\max\{1,|\tau-1|\}\sqrt{\langle|\mathcal{P}|^2\xi,\xi\rangle\langle|\mathcal{Q}|^2\xi,\xi\rangle} + |\langle\mathcal{P}\xi,\mathcal{Q}\xi\rangle|\right)$
$= \langle(|\mathcal{P}|^2+|\mathcal{Q}|^2)\xi,\xi\rangle + \frac{2}{|\tau|}\left(\max\{1,|\tau-1|\}\sqrt{\langle|\mathcal{P}|^2\xi,\xi\rangle\langle|\mathcal{Q}|^2\xi,\xi\rangle} + |\langle\mathcal{Q}^*\mathcal{P}\xi,\xi\rangle|\right)$
$\leq \langle(|\mathcal{P}|^2+|\mathcal{Q}|^2)\xi,\xi\rangle + \frac{1}{|\tau|}\max\{1,|\tau-1|\}\langle(|\mathcal{P}|^2+|\mathcal{Q}|^2)\xi,\xi\rangle + \frac{2}{|\tau|}|\langle\mathcal{Q}^*\mathcal{P}\xi,\xi\rangle|$
(by the arithmetic − geometric mean inequality)
$\leq \left(1+\frac{1}{|\tau|}\max\{1,|\tau-1|\}\right)\langle(|\mathcal{P}|^2+|\mathcal{Q}|^2)\xi,\xi\rangle + \frac{2}{|\tau|}|\langle\mathcal{Q}^*\mathcal{P}\xi,\xi\rangle|$
$\leq \left(1+\frac{1}{|\tau|}\max\{1,|\tau-1|\}\right)\||\mathcal{P}|^2+|\mathcal{Q}|^2\| + \frac{2}{|\tau|}\omega(\mathcal{Q}^*\mathcal{P}).$

Identifying the supremum over all vectors $\xi,\zeta\in\mathcal{K}$ using $\langle\xi,\zeta\rangle = q$ and $\|\xi\| = \|\zeta\| = 1$ allows us to

$$\omega_q^2(\mathcal{P}+\mathcal{Q}) \leq \left(1+\frac{1}{|\tau|}\max\{1,|\tau-1|\}\right)\||\mathcal{P}|^2+|\mathcal{Q}|^2\| + \frac{2}{|\tau|}\omega(\mathcal{Q}^*\mathcal{P}).$$

Using Theorem 2.19 and Theorem 2.1 of [20], we have

Corollary 2.20: *With υ being a nonzero complex number, let $\mathcal{P},\mathcal{Q}\in\mathcal{L}(\mathcal{K})$, and $q\in[0,1]$ be considered. Next,*

$$\frac{1}{4}\left(\frac{q}{2-q^2}\right)\|\mathcal{P}+\mathcal{Q}\|^2 \leq \left(1+\frac{1}{|\upsilon|}\max\{1,|\upsilon-1|\}\right)\||\mathcal{P}|^2+|\mathcal{Q}|^2\| + \frac{2}{|\upsilon|}\omega(\mathcal{Q}^*\mathcal{P})$$

As a consequence of Lemma 2.18, we have

Proposition 2.21: Let $\xi,\zeta,\eta\in\mathcal{K}$ with $\|\xi\|=1$, τ a nonzero complex number and $\vartheta = max\{1,|\tau-1|\}$. Then

$$|\langle\zeta,\xi\rangle\langle\xi,\eta\rangle|^2 \leq \frac{\vartheta^2}{|\tau|^2}\|\zeta\|^2\|\eta\|^2 + \frac{2\vartheta+1}{|\tau|^2}\|\zeta\|\|\eta\||\langle\zeta,\eta\rangle|. \tag{2.19}$$

Theorem 2.22: *Let $\Psi = \begin{bmatrix}\mathcal{J} & \mathcal{D}\\ \mathcal{G} & \mathcal{M}\end{bmatrix}\in\mathcal{L}(\mathcal{K}\oplus\mathcal{K})$, $q\in(0,1)$, τ a nonzero complex number and $\vartheta = max\{1,|\tau-1|\}$. Then*

$$\frac{1}{8}\omega_q^4\left(\begin{bmatrix}\mathcal{J} & \mathcal{D}\\ \mathcal{G} & \mathcal{M}\end{bmatrix}\right) \leq \frac{q^4\vartheta^2}{|\tau|^2}\max\{\||\mathcal{J}|^4+|\mathcal{J}^*|^4\|,\||\mathcal{M}|^4+|\mathcal{M}^*|^4\|\}$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\max\{\||\mathcal{J}|^2+|\mathcal{J}^*|^2\|,\||\mathcal{M}|^2+|\mathcal{M}^*|^2\|\}\max\{\omega(\mathcal{J}^2),\omega(\mathcal{M}^2)\}$$
$$+2Æ^2\max\{\|\mathcal{J}\|^4,\|\mathcal{M}\|^4\} + \frac{q^4\vartheta^2}{|\tau|^2}\max\{\||\mathcal{G}|^4+|\mathcal{D}^*|^4\|,\||\mathcal{D}|^4+|\mathcal{G}^*|^4\|\}$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\max\{\||\mathcal{G}|^2+|\mathcal{D}^*|^2\|,\||\mathcal{D}|^2+|\mathcal{G}^*|^2\|\}\max\{\omega(\mathcal{G}\mathcal{D}),\omega(\mathcal{D}\mathcal{G})\}$$
$$+2Æ^2\max\{\|\mathcal{D}\|^4,\|\mathcal{G}\|^4\},$$

where $Æ = 1 - q^2 + q\sqrt{1-q^2}$.

Proof: Let $\Theta = \begin{bmatrix} \mathcal{J} & O \\ O & \mathcal{M} \end{bmatrix}$ and $\Xi = \begin{bmatrix} O & \mathcal{D} \\ \mathcal{G} & O \end{bmatrix}$, and let $\hbar = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \wp = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ such that $\langle \hbar, \wp \rangle = q$. Then, $\wp$ may be written as $\wp = q\hbar + \sqrt{1-q^2}\aleph$, where $\langle \hbar, \aleph \rangle = 0$ and $\|\aleph\| = 1$. For each $\Upsilon \in \mathcal{L}(\mathcal{K} \oplus \mathcal{K})$ in this configuration, we have that

$$|\langle \Upsilon\hbar, \wp \rangle| \leq q|\langle \Upsilon\hbar, \hbar \rangle| + \sqrt{1-q^2}|\langle \Upsilon\hbar, \aleph \rangle|. \tag{2.20}$$

Assume that F is any orthonormal set that contains $\hbar$ and $\aleph$ in $\mathcal{K} \oplus \mathcal{K}$. It follows from "Bessel's Inequality"

$$\sum_{e \in F \setminus \{\hbar\}} |\langle \Upsilon\hbar, e \rangle|^2 + |\langle \Upsilon\hbar, \hbar \rangle|^2 \leq \|\Upsilon\hbar\|^2$$
$$\Rightarrow |\langle \Upsilon\hbar, \aleph \rangle|^2 \leq \sum_{e \in F \setminus \{\hbar\}} |\langle \Upsilon\hbar, e \rangle|^2 \leq \|\Upsilon\hbar\|^2 - |\langle \Upsilon\hbar, \hbar \rangle|^2.$$

Using the above relation (2.20), we get

$$\begin{aligned}
&|\langle \Upsilon\hbar, \wp \rangle|^2 \leq q|\langle \Upsilon\hbar, \hbar \rangle| + \sqrt{1-q^2}(\|\Upsilon\hbar\|^2 - |\langle \Upsilon\hbar, \hbar \rangle|^2)^{\frac{1}{2}} \\
&\Rightarrow |\langle \Upsilon\hbar, \wp \rangle|^2 \leq \left(q|\langle \Upsilon\hbar, \hbar \rangle| + \sqrt{1-q^2}(\|\Upsilon\hbar\|^2 - |\langle \Upsilon\hbar, \hbar \rangle|^2)^{\frac{1}{2}}\right)^2 \\
&\quad = q^2|\langle \Upsilon\hbar, \hbar \rangle|^2 + 2q\sqrt{1-q^2}|\langle \Upsilon\hbar, \hbar \rangle|(\|\Upsilon\hbar\|^2 - |\langle \Upsilon\hbar, \hbar \rangle|^2)^{\frac{1}{2}} \\
&\quad +(1-q^2)(\|\Upsilon\hbar\|^2 - |\langle \Upsilon\hbar, \hbar \rangle|^2) \\
&\quad = q^2|\langle \Upsilon\hbar, \hbar \rangle|^2 + Æ\|\Upsilon\hbar\|^2.
\end{aligned} \tag{2.21}$$

Now,

$$\begin{aligned}
&\frac{1}{8}|\langle \Psi\hbar, \wp \rangle|^4 \leq (|\langle \Theta\hbar, \wp \rangle| + |\langle \Xi\hbar, \wp \rangle|)^4 \\
&\leq (|\langle \Theta\hbar, \wp \rangle|^4 + |\langle \Xi\hbar, \wp \rangle|^4) \text{(by Lemma 2.3)} \\
&\leq (q^2|\langle \Theta\hbar, \hbar \rangle|^2 + Æ\|\Theta\hbar\|^2)^2 \\
&+(q^2|\langle \Xi\hbar, \hbar \rangle|^2 + Æ\|\Xi\hbar\|^2)^2 \text{(by the inequality (2.21))} \\
&\leq 2(q^4|\langle \Theta\hbar, \hbar \rangle|^4 + Æ^2\|\Theta\hbar\|^4) \\
&+2(q^4|\langle \Xi\hbar, \hbar \rangle|^4 + Æ^2\|\Xi\hbar\|^4) \text{(by Lemma 2.3)} \\
&\leq 2q^4|\langle \Theta\hbar, \hbar \rangle\langle \hbar, \Theta^*\hbar \rangle|^2 + 2Æ^2\|\Theta\|^4 \\
&+2q^4|\langle \Xi\hbar, \hbar \rangle\langle \hbar, \Xi^*\hbar \rangle|^2 + 2Æ^2\|\Xi\|^4 \\
&\leq 2q^4\left(\frac{\vartheta^2}{|\tau|^2}\|\Theta\hbar\|^2\|\Theta^*\hbar\|^2 + \frac{1+2\vartheta}{|\tau|^2}\|\Theta\hbar\|\|\Theta^*\hbar\||\langle \Theta\hbar, \Theta^*\hbar \rangle|\right) + 2Æ^2\|\Theta\|^4 \\
&+2q^4\left(\frac{\vartheta^2}{|\tau|^2}\|\Xi\hbar\|^2\|\Xi^*\hbar\|^2 + \frac{1+2\vartheta}{|\tau|^2}\|\Xi\hbar\|\|\Xi^*\hbar\||\langle \Xi\hbar, \Xi^*\hbar \rangle|\right) + 2Æ^2\|\Xi\|^4 \\
&\text{(by Proposition 2.21)} \\
&\leq 2q^4\left(\frac{\vartheta^2}{|\tau|^2}\langle |\Theta|^2\hbar, \hbar \rangle\langle |\Theta^*|^2\hbar, \hbar \rangle + \frac{1+2\vartheta}{|\tau|^2}\sqrt{\langle |\Theta|^2\hbar, \hbar \rangle\langle |\Theta^*|^2\hbar, \hbar \rangle}|\langle \Theta^2\hbar, \hbar \rangle|\right) \\
&+2Æ^2\|\Theta\|^4 \\
&+2q^4\left(\frac{\vartheta^2}{|\tau|^2}\langle |\Xi|^2\hbar, \hbar \rangle\langle |\Xi^*|^2\hbar, \hbar \rangle + \frac{1+2\vartheta}{|\tau|^2}\sqrt{\langle |\Xi|^2\hbar, \hbar \rangle\langle |\Xi^*|^2\hbar, \hbar \rangle}|\langle \Xi^2\hbar, \hbar \rangle|\right) \\
&+2Æ^2\|\Xi\|^4
\end{aligned}$$

$$\leq \frac{q^4\vartheta^2}{|\tau|^2}\langle(|\Theta|^4+|\Theta^*|^4)\hbar,\hbar\rangle+\frac{(1+2\vartheta)q^4}{|\tau|^2}\langle(|\Theta|^2+|\Theta^*|^2)\hbar,\hbar\rangle|\langle\Theta^2\hbar,\hbar\rangle|$$
$$+2Æ^2\|\Theta\|^4$$
$$+\frac{q^4\vartheta^2}{|\tau|^2}\langle(|\Xi|^4+|\Xi^*|^4)\hbar,\hbar\rangle+\frac{(1+2\vartheta)q^4}{|\tau|^2}\langle(|\Xi|^2+|\Xi^*|^2)\hbar,\hbar\rangle|\langle\Xi^2\hbar,\hbar\rangle|$$
$$+2Æ^2\|\Xi\|^4$$
$$=\frac{q^4\vartheta^2}{|\tau|^2}\langle\begin{bmatrix}|\mathcal{J}|^4+|\mathcal{G}^*|^4 & O\\ O & |\mathcal{M}|^4+|\mathcal{M}^*|^4\end{bmatrix}\hbar,\hbar\rangle$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\langle\begin{bmatrix}|\mathcal{G}|^2+|\mathcal{G}^*|^2 & O\\ O & |\mathcal{G}|^2+|\mathcal{G}^*|^2\end{bmatrix}\hbar,\hbar\rangle\langle\begin{bmatrix}\mathcal{G}^2 & O\\ O & \mathcal{G}^2\end{bmatrix}\hbar,\hbar\rangle$$
$$+2Æ^2\left\|\begin{bmatrix}\mathcal{G} & O\\ O & \mathcal{G}\end{bmatrix}\right\|^4$$
$$+\frac{q^4\vartheta^2}{|\tau|^2}\langle\begin{bmatrix}|\mathcal{D}|^4+|\mathcal{G}^*|^4 & O\\ O & |\mathcal{G}|^4+|\mathcal{G}^*|^4\end{bmatrix}\hbar,\hbar\rangle$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\langle\begin{bmatrix}|C|^2+|\mathcal{G}^*|^2 & O\\ O & |\mathcal{G}|^2+|C^*|^2\end{bmatrix}\hbar,\hbar\rangle\ \langle\begin{bmatrix}\mathcal{G}\mathcal{D} & O\\ O & \mathcal{D}\mathcal{G}\end{bmatrix}\hbar,\hbar\rangle$$
$$+2Æ^2\left\|\begin{bmatrix}O & \mathcal{G}\\ \mathcal{D} & O\end{bmatrix}\right\|^4$$
$$\leq\frac{q^4\vartheta^2}{|\tau|^2}\max\{\||\mathcal{J}|^4+|\mathcal{J}^*|^4\|,\||\mathcal{M}|^4+|\mathcal{M}^*|^4\|\}$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\max\{\||\mathcal{J}|^2+|\mathcal{J}^*|^2\|,\||\mathcal{M}|^2+|\mathcal{M}^*|^2\|\}\max\{\omega(\mathcal{J}^2),\omega(\mathcal{M}^2)\}$$
$$+2Æ^2\max\{\|\mathcal{J}\|^4,\|\mathcal{M}\|^4\}+\frac{q^4\vartheta^2}{|\tau|^2}\max\{\||\mathcal{D}|^4+|\mathcal{G}^*|^4\|,\||\mathcal{G}|^4+|\mathcal{J}^*|^4\|\}$$
$$+\frac{(1+2\vartheta)q^4}{|\tau|^2}\max\{\||\mathcal{D}|^2+|\mathcal{G}^*|^2\|,\||\mathcal{G}|^2+|\mathcal{D}^*|^2\|\}\max\{\omega(\mathcal{G}\mathcal{D}),\omega(\mathcal{D}\mathcal{G})\}$$
$$+2Æ^2\max\{\|\mathcal{G}\|^4,\|\mathcal{D}\|^4\}.$$

The evidence is finished now.

Based on Theorem 2.22's proof, we have

Corollary 2.23: *Let* $\Upsilon=\begin{bmatrix}O & \mathcal{G}\\ \mathcal{D} & O\end{bmatrix}$ *and* $q\in(0,1)$. *Then*

$$\omega_q^2(\Upsilon)\leq q^2\max\{\omega(\mathcal{G}\mathcal{D}),\omega(\mathcal{D}\mathcal{G})\}+Æ\max\{\|\mathcal{G}\|^2,\|\mathcal{D}\|^2\},$$

where $Æ=1-q^2+q\sqrt{1-q^2}$.

Lemma 2.24 ([23]): *Let* $\xi,\zeta,\eta\in\mathcal{K}$. *Then*

$$|\langle\xi,\zeta\rangle|^2+|\langle\xi,\eta\rangle|^2\leq\|\xi\|^2(|\langle\zeta,\zeta\rangle|^2+|\langle\eta,\eta\rangle|^2+2|\langle\zeta,\eta\rangle|^2)^{\frac{1}{2}}. \quad (2.22)$$

Theorem 2.25: *Let* $\mathcal{D}\in\mathcal{L}(\mathcal{K})$, $q\in(0,1)$ *and* $p\geq 2$. *Then*

$$\omega_q^{2p}(\mathcal{D})\leq q^{2p}2^{p-1}(\||\mathcal{D}|^{2p}+|\mathcal{D}^*|^{2p}\|+\omega^p(\mathcal{D}^2))+2^{p-1}Æ^{2p}\|\mathcal{D}\|^{2p},$$

where $Æ=1-q^2+q\sqrt{1-q^2}$

Proof: Let $\xi,\zeta \in \mathcal{K}$ with $\|\xi\| = \|\zeta\| = 1$ and $\langle \xi,\zeta\rangle = q$. Then

$$\frac{1}{2^p}|\langle \mathcal{D}\xi,\zeta\rangle|^{2p} = \left(\frac{2|\langle \mathcal{D}\xi,\zeta\rangle|^2}{4}\right)^p \leq \left(\frac{2q|\langle \mathcal{D}\xi,\xi\rangle|^2+2Æ\|\mathcal{D}\|^2}{4}\right)^p$$

$$= \left(\frac{q^2(|\langle \mathcal{D}\xi,\xi\rangle|^2+|\langle \xi,\mathcal{D}^*\xi\rangle|^2)+2Æ^2\|\mathcal{D}\|^2}{4}\right)^p$$

$$\leq \left(\frac{q^2\left(|\langle \mathcal{D}\xi,\mathcal{D}\xi\rangle|^2+|\langle \mathcal{D}^*\xi,\mathcal{D}^*\xi\rangle|^2+2|\langle \mathcal{D}\xi,\mathcal{D}^*\xi\rangle|^2\right)^{\frac{1}{2}}+2Æ^2\|\mathcal{D}\|^2}{4}\right)^p$$

(by Lemma 2.24)

$$\leq 2^{p-1}\left(\frac{q^2\left(|\langle \mathcal{D}\xi,\mathcal{D}\xi\rangle|^2+|\langle \mathcal{D}^*\xi,\mathcal{D}^*\xi\rangle|^2+2|\langle \mathcal{D}\xi,\mathcal{D}^*\xi\rangle|^2\right)^{\frac{1}{2}}}{4}\right)^p + 2^{p-1}\left(\frac{2Æ^2\|\mathcal{D}\|^2}{4}\right)^p$$

$$\leq \frac{q^{2p}}{2}\left(\frac{|\langle \mathcal{D}\xi,\mathcal{D}\xi\rangle|^p+|\langle \mathcal{D}^*\xi,\mathcal{D}^*\xi\rangle|^p+2|\langle \mathcal{D}\xi,\mathcal{D}^*\xi\rangle|^p}{4}\right) + \frac{1}{2}Æ^{2p}\|\mathcal{D}\|^{2p}$$

$$\leq \frac{q^{2p}}{2}\left(\frac{1}{4}\langle |\mathcal{D}|^{2p}\xi,\xi\rangle + \frac{1}{4}\langle |\mathcal{D}^*|^{2p}\xi,\xi\rangle + \frac{1}{2}|\langle \mathcal{D}^2\xi,\xi\rangle|^p\right) + \frac{1}{2}Æ^{2p}\|\mathcal{D}\|^{2p}$$

$$\leq \frac{q^{2p}}{2}\left(\frac{1}{4}\langle (|\mathcal{D}|^{2p}+|\mathcal{D}^*|^{2p})\xi,\xi\rangle + \frac{1}{2}|\langle \mathcal{D}^2\xi,\xi\rangle|^p\right) + \frac{1}{2}Æ^{2p}\|\mathcal{D}\|^{2p}.$$

Finding the supremum over all vectors $\xi,\zeta \in \mathcal{K}$ using $\langle \xi,\zeta\rangle = q$ and $\|\xi\| = \|\zeta\| = 1$ yields

$$\omega_{\mathrm{q}}^{2p}(\mathcal{D}) \leq q^{2p}2^{p-1}(\||\mathcal{D}|^{2p} + |\mathcal{D}^*|^{2p}\| + \omega^p(\mathcal{D}^2)) + 2^{p-1}Æ^{2p}\|\mathcal{D}\|^{2p}.$$

3. Conclusion and future Work

In conclusion, this study introduces novel upper bounds on the q-numerical radius, which makes a substantial contribution to the field of bounded linear operators in complex Hilbert spaces. Particularly in q -numerical radius inequalities for operator sums and two by two operator matrices, our contributions add to the body of knowledge and provide insightful information. The versatility and applicability of our derived results provide a comprehensive understanding of the q-numerical radius in various scenarios, extending beyond theoretical considerations to potential applications in mathematics and related fields. Future directions include extending our findings to higher dimensions, exploring connections with other operator norms, and investigating practical applications in quantum mechanics, signal processing, and optimization problems. This research establishes a foundation for deeper exploration, paving the way for further investigations and applications in diverse scientific and mathematical domains.

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