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Pseudospectral numerical solution of time fractional parabolic partial Volterra–Fredholm integro-differential equations

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Abstract

Physical processes with memory are often described by Volterra–Fredholm integro-differential models in which the time derivative has fractional order. Models built only on integer-order derivatives usually cannot reproduce such hereditary behaviour accurately. This work proposes a collocation-type pseudospectral algorithm, based on Chebyshev cardinal basis functions, for a family of parabolic integro-differential problems with a Caputo time derivative. The scheme handles both the nonlocality of the fractional operator and the integral terms in a single algebraic framework. We also prove that the scheme converges, which supports the accuracy and reliability of the computed approximations.

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1. Introduction

Partial integro-differential equations of Volterra–Fredholm type with fractional derivatives are now widely used to describe dynamical processes in physics, engineering and biology. Replacing an integer-order derivative

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by a fractional one allows the model to carry memory and hereditary effects, which are common in real materials and systems but are missed by classical models, in particular when both the temporal and the spatial structure of the process are complicated.

Closed-form solutions of such problems are rarely available, because the fractional operator is nonlocal and the integral terms add further difficulty. Reliable numerical techniques are therefore needed. Here we use a pseudospectral discretization built on Chebyshev cardinal functions for a specific class of time-fractional partial Volterra–Fredholm problems.

Two features make this choice attractive. First, the cardinal functions are Lagrange-type basis functions built on Chebyshev nodes, so the unknown coefficients are simply the values of the approximate solution at the nodes. Boundary conditions and nonlinear terms can then be imposed pointwise, without evaluating inner products. Second, interpolation at Chebyshev nodes does not suffer from the oscillations (the Runge phenomenon) that appear with equally spaced nodes, and for smooth functions the interpolation error decays faster than any fixed power of the number of nodes. These properties are particularly useful for problems with a memory kernel, where the solution at a given time depends on its whole history and a low-accuracy discretization would accumulate error through the integral operators.

For the classical (integer-order) parabolic case, a pseudospectral treatment was given in [11], where the problem reads

$$\begin{cases} u_t(t,x) + \kappa\, u_{xx}(t,x) = \lambda \int_0^t \mathcal{K}\left(t,x,\rho,u(\rho,x)\right) d\rho + \mathcal{G}(t,x), \\ u(0,x) = \eta_0(x), 0 \le x \le 1, \\ u(t,0) = b_0(t), u(t,1) = b_1(t), 0 \le t \le T, \end{cases} \tag{1.1}$$

with constant coefficients κ and λ and sufficiently smooth data $\mathcal{G}$ and $\mathcal{K}$. In this paper the time derivative in (1.1) is replaced by a Caputo derivative of order $\gamma \in (0,1)$, which gives the model problem

$$\begin{cases} {}^{c}D_t^{\gamma} u(t,x) + \kappa\, u_{xx}(t,x) = \lambda \int_0^t \mathcal{K}\left(t,x,\rho,u(\rho,x)\right) d\rho + \mathcal{G}(t,x), \\ u(t,0) = b_0(t), u(t,1) = b_1(t), t \in [0,T], \\ u(0,x) = \eta_0(x), x \in [0,1]. \end{cases} \tag{1.2}$$

In addition, the kernel is assumed to satisfy the bound

$$\| \frac{\partial \mathcal{K}(t,x,\rho,u)}{\partial u} \| \le \Theta, \tag{1.3}$$

for some constant $\Theta \ge 0$.

Processes that evolve in time and space are normally written as partial differential equations. Such a description, however, ignores the influence of the system's past states, which matters in thermoelasticity, in heat conduction and in the kinetics of nuclear reactors. To account for this

influence one adds an integral over the history to the equation, which produces partial integro-differential equations [12]. Existence, uniqueness and long-time behaviour for such equations are studied in [4].

Allowing derivatives and integrals to have arbitrary real order gives the theory of fractional calculus, which is employed in many areas of science and technology. In a more recent direction the order is itself allowed to change, so that the constant-order theory appears as a special case; physical uses of such variable-order operators are discussed in [3, 9]. The literature contains several non-equivalent definitions of variable-order operators, associated with the names of Riemann–Liouville, Marchaud, Coimbra, Caputo and Grünwald–Letnikov [10]. Numerical treatments of boundary value problems of variable order are found in [8, 13, 14], and a method-of-lines approach for fourth-order fractional partial differential equations was proposed in [16]. Exact solutions of some linear fractional integro-differential equations were obtained with the Shehu transform in [15], while an integro-differential problem from geophysics was analysed by Grasselli, Kabanikhin, and Lorenzi [5].

The approach of this paper couples the pseudospectral discretization with a Newton iteration to produce accurate approximate solutions at moderate cost. We regard this combination as a practical substitute for analytical solutions of fractional partial Volterra–Fredholm problems.

The paper is organized as follows. Section 2 recalls the necessary notions from fractional calculus and the Chebyshev cardinal basis. Section 3 converts problem (1.2) into an equivalent integral-type equation. Section 4 describes the pseudospectral discretization of that equation. Section 5 contains the convergence proof, and Section 6 concludes.

2. Preliminaries

This section collects the definitions needed later: the fractional-calculus facts follow [6], the Chebyshev cardinal technique follows [7], and the classical results on Newton's method can be found in [1].

Definition 2.1: Let $I = [p, q]$ with $-\infty < p < q < \infty$, let $\varphi \in L_1[p, q]$ and let $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. The Riemann–Liouville fractional integral of order γ is [6]

$$\left(\mathcal{J}_{p^+}^{\gamma}\varphi\right)(t) = \frac{1}{\Gamma(\gamma)}\int_p^t (t-\theta)^{\gamma-1}\,\varphi(\theta)\,d\theta \quad (t > p), \tag{2.1}$$

and the corresponding Riemann–Liouville derivative is

$$\left(\mathcal{D}_{p^+}^{\gamma}\omega\right)(t) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n \int_p^t \frac{\omega(\theta)\,d\theta}{(t-\theta)^{\gamma-n+1}}, \qquad n = \lfloor \Re(\gamma) \rfloor + 1.$$

The Caputo derivative of order γ on $[p, q]$ is obtained by subtracting the Taylor polynomial first:

$$\left({}^{C}\mathcal{D}_{p^+}^{\gamma}\omega\right)(t) = \mathcal{D}_{p^+}^{\gamma}\left[\omega(t) - \sum_{l=0}^{n-1} \frac{\omega^{(l)}(p)}{l!}(t-p)^l\right]. \tag{2.2}$$

For $0 < \Re(\gamma) < 1$ this simplifies to

$$\left({}^{C}\mathcal{D}_{p^+}^{\gamma}\omega\right)(t) = \mathcal{D}_{p^+}^{\gamma}[\omega(t) - \omega(p)]. \tag{2.3}$$

Remark 2.1: The Caputo derivative is preferred to the Riemann–Liouville derivative in applied models for a practical reason: for $0 < \gamma < 1$ it annihilates constants, and the initial value $u(0,x) = \eta_0(x)$ that appears in (1.2) has the usual physical meaning of an initial state. With the Riemann–Liouville derivative the initial condition would have to be prescribed for a fractional integral of u, which is rarely measurable. Moreover, for a sufficiently regular function one has $\mathcal{I}_{0^+}^{\gamma}\,{}^{C}\mathcal{D}_{0^+}^{\gamma}u = u - u(0)$, and this identity is exactly what is used in Section 3 to pass from the differential form to the integral form of the problem.

2.1. *Chebyshev cardinal functions*

Let m be a positive integer. On $[-1,1]$ the Chebyshev cardinal functions of order m are [2]

$$\Lambda_k(\xi) = \frac{T_{m+1}(\xi)}{T_{m+1}'(\xi_k)\,(\xi - \xi_k)}, \qquad k = 1,2,\dots,m+1, \tag{2.4}$$

where $T_{m+1}(\xi) = \cos\big((m+1)\arccos\xi\big)$ is the Chebyshev polynomial of degree $m+1$ and $\xi_k = \cos\left(\frac{(2k-1)\pi}{2(m+1)}\right), k = 1,\dots,m+1$, are its zeros; the prime denotes differentiation with respect to ξ. The affine map $\zeta = (\xi+1)T/2$ transfers these functions to $[0,T]$, with nodes ζ_i the images of ξ_i. A function h on $[0,T]$ is then interpolated by

$$h(t) \approx \sum_{i=1}^{m+1} h(\zeta_i)\Lambda_i(\zeta) = \mathbf{h}^T\boldsymbol{\Xi}(\zeta), \tag{2.5}$$

where

$$\mathbf{h} = [h(\zeta_1),\dots,h(\zeta_{m+1})]^T, \qquad \boldsymbol{\Xi}(\zeta) = [\Lambda_1(\zeta),\dots,\Lambda_{m+1}(\zeta)]^T. \tag{2.6}$$

The derivative of the basis vector can be expressed through the basis itself:

$$\boldsymbol{\Xi}'(\zeta) = [\Lambda'_1(\zeta),\dots,\Lambda'_{m+1}(\zeta)]^T = \mathbf{D}\,\boldsymbol{\Xi}(\zeta), \tag{2.7}$$

where $\mathbf{D}$ is the $(m+1)\times(m+1)$ differentiation matrix of the cardinal basis. Because each Λ_j' is interpolated at the nodes,

$$\Lambda'_j(\zeta) \approx \sum_{i=1}^{m+1} \Lambda'_j(\zeta_i)\Lambda_i(\zeta), \tag{2.8}$$

and therefore

$$\mathbf{D} = \begin{bmatrix} \Lambda'_1(\zeta_1) & \cdots & \Lambda'_1(\zeta_{m+1}) \\ \vdots & & \vdots \\ \Lambda'_{m+1}(\zeta_1) & \cdots & \Lambda'_{m+1}(\zeta_{m+1}) \end{bmatrix}. \tag{2.9}$$

A short comment on the construction is useful before the next section. Because $\Lambda_k(\xi_j) = \delta_{kj}$, the coefficient of Λ_i in (2.5) equals the value of the function at the node ζ_i. Consequently, differentiating an interpolant amounts to multiplying the vector of nodal values by the matrix **D**, and a differential or integral equation is transformed into a matrix equation that is evaluated at the nodes only. The same idea is applied below in two variables, with one basis in x and another in t.

3. Equivalent integral formulation

Problem (1.2) can be rewritten as an equation with a nonlinear operator,

$$ {}^{C}\mathcal{D}_{0+}^{\gamma} u(t,x) = \mathcal{F}u(t,x) + \mathcal{G}(t,x), \gamma \in (0,1), \tag{3.1} $$

that is,

$$ \mathcal{D}_{0+}^{\gamma}[u(\theta,x) - u(0,x)](t) = \mathcal{F}u(t,x) + \mathcal{G}(t,x), \tag{3.2} $$

with $\mathcal{F}u(t,x) = -\kappa\, u_{xx}(t,x) + \lambda \int_0^t \mathcal{K}\left(t,x,\rho,u(\rho,x)\right) d\rho$. Applying $\mathcal{I}_{0+}^{\gamma}$ to both sides gives

$$ \mathcal{I}_{0+}^{\gamma}\mathcal{D}_{0+}^{\gamma}[u(\theta,x) - u(0,x)](t) = \mathcal{I}_{0+}^{\gamma}\big(\mathcal{F}u(t,x)\big) + \mathcal{I}_{0+}^{\gamma}\big(\mathcal{G}(t,x)\big), \tag{3.3} $$

so that

$$ u(t,x) = \eta_0(x) + \mathcal{I}_{0+}^{\gamma}\big(\mathcal{F}u(t,x)\big) + \mathcal{I}_{0+}^{\gamma}\big(\mathcal{G}(t,x)\big), \tag{3.4} $$

or, written out,

$$ \begin{aligned} u(t,x) = \eta_0(x) &+ \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\mathcal{F}u(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\mathcal{G}(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta. \end{aligned} \tag{3.5} $$

This proves the following statement.

Lemma 3.1: *Problem (1.2) is equivalent to the nonlinear integral equation*

$$ \begin{aligned} u(t,x) = \eta_0(x) &+ \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\mathcal{F}u(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\mathcal{G}(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta \end{aligned} \tag{3.6} $$

together with the boundary conditions

$$ u(t,0) = b_0(t), u(t,1) = b_1(t), 0 \le t \le T. \tag{3.7} $$

Remark 3.1: *The equivalence stated in the lemma holds for functions regular enough for the manipulations above to be legitimate, for instance for u continuous with u_{xx} continuous on $[0,1] \times [0,T]$. Since the weakly singular kernel $(t-\theta)^{\gamma-1}$ is integrable for $\gamma > 0$, every integral in (3.6) is well defined*

for such u. The integral form is also more convenient numerically than (1.2): it contains no fractional derivative, so only integrals of known basis functions have to be evaluated, and these can be computed once, independently of the particular problem.

4. Pseudospectral discretization

We now apply the Chebyshev cardinal basis to (3.6)–(3.7). Define the operators

$$\mathcal{P}(u)(t,x) = u(t,x) + \frac{\kappa}{\Gamma(\gamma)} \int_0^t \frac{u_{xx}(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta, \tag{4.1}$$

$$\mathcal{W}(u)(t,x) = -\frac{\lambda}{\Gamma(\gamma)} \int_0^t \frac{\int_0^\theta \mathcal{K}\left(t,x,\rho,u(\rho,x)\right) d\rho}{(t-\theta)^{1-\gamma}}\, d\theta, \tag{4.2}$$

and the right-hand side

$$\Upsilon(t,x) = \eta_0(x) + \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\mathcal{G}(\theta,x)}{(t-\theta)^{1-\gamma}}\, d\theta. \tag{4.3}$$

Then (3.6) takes the compact form

$$(\mathcal{P} + \mathcal{W})(u) = \Upsilon. \tag{4.4}$$

We seek an approximation of the form

$$u(t,x) \approx \hat{u}(t,x) = \sum_{i=1}^{m+1} \sum_{k=1}^{m+1} \hat{u}_{i,k}\, \Lambda_i(x) \Lambda_k(t). \tag{4.5}$$

Collecting the coefficients $\hat{u}_{i,k}$ in the $(m+1) \times (m+1)$ matrix $\mathbf{U}$ we have

$$\hat{u} = \mathbf{\Lambda}^T(x)\, \mathbf{U}\, \mathbf{\Lambda}(t), \tag{4.6}$$

where $\mathbf{\Lambda}(\cdot)$ denotes the vector of cardinal functions. Differentiating with respect to x gives

$$\hat{u}_x(t,x) = \sum_{i=1}^{m+1} \sum_{k=1}^{m+1} \hat{u}_{i,k}\, \Lambda_i'(x) \Lambda_k(t) = \mathbf{\Lambda'}^T(x)\, \mathbf{U}\, \mathbf{\Lambda}(t). \tag{4.7}$$

Let $\mathbf{P} \in \mathbb{R}^{(m+1)\times(m+1)}$ be the differentiation matrix with entries $P_{i,j} = \Lambda_i'(x_j)$, so that $\mathbf{\Lambda'}(x) = \mathbf{P}\mathbf{\Lambda}(x)$. Then

$$\hat{u}_x(t,x) = \mathbf{\Lambda}^T(x)\, \mathbf{P}^T \mathbf{U}\, \mathbf{\Lambda}(t), \tag{4.8}$$

and, by repeating the argument,

$$\hat{u}_{xx}(t,x) = \mathbf{\Lambda}^T(x)\, (\mathbf{P}^T)^2\, \mathbf{U}\, \mathbf{\Lambda}(t). \tag{4.9}$$

Inserting (4.6) and (4.9) into (4.1) we obtain

$$\mathcal{P}(\hat{u})(t,x) \quad \approx \mathbf{\Lambda}^T(x)\mathbf{U}\mathbf{\Lambda}(t) + \frac{\kappa}{\Gamma(\gamma)} \int_0^t \frac{\mathbf{\Lambda}^T(x)(\mathbf{P}^T)^2\mathbf{U}\mathbf{\Lambda}(\theta)}{(t-\theta)^{1-\gamma}}\, d\theta$$

$$= \mathbf{\Lambda}^T(x)\mathbf{U}\mathbf{\Lambda}(t) + \frac{\kappa}{\Gamma(\gamma)}\mathbf{\Lambda}^T(x)(\mathbf{P}^T)^2\mathbf{U}\int_0^t \frac{\mathbf{\Lambda}(\theta)}{(t-\theta)^{1-\gamma}}\,d\theta.$$

The remaining time integral is itself expanded in the basis:

$$\frac{1}{\Gamma(\gamma)}\int_0^t \frac{\mathbf{\Lambda}(\theta)}{(t-\theta)^{1-\gamma}}\,d\theta = \mathcal{J}_1\mathbf{\Lambda}(t) = \sigma(t) \approx \mathbf{S}^T\mathbf{\Lambda}(t), \tag{4.10}$$

with

$$\mathbf{S} = [\sigma(t_1), \dots, \sigma(t_{m+1})]^T, \tag{4.11}$$

and, for the ordinary integral,

$$\begin{aligned} &\int_0^t \mathbf{\Lambda}\,(\theta)\,d\theta = \mathcal{J}_2\mathbf{\Lambda}(t) = \varpi(t) \approx \mathbf{Z}^T\mathbf{\Lambda}(t), \\ &\mathbf{Z} = [\varpi(t_1), \dots, \varpi(t_{m+1})]^T. \end{aligned} \tag{4.12}$$

Consequently

$$\mathcal{P}(\hat{u})(t,x) \approx \mathbf{\Lambda}^T(x)\mathbf{U}\mathbf{\Lambda}(t) + \kappa\,\mathbf{\Lambda}^T(x)(\mathbf{P}^T)^2\mathbf{U}\mathbf{S}^T\mathbf{\Lambda}(t). \tag{4.13}$$

For the operator $\mathcal{W}$ we replace u by $\hat{u}$ in (4.2):

$$\mathcal{W}(\hat{u})(t,x) \approx -\frac{\lambda}{\Gamma(\gamma)}\int_0^t \frac{\int_0^\theta \mathcal{K}\left(t,x,\rho,\hat{u}(\rho,x)\right)d\rho}{(t-\theta)^{1-\gamma}}\,d\theta. \tag{4.14}$$

Suppose that $\mathcal{K}\big(t,x,\rho,\hat{u}(\rho,x)\big) \approx \mathbf{\Lambda}^T(x)\mathbf{H}\mathbf{\Lambda}(\rho)$, where the matrix $\mathbf{H}$ depends on t and on $\mathbf{U}$. Using the integration operators $\mathcal{J}_1$ and $\mathcal{J}_2$ we find

$$\begin{aligned} \mathcal{W}(\hat{u})(t,x) \quad &\approx -\frac{\lambda}{\Gamma(\gamma)}\int_0^t \frac{\int_0^\theta \mathbf{\Lambda}^T\,(x)\mathbf{H}\mathbf{\Lambda}(\rho)\,d\rho}{(t-\theta)^{1-\gamma}}\,d\theta \\ &= -\frac{\lambda}{\Gamma(\gamma)}\int_0^t \mathbf{\Lambda}^T\,(x)\mathbf{H}\mathcal{J}_2\mathbf{\Lambda}(\theta)\,\frac{d\theta}{(t-\theta)^{1-\gamma}} \\ &= -\frac{\lambda}{\Gamma(\gamma)}\mathbf{\Lambda}^T(x)\mathbf{H}\mathcal{J}_2\mathcal{J}_1\mathbf{\Lambda}(t) \\ &= o(t,x) \approx \mathbf{\Lambda}^T(x)\mathbf{O}\mathbf{\Lambda}(t), \end{aligned} \tag{4.15}$$

where $\mathbf{O} := [o(t_i,t_k)]_{i,k=1}^{m+1}$. Combining (4.13) and (4.15) in (4.4) gives

$$\mathbf{\Lambda}^T(x)(\mathbf{U} + \kappa(\mathbf{P}^T)^2\mathbf{U}\mathbf{S}^T + \mathbf{O})\mathbf{\Lambda}(t) = \mathbf{\Lambda}^T(x)\mathbf{\Phi}\mathbf{\Lambda}(t), \tag{4.16}$$

where $\mathbf{\Phi}$ is the matrix of the approximation of Y. Since the cardinal functions are linearly independent, the coefficient matrices must agree:

$$\mathbf{U} + \kappa(\mathbf{P}^T)^2\mathbf{U}\mathbf{S}^T + \mathbf{O} = \mathbf{\Phi}. \tag{4.17}$$

Equivalently, we look for a zero of

$$\mathbf{\Psi}(\mathbf{U}) := \mathbf{U} + \kappa(\mathbf{P}^T)^2\mathbf{U}\mathbf{S}^T + \mathbf{O} - \mathbf{\Phi} = \mathbf{0}. \tag{4.18}$$

The first and last rows of this system are replaced by the boundary conditions:

$$[\mathbf{\Psi}(\mathbf{U})]_{1,i} = [\mathbf{\Lambda}^T(0)\mathbf{U}]_i - b_0(t_i),$$
$$[\mathbf{\Psi}(\mathbf{U})]_{m+1,i} = [\mathbf{\Lambda}^T(1)\mathbf{U}]_i - b_1(t_i), \quad i = 1, \dots, m+1. \tag{4.19}$$

The result is a nonlinear algebraic system of size $(m+1) \times (m+1)$ for the unknowns $\hat{u}_{i,k}$, which we solve by Newton's method; the approximate solution is then recovered from (4.5).

For the reader's convenience we summarize the algorithm in the order in which it is carried out.

1. Choose the order m and compute the Chebyshev nodes ξ_k and their images ζ_k in the required intervals.
2. Assemble the differentiation matrix $\mathbf{P}$ and the integration matrices $\mathcal{J}_1$ and $\mathcal{J}_2$; they depend only on m, γ and T, so they can be reused for different data.
3. Evaluate the right-hand side Υ at the nodes to obtain the matrix $\mathbf{\Phi}$.
4. Form the residual $\mathbf{\Psi}(\mathbf{U})$ of (4.18) and replace its first and last rows by the boundary conditions (4.19).
5. Solve $\mathbf{\Psi}(\mathbf{U}) = \mathbf{0}$ with Newton's method, starting, for example, from the initial profile $u(0,x) = \eta_0(x)$ repeated in time or from a few steps of a fixed-point iteration.
6. Reconstruct $\hat{u}(t,x)$ from (4.5) and, if needed, evaluate it at any point of the domain.

Each Newton step requires the solution of a linear system whose size is $(m+1)^2$. Because the unknowns are coupled through the integral operators, the matrices involved are dense, and the cost of the method grows quickly with m; the high accuracy obtained already for moderate m is what makes the approach competitive.

5. Convergence analysis

Let Υ be continuous on $J = [0,1] \times [0,T]$ and let $\hat{\Upsilon}$ be its Chebyshev cardinal interpolant. Denote by $\hat{u}$ the pseudospectral solution of (4.4), by $\epsilon = u - \hat{u}$ the error, and by $\mathbf{\Delta}$ the matrix with entries $\epsilon_{i,j}$ in the basis expansion of ϵ. The next result is taken from [11].

Theorem 5.1: *For a sufficiently smooth function $\Upsilon: J \to \mathbb{R}^2$ the interpolation error satisfies*

$$\|\Upsilon - \hat{\Upsilon}\| \approx O(2^{-2m}). \tag{5.1}$$

Theorem 5.2: *The scheme of Section 4 for (3.6)–(3.7) converges whenever*

$$\|\mathbf{\Delta}\| \le \frac{\kappa}{\Gamma(\gamma)} \|(\mathbf{P}^T)^2 \mathbf{\Delta} \mathbf{S}^T\| + \frac{\Theta\lambda}{\Gamma(\gamma)} \|\mathbf{\Delta} \mathbf{S}^T \mathbf{Z}^T\| + \|\mathbf{\varrho}\| \tag{5.2}$$

and the constant

$$1-\frac{\kappa}{\Gamma(\gamma)}\|(\mathbf{P}^T)^2\|\|\mathbf{S}^T\|-\frac{\Theta\lambda}{\Gamma(\gamma)}\|\mathbf{S}^T\|\|\mathbf{Z}^T\| \tag{5.3}$$

is positive (in particular, nonzero).

Proof: Let $\hat{u}$ solve $(\mathcal{P}+\mathcal{W})(\hat{u})=\hat{Y}$, and let $\epsilon=u-\hat{u}$. Subtracting this relation from (4.4) yields the error equation

$$(\mathcal{P}+\mathcal{W})(\epsilon)=Y-\hat{Y}, \tag{5.4}$$

which, written out, is

$$\begin{aligned}\epsilon(t,x)\quad &+\frac{\kappa}{\Gamma(\gamma)}\int_0^t\frac{\epsilon_{xx}(\theta,x)}{(t-\theta)^{1-\gamma}}\,d\theta\\ &=\frac{\lambda}{\Gamma(\gamma)}\int_0^t\frac{1}{(t-\theta)^{1-\gamma}}\int_0^\theta[\mathcal{K}\big(t,x,\rho,u(\rho,x)\big)\\ &-\mathcal{K}\big(t,x,\rho,\hat{u}(\rho,x)\big)]d\rho\,d\theta+Y-\hat{Y}.\end{aligned} \tag{5.5}$$

By the mean value theorem there is a function $\tilde{u}(t,x)$ lying between u and $\hat{u}$ such that

$$\begin{aligned}&\epsilon(t,x)+\frac{\kappa}{\Gamma(\gamma)}\int_0^t\frac{\epsilon_{xx}(\theta,x)}{(t-\theta)^{1-\gamma}}\,d\theta\\ &-\frac{\lambda}{\Gamma(\gamma)}\int_0^t\frac{\int_0^\theta\frac{\partial\mathcal{K}\big(t,x,\rho,\tilde{u}(\rho,x)\big)}{\partial u}\epsilon(\rho,x)\,d\rho}{(t-\theta)^{1-\gamma}}\,d\theta=Y-\hat{Y}.\end{aligned} \tag{5.6}$$

Applying the bound (1.3) we get

$$\begin{aligned}&\epsilon(t,x)+\frac{\kappa}{\Gamma(\gamma)}\int_0^t\frac{\epsilon_{xx}(\theta,x)}{(t-\theta)^{1-\gamma}}\,d\theta\\ &\leq\frac{\Theta\lambda}{\Gamma(\gamma)}\int_0^t\frac{\int_0^\theta\epsilon\,(\rho,x)\,d\rho}{(t-\theta)^{1-\gamma}}\,d\theta+Y-\hat{Y}.\end{aligned} \tag{5.7}$$

Expand the error in the cardinal basis,

$$\epsilon(t,x)\approx\mathbf{\Lambda}^T(x)\mathbf{\Delta}\mathbf{\Lambda}(t), \tag{5.8}$$

with $\mathbf{\Delta}=\left[\epsilon_{i,k}\right]_{i,k=1}^{m+1}$, and let $\mathfrak{e}$ be the coefficient matrix of $Y-\hat{Y}$. Then (5.7) becomes

$$\begin{aligned}&\mathbf{\Lambda}^T(x)\mathbf{\Delta}\mathbf{\Lambda}(t)+\frac{\kappa}{\Gamma(\gamma)}\mathbf{\Lambda}^T(x)(\mathbf{P}^T)^2\mathbf{\Delta}\mathbf{S}^T\mathbf{\Lambda}(t)\leq\\ &\frac{\Theta\lambda}{\Gamma(\gamma)}\mathbf{\Lambda}^T(x)\mathbf{\Delta}\mathbf{S}^T\mathbf{Z}^T\mathbf{\Lambda}(t)+\mathbf{\Lambda}^T(x)\mathfrak{e}\mathbf{\Lambda}(t).\end{aligned} \tag{5.9}$$

By linear independence of the basis, a comparison of coefficients gives

$$\mathbf{\Delta}\leq\frac{\kappa}{\Gamma(\gamma)}(\mathbf{P}^T)^2\mathbf{\Delta}\mathbf{S}^T+\frac{\Theta\lambda}{\Gamma(\gamma)}\mathbf{\Delta}\mathbf{S}^T\mathbf{Z}^T+\mathfrak{e} \tag{5.10}$$

Taking norms and using (5.2),

$$\|\mathbf{\Delta}\| \leq \frac{\kappa}{\Gamma(\gamma)} \|(\mathbf{P}^T)^2\| \|\mathbf{S}^T\| \|\mathbf{\Delta}\| + \frac{\Theta\lambda}{\Gamma(\gamma)} \|\mathbf{S}^T\| \|\mathbf{Z}^T\| \|\mathbf{\Delta}\| + \|\mathbf{\varrho}\|, \quad (5.11)$$

which can be rearranged as

$$\left(1 - \frac{\kappa}{\Gamma(\gamma)} \|(\mathbf{P}^T)^2\| \|\mathbf{S}^T\| - \frac{\Theta\lambda}{\Gamma(\gamma)} \|\mathbf{S}^T\| \|\mathbf{Z}^T\|\right) \|\mathbf{\Delta}\| \leq \|\mathbf{\varrho}\|. \quad (5.12)$$

For smooth data Y, Theorem 5.1 gives $\|\mathbf{\varrho}\| \to 0$ as $m \to \infty$. Combined with (5.3) this implies

$$\|\mathbf{\Delta}\| \to 0 \quad \text{as} \quad m \to \infty, \quad (5.13)$$

i.e., the method converges.

Remark 5.1: The two conditions of the second theorem have a simple interpretation. Condition (5.3) says that the quantity $\frac{\kappa}{\Gamma(\gamma)} \|(\mathbf{P}^T)^2\| \|\mathbf{S}^T\| + \frac{\Theta\lambda}{\Gamma(\gamma)} \|\mathbf{S}^T\| \|\mathbf{Z}^T\|$ must be smaller than one, which is a contraction-type requirement: it holds when the diffusion coefficient κ, the intensity λ of the memory term, or the Lipschitz constant Θ of the kernel is not too large relative to $\Gamma(\gamma)$. Under this requirement the error of the discrete scheme is controlled by the interpolation error of the data, as shown by (5.12). In practice the norms appearing in (5.3) can be computed explicitly for a given m, so the condition can be verified numerically before the scheme is applied.

Remark 5.2: The estimate (5.1) is a statement about the interpolation of smooth functions. It does not claim that the solution of (1.2) is smooth: for fractional equations the solution may behave like a power of t near $t = 0$, and then the rate predicted by (5.1) is attained only after the data or the time variable have been suitably regularized. This limitation applies to all global spectral methods in time and is the main reason why a numerical test with a known solution is recommended in practice.

6. Conclusion

We have constructed a pseudospectral algorithm based on Chebyshev cardinal functions for a class of time-fractional parabolic Volterra–Fredholm integro-differential equations. The fractional problem is first rewritten as an integral equation and then reduced to a nonlinear algebraic system, which is solved with Newton's iteration. The scheme deals directly with the nonlocal fractional derivative and with the memory carried by the integral terms. The convergence proof shows that the error tends to zero under suitable conditions on the kernel and on the smoothness of the data, and the interpolation estimate indicates exponential decay of the error as the number of collocation points grows. Owing to the flexibility and efficiency of the cardinal basis, the technique is a promising candidate for more general fractional integro-differential models in science and

engineering. Possible extensions include the variable-order case, systems of such equations, problems with a nonlinear dependence on the fractional derivative, and higher space dimensions, for which tensor-product versions of the same basis can be used.

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