

Perturbations in dynamical systems for EFL learning

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Abstract

We study mathematical perturbed equations that can be used to describe how learner motivation, engagement, and proficiency develop together in English as a Foreign Language (EFL) contexts. The basic system uses first-order ordinary differential equations to show how motivation and engagement influence each other, while proficiency grows in a logistic way but also declines if not reinforced. To try to make the system closer to classroom situations, we add a small perturbation term that represents changes caused by teaching methods or other external factors. We analyze the stability and behavior of the system. Numerical examples show that the equations remain stable even under small perturbations. Our results suggest

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that mathematical systems with perturbations provide useful tools for understanding how motivation and learning progress in second language acquisition.

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1. Introduction

In the domain of second language acquisition (SLA), the interplay between motivation, engagement, and proficiency has long been recognized as a central determinant of learners' progress. Earlier approaches often relied on static, cross-sectional models, and more recent developments emphasize the evolution and character of learning processes [4, 9, 13]. Learners' emotional and cognitive states vary over time, interacting with classroom context and pedagogical strategies, leading to developmental patterns that linear statistical models cannot fully represent.

Motivation has been a foundational construct of SLA research since the model of Gardner [7], which introduced the distinction between integrative and instrumental orientations. Later, the emergence of Self Determination Theory (SDT) [14] shifted the focus toward autonomy, competence, and relatedness as key drivers of persistence. More recently, Dornyei in his L2 Motivational Self System [5] framed motivation as the alignment between learners' actual and ideal L2 selves, showing the role of identity construction in sustaining effort. These perspectives emphasize that motivation is neither static nor uniform but evolves on time.

Parallel to motivation, engagement has gained prominence as a mediating construct linking motivational dispositions to actual learning outcomes. It is defined as a multidimensional construct encompassing behavioral, emotional, and cognitive involvement [6]. In SLA contexts, engagement determines the degree of sustained practice and strategy use, both of which are crucial for constructing proficiency [10]. Empirical studies show that motivation and engagement reinforce each other over time: motivated learners are more engaged, and engaged learners often report higher motivation [12].

Language proficiency develops through nonlinear trajectories. Initial stages often yield to rapid growth, followed by plateaus or even regressions. These patterns resemble logistic learning curves [11], where the rate of improvement slows as learners approach higher levels of competence, and forgetting effects emerge in the absence of reinforcement. Computational models of practice and forgetting, such as spaced-repetition algorithms [15], provide evidence that language proficiency can be modeled using differential or difference equations.

Dynamic Systems Theory (DST) presents a natural framework for integrating these strands and other phenomena [1, 2, 8]. In their work, Larsen-Freeman [9] and Bot, Lowie, and Verspoor [4] proposed that language development should be seen as a self-organizing system with attractors, feedback

loops, and sensitivity to initial conditions. This perspective has motivated studies employing time-series analyses, ecological momentary assessments, and complexity-informed designs [16]. Yet, relatively few contributions have attempted to formalize SLA processes using explicit systems of differential equations.

The present paper deals with advancing this field of inquiry by analyzing a nonlinear coupled problem of ordinary differential equations (ODEs) that can capture the interaction between motivation, engagement, and proficiency in an English as a Foreign Language (EFL) setting. Our system integrates a bidirectional coupling between motivation and engagement, a nonlinear growth equation for proficiency with forgetting, and a regular perturbation term, involving an infinitesimal ε , representing structured pedagogical interventions such as motivational guidance, feedback mechanisms, or technology-enhanced support.

The novelty of our contribution lies in combining SLA theory with tools from DST, specifically regular perturbation theory, to study how small structured external influences alter the trajectory of learner development. By comparing the unperturbed and perturbed cases, we show how even weak interventions can shift equilibrium, improve stability, and accelerate proficiency growth. This approach opens a pathway toward quantitative, simulation-based predictions that complement existing qualitative DST perspectives in SLA.

2. The Equations

In order to capture the dynamic and interaction between psychological and learning variables in an EFL context, we propose a mathematical system of coupled ODEs. This system is inspired by previous works in applied linguistics that use structural tables and cross-lagged panel designs to study the longitudinal relation between motivation, engagement, and learning outcomes, see [3, 12]. Here, we formulate the problem in continuous time through the language of dynamical systems.

The proposed ODEs take the form:

$$\begin{cases} \dot{M}(t) = \alpha E(t) - \gamma M(t) + \varepsilon f(M, E, P), \\ \dot{E}(t) = cM(t) - \delta E(t) + \varepsilon g(M, E, P), \\ \dot{P}(t) = \lambda M(t)E(t) - \mu P(t) + \varepsilon h(M, E, P), \end{cases} \quad (1)$$

where $M(t)$ denotes motivation, $E(t)$ denotes engagement, and $P(t)$ denotes language proficiency at t .

The first equation can describe the evolution of learner motivation. Motivation is influenced by $\alpha E(t)$, reflecting empirical results that active participation in learning activities can boost self-determined motivation [3]. At the same time, motivation can decay at rate γ if not sustained by contextual support. The perturbation term $\varepsilon f(M, E, P)$ is to capture external inputs such as teacher encouragement, peer collaboration, or digital nudges that adjust motivational trajectories.

The second equation is for learner engagement. It grows with motivation, represented by $cM(t)$, consistent with the idea that motivated students are to invest effort and persist in tasks [12]. It simultaneously decays at rate δ when not maintained. The perturbation term $\varepsilon g(M, E, P)$ represents situational factors such as classroom climate, task design, or assessment incentives that can sustain or disrupt engagement independently of motivation.

The third equation describes the evolution of linguistic proficiency. It grows proportionally to the product $M(t)E(t)$ with rate λ , meaning that only when learners are both motivated and engaged does effective learning take place. The decay term $-\mu P(t)$ is for attrition or forgetting, well-documented in second language acquisition [4]. The perturbation $\varepsilon h(M, E, P)$ represents structured instructional interventions; targeted feedback, adaptive learning technologies that enhance proficiency growth beyond the baseline interaction of motivation and engagement.

This system captures the relation between motivation and engagement observed in applied linguistics research. It links psychological variables to measurable outcomes (language proficiency), reflecting theories of self-determination and engagement. It incorporates natural decay terms to reflect the empirically observed decline of motivation, engagement, or proficiency in the absence of support. It allows for external perturbations (εf_i), which model pedagogical interventions such as scaffolding, digital support, or feedback loops.

In this way, system (1) situates itself between purely statistical structural problems used in applied linguistics and nonlinear dynamical problems from mathematical psychology and education [4, 16]. The goal is to find dynamical results and also to explore new qualitative behaviors that may emerge from the interaction of these key variables in the EFL context.

3. Results

3.1 Unperturbed Environment ($\varepsilon = 0$)

Consider the nonlinear system describing motivation $M(t)$, engagement $E(t)$, and proficiency $P(t)$ with no external perturbation:

$$\begin{cases} \dot{M} = \alpha E - \gamma M, \\ \dot{E} = cM - \delta E, \\ \dot{P} = \lambda ME - \mu P, \end{cases} \quad (1)$$

where $\alpha, \gamma, c, \delta, \lambda, \mu > 0$ are constants.

Analysis of the (M, E) subsystem:

The first two equations form a linear system:

$$\dot{X} = AX, X = \begin{pmatrix} M \\ E \end{pmatrix}, A = \begin{pmatrix} -\gamma & \alpha \\ c & -\delta \end{pmatrix}.$$

* The origin $(M, E) = (0, 0)$ is the unique equilibrium.

** Eigenvalues: The characteristic polynomial is

$$\lambda^2 + (\gamma + \delta)\lambda + (\gamma\delta - \alpha c) = 0.$$

Denote $\Delta := \gamma\delta - \alpha c$.

Then,

$$\lambda_{1,2} = \frac{-(\gamma+\delta) \pm \sqrt{(\gamma-\delta)^2 + 4\alpha c}}{2}.$$

Stability:

- $\Delta > 0$ ($\alpha c < \gamma\delta$): both eigenvalues negative $\Rightarrow (M, E) \rightarrow 0$ (stable decay).
- $\Delta < 0$ ($\alpha c > \gamma\delta$): one eigenvalue positive $\Rightarrow (M, E)$ grows (mutual amplification, self-amplifying growth).
- $\Delta = 0$ ($\alpha c = \gamma\delta$): one zero eigenvalue $\Rightarrow$ slow neutral mode, sensitive to nonlinear effects.

Solution of (M, E) : For $\Delta \neq 0$, let $\lambda_1 \neq \lambda_2$ and corresponding eigenvectors $v^{(1)}, v^{(2)}$. Then, we have the solution:

$$X(t) = C_1 v^{(1)} e^{\lambda_1 t} + C_2 v^{(2)} e^{\lambda_2 t},$$

with constants C_1, C_2 are in $\mathbb{R}$.

Proficiency equation: Given $(M(t), E(t))$, the nonlinear equation for proficiency is

$$\dot{P} + \mu P = \lambda M(t) E(t).$$

Solution: We have:

$$P(t) = e^{-\mu t} \left[P(0) + \lambda \int_0^t M(s) E(s) e^{\mu s} ds \right].$$

Interpretation: We have the following three scenarios:

- If $(M, E) \rightarrow (0, 0)$, then $P(t) \rightarrow 0$: forgetting dominates and there is no sustained learning.
- If $(M, E) \rightarrow (M^*, E^*) > 0$, then $P(t)$ converges to a positive value: sustained proficiency emerges.
- If (M, E) grow, then $P(t)$ grows nonlinearly.

EFL interpretation:

- **Stable decay** ($\alpha c < \gamma\delta$): Without external support, learners' motivation and engagement decrease, leading to reduced proficiency over time.
- **Mutual amplification** ($\alpha c > \gamma\delta$): Small initial motivation or engagement may grow via feedback, producing increased proficiency. Real classroom constraints (fatigue, cognitive limits) may require nonlinear saturation to prevent unrealistic unbounded growth.
- **Critical regime** ($\alpha c = \gamma\delta$): Growth is slow; small interventions can significantly affect long-term learning trajectories.

This analysis shows that even without perturbations, the nonlinear system captures essential dynamics of motivation, engagement, and proficiency in EFL learning.

Perturbed Environment ($0 < \varepsilon \ll 1$)

We consider the perturbed system:

$$\begin{cases} \dot{M} = \alpha E - \gamma M + \varepsilon f(M, E, P), \\ \dot{E} = cM - \delta E + \varepsilon g(M, E, P), \\ \dot{P} = \lambda ME - \mu P + \varepsilon h(M, E, P), \end{cases} \quad (2)$$

where f, g, h are bounded functions representing small pedagogical interventions such as temporary teacher guidance, corrective feedback, and short digital learning reminders

Perturbation expansion

We expand each variable in powers of ε :

$$\begin{aligned} M(t) &= M_0(t) + \varepsilon M_1(t) + \varepsilon^2 M_2(t) + \dots, \\ E(t) &= E_0(t) + \varepsilon E_1(t) + \varepsilon^2 E_2(t) + \dots, \\ P(t) &= P_0(t) + \varepsilon P_1(t) + \varepsilon^2 P_2(t) + \dots, \end{aligned}$$

where (M_0, E_0, P_0) solves the unperturbed nonlinear system:

$$\begin{cases} \dot{M}_0 = \alpha E_0 - \gamma M_0, \\ \dot{E}_0 = cM_0 - \delta E_0, \\ \dot{P}_0 = \lambda M_0 E_0 - \mu P_0. \end{cases}$$

Then, collecting terms of order ε , we obtain the linearized system around (M_0, E_0, P_0) :

$$\begin{cases} \dot{M}_1 = \alpha E_1 - \gamma M_1 + f(M_0, E_0, P_0), \\ \dot{E}_1 = cM_1 - \delta E_1 + g(M_0, E_0, P_0), \\ \dot{P}_1 = \lambda(M_0 E_1 + M_1 E_0) - \mu P_1 + h(M_0, E_0, P_0). \end{cases}$$

Now, suppose (M_0^*, E_0^*, P_0^*) is the unperturbed equilibrium. The first-order equilibrium point (M_1^*, E_1^*, P_1^*) satisfies:

$$\begin{cases} 0 = \alpha E_1^* - \gamma M_1^* + f(M_0^*, E_0^*, P_0^*), \\ 0 = cM_1^* - \delta E_1^* + g(M_0^*, E_0^*, P_0^*), \\ 0 = \lambda(M_0^* E_1^* + M_1^* E_0^*) - \mu P_1^* + h(M_0^*, E_0^*, P_0^*). \end{cases}$$

3.2.1 EFL Interpretation

- Minor interventions (εf ; f is bounded) can modify motivation and engagement, leading to a nonlinear increase in proficiency.
- Within the stable decay regime ($\alpha c < \gamma \delta$), interventions serve to maintain motivation and engagement, reducing forgetting.
- In the mutual amplification regime ($\alpha c > \gamma \delta$), interventions enhance the rate of growth, resulting in faster improvement in proficiency.

- Near-critical conditions ($\alpha c \approx \gamma \delta$) are highly sensitive: small interventions generate significant long-term impacts on motivation, engagement, and proficiency.
- The nonlinear term λME guarantees that proficiency increases only when both motivation and engagement are sufficient, reflecting realistic EFL learning dynamics.

Observations: The perturbed system preserves nonlinear dynamics, incorporates small pedagogical effects, and provides a framework to understand how minor interventions can amplify long-term learning outcomes in EFL classrooms.

3.3 Numerical illustrations

To visualize the behavior of our system, we utilize numerical simulations. We explore the three regimes identified in our analysis: stable decay, mutual amplification, and critical regime.

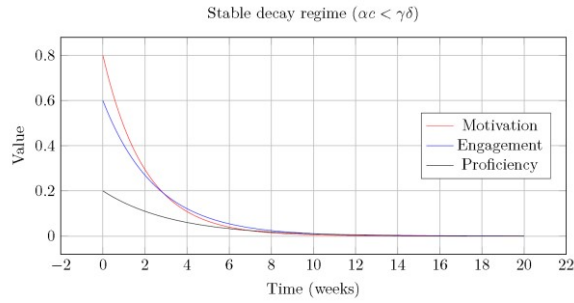


Figure 1

In the stable decay regime ($\alpha = 0.3$, $c = 0.4$, $\gamma = 0.8$, $\delta = 0.7$), all variables decay to zero without external support, representing learner disengagement over time.

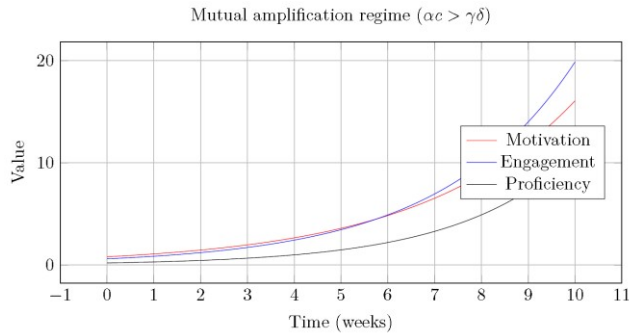
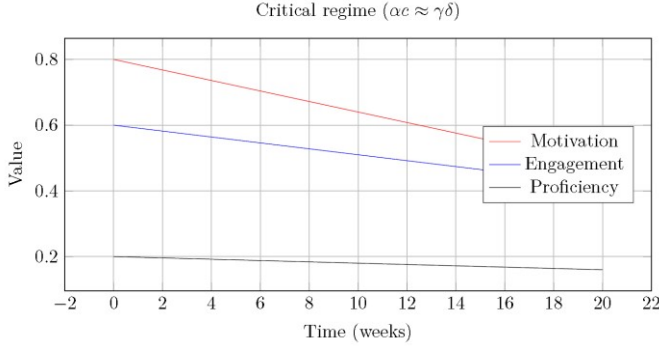
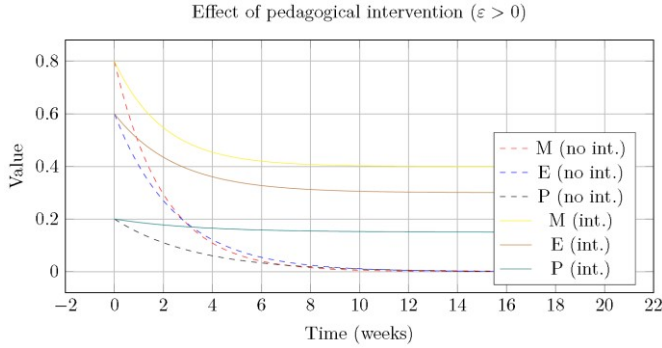


Figure 2

In the mutual amplification regime ($\alpha = 0.8$, $c = 0.9$, $\gamma = 0.4$, $\delta = 0.3$), small initial motivation and engagement lead to self-reinforcing growth, representing a virtuous cycle of learning.

**Figure 3**

In the critical regime ($\alpha = 0.5$, $c = 0.5$, $\gamma = 0.5$, $\delta = 0.5$), the system presents slow dynamics and is highly sensitive to initial conditions and perturbations.

**Figure 4**

Effect of a small constant pedagogical intervention ($\varepsilon = 0.1$, $f_1 = 0.3$, $f_2 = 0.2$, $f_3 = 0.1$) in the stable decay regime. The intervention shifts the equilibrium to positive values, preventing complete disengagement.

The numerical results confirm our analysis:

- * In the stable decay regime, see Figure 1, all variables approach zero without intervention.
- * In the mutual amplification regime, see Figure 2, small initial values grow exponentially.
- * In the critical regime, see Figure 3, the system shows slow dynamics.
- * Small pedagogical interventions can significantly alter system behavior, particularly in the stable decay and critical regimes, it is represented by Figure 4.

Conclusion

This work has investigated a nonlinear differential system to study the interactions between motivation, engagement, and proficiency in EFL learning

contexts. By combining principles from applied linguistics and dynamical system theory, we have given a problem that captures the dynamics of these three variables. Our results present regimes with pedagogical implications: the stable decay regime where motivation and engagement diminish without external support, the mutual amplification regime where small initial motivation can serve as trigger for self-reinforcing growth, and the critical regime where the system shows high sensitivity to interventions.

The perturbation analysis proves that structured pedagogical support, represented by ε , can alter system trajectories, particularly in the critical regime where well timed interventions produce disproportionate benefits. These results provide a mathematical justification for targeted instructional strategies and provide understanding into the timing of educational support.

Future work should focus on validation with real-world EFL learning data, incorporation of additional psychological factors, and development of stochastic elements to better capture the complexity of authentic learning environments.

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