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On the quasiconvex functions

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Abstract

The quasiconvex functions are considered, and some properties of these functions are stated and proved in this paper.

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1. Introduction

Recall that a function $\mu: Z \to \mathbb{R}$, where the set Z is nonempty and convex in $\mathbb{R}^n$, is called *convex* if for every $z_1,\ z_2 \in Z,\ \delta \in [0,1]$ we have that

$$\mu(\delta z_1 + (1-\delta)z_2) \leq \delta\mu(z_1) + (1-\delta)\mu(z_2), \tag{1}$$

and *concave*, if for every $z_1,\ z_2 \in Z$ and every $\delta \in [0,1]$,

$$\mu(\delta z_1 + (1-\delta)z_2) \geq \delta\mu(z_1) + (1-\delta)\mu(z_2). \tag{2}$$

Definition 1: Consider the set Z, which is nonempty and convex in $\mathbb{R}^n$, a convex function $\mu: Z \to \mathbb{R}$, and a real constant β. Define the following sets for the function μ:

$$L^{\leq}(\mu,\beta) \overset{\text{def}}{=} \{z \in Z: \mu(z) \leq \beta\}, \tag{3}$$

$$L^{=}(\mu,\beta) \overset{\text{def}}{=} \{z \in Z: \mu(z) = \beta\}, \tag{4}$$

$$L^{\geq}(\mu,\beta) \overset{\text{def}}{=} \{z \in Z: \mu(z) \geq \beta\}, \tag{5}$$

as well as the strict versions $L^{<}(\mu,\beta)$ and $L^{>}(\mu,\beta)$ (with strict inequalities in the definitions (3) and (5), respectively).

The set defined by (3) is known as the *Lebesgue set*.

The following result holds true.

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Theorem 1: *Let the set Z be nonempty and convex in $\mathbb{R}^n$ and the function $\mu: Z \to \mathbb{R}$ be convex. Then the Lebesgue set $L^{\leq}(\mu, \beta)$ of μ is convex.*

Proof: Let $z_1, z_2 \in L^{\leq}(\mu, \beta)$. By definition of $L^{\leq}(\mu, \beta)$,

$$z_1 \in Z, z_2 \in Z, \mu(z_1) \leq \beta, \mu(z_2) \leq \beta.$$

Let $\delta \in [0,1]$. Then convexity of the set Z implies $\delta z_1 + (1-\delta) z_2 \in Z$, and convexity of function μ and $\delta \geq 0,\ 1-\delta \geq 0$ imply

$$\mu(\delta z_1 + (1-\delta) z_2) \leq \delta \mu(z_1) + (1-\delta)\mu(z_2) \leq \delta\beta + (1-\delta)\beta \equiv \beta.$$

We have obtained $\delta z_1 + (1-\delta) z_2 \in Z$ and $\mu(\delta z_1 + (1-\delta) z_2) \leq \beta$.

This conclusion and definition of $L^{\leq}(\mu, \beta)$ imply that $\delta z_1 + (1-\delta) z_2 \in L^{\leq}(\mu, \beta)$ for every $z_1, z_2 \in L^{\leq}(\mu, \beta)$ and $\delta \in [0,1]$. Hence, the Lebesgue set $L^{\leq}(\mu, \beta)$ by definition is convex.

In the general case, the converse statement is not true.

However, there is a class of *convex like* (*almost convex*) functions, for which the converse statement of Theorem 1 holds true.

In reference titles [1]-[33], the related topics are considered and studied.

In Section 2, the definition and some properties of quasiconvex and quasiconcave functions are presented. In Section 3, open problem for research is outlined.

2. Quasiconvex and quasiconcave functions

Definition 2: A function $\mu: Z \to \mathbb{R}$, where the set $Z \subseteq \mathbb{R}^n$ is nonempty and convex, is called

- *quasiconvex*: if $\forall\ z_1, z_2 \in Z$ and $\forall\ \delta \in (0,1)$ we have that
$$\mu(\delta z_1 + (1-\delta) z_2) \leq \max\{\mu(z_1), \mu(z_2)\}, \tag{6}$$
- *quasiconcave*: if $-f$ is quasiconvex, that is, if $\forall\ z_1, z_2 \in Z$ and $\forall\ \delta \in (0,1)$ we have that
$$\mu(\delta z_1 + (1-\delta) z_2) \geq \min\{\mu(z_1), \mu(z_2)\}, \tag{7}$$
- *strictly quasiconvex*: if the inequality in (6) is strict;
- *strictly quasiconcave*: if the inequality in (7) is strict.

Inequality (6) can be expressed equivalently as follows

$$\mu(z_1) \leq \mu(z_2) \text{ implies } \mu(z_0) \leq \mu(z_2) \text{ for every } z_0 \in (z_1, z_2).$$

The function μ, which is both quasiconvex and quasiconcave, is *quasilinear*.

Each convex function turns out to be a quasiconvex function as well.

Indeed, if $\mu(z)$ is convex then, using (1), we get

$$\mu(\delta z_1 + (1-\delta) z_2) \overset{(1)}{\leq} \delta\mu(z_1) + (1-\delta)\mu(z_2) \leq$$
$$\leq \delta \max\{\mu(z_1), \mu(z_2)\} + (1-\delta)\max\{\mu(z_1), \mu(z_2)\} =$$

$$= [\delta + (1-\delta)]\max\{\mu(z_1), \mu(z_2)\} = \max\{\mu(z_1), \mu(z_2)\},$$

that is, $\mu(z)$ is quasiconvex according to Definition 2.

Quasiconvex and quasiconcave functions have some interesting properties.

It turns out that for quasiconvex functions, both statement of Theorem 1 and the converse statement of Theorem 1 hold true.

Theorem 2: *Let the set Z be nonempty and convex in $\mathbb{R}^n$. A function $\mu: Z \to \mathbb{R}$ is quasiconvex if and only if its Lebesgue set $L^{\leq}(\mu, \beta)$ is convex for each value of $\beta \in \mathbb{R}$.*

Proof: *Necessity.* Let function μ be quasiconvex, $z_1, z_2 \in L^{\leq}(\mu, \beta)$, and $\delta \in (0,1)$. Therefore $z_1, z_2 \in Z$ and $\mu(z_1) \leq \beta$, $\mu(z_2) \leq \beta$ in accordance with the definition of $L^{\leq}(\mu, \beta)$.

Consider the point $z_0 \stackrel{\text{def}}{=} \delta z_1 + (1-\delta) z_2$. Since the set Z is convex then $z_0 \in Z$.

Because function μ is quasiconvex by assumption, Definition 2 implies

$$\mu(z_0) \equiv \mu(\delta z_1 + (1-\delta) z_2) \leq \max\{\mu(z_1), \mu(z_2)\} \leq \max\{\beta, \beta\} = \beta. \tag{8}$$

From $z_0 \in Z$, (8) and Definition 1 of the Lebesgue set $L^{\leq}(\mu, \beta)$ we get $\delta z_1 + (1-\delta) z_2 \in L^{\leq}(\mu, \beta)$ for every $z_1, z_2 \in L^{\leq}(\mu, \beta)$, $\delta \in (0,1)$.

Hence $L^{\leq}(f, \beta)$ by definition is convex.

Sufficiency. Conversely, let $L^{\leq}(\mu, \beta)$ be a convex set for each value of $\beta \in \mathbb{R}$, where $\mu: Z \to \mathbb{R}$, and the set Z is nonempty and convex in $\mathbb{R}^n$.

Let $z_1, z_2 \in Z$ be arbitrarily chosen and fixed after that. Set $\beta = \max\{\mu(z_1), \mu(z_2)\}$. Therefore $z_1, z_2 \in L^{\leq}(\mu, \beta)$ for this choice of β in accordance with definition of $L^{\leq}(\mu, \beta)$.

By assumption, the Lebesgue set $L^{\leq}(\mu, \beta)$ is convex for each value of $\beta \in \mathbb{R}$, including for the value of β chosen above. Therefore $z_0 \stackrel{\text{def}}{=} \delta z_1 + (1-\delta) z_2 \in L^{\leq}(\mu, \beta)$ for every $z_1, z_2 \in Z$ and every $\delta \in (0,1)$.

Hence

$$\mu(z_0) \equiv \mu(\delta z_1 + (1-\delta) z_2) \leq \beta \stackrel{\text{def}}{=} \max\{\mu(z_1), \mu(z_2)\}, \tag{9}$$

where for the inequality in (9), the definition of $L^{\leq}(\mu, \beta)$ is used.

Since z_1, z_2 have been arbitrarily chosen, then (9) implies that μ is a quasiconvex function by Definition 2.

Conclusion

In this paper, some results concerning quasiconvex functions are stated and proved.

Open problem for research is to establish other properties, relations and applications of quasiconvex functions.

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