

On the graphical solution of matrix games

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Abstract

The graphical solution of 2×2 , $2 \times n$, $m \times 2$ matrix games is considered, and examples for applying this approach are presented.

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1. Introduction

In this paper, matrix games of dimension 2×2 , $2 \times n$, $m \times 2$ are considered, and the graphical solution of such games is discussed.

This paper is continuation of papers [17] and [18] by the same author.

Recall [17] that the matrix

$$A = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in \mathbf{R}^{m,n}$$

is called the *payment (payoff) matrix* of the first player I of the game.

Since the interests are opposite, then $-A = (-a_{ij})_{m \times n}$ turns out to be the payoff matrix of the second player II.

Denote

$$\alpha = \max_i \min_j a_{ij}, \quad \beta = \min_j \max_i a_{ij}$$

(minimum per rows, maximum per columns for entries a_{ij} of the payoff matrix A).

In Section 2, graphical solution of matrix games 2×2 , $2 \times n$, and $m \times 2$ is discussed. In Section 3, contents of the paper is summarized.

Topics, considered in this paper, and related to them are considered in reference titles [1]–[20], etc.

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2. Graphical solution of matrix games 2×2 , $2 \times n$, $m \times 2$

Graphical solution of matrix games is applicable only to games, in which at least one of players has only 2 strategies.

a) *Case 2×2 .* Consider matrix game with payoff matrix

player I \ player II	y_1	$y_2 = 1 - y_1$
x_1	a_{11}	a_{12}
$x_2 = 1 - x_1$	a_{21}	a_{22}

where x_1 and x_2 are the probabilities (frequencies), with which player I chooses his pure strategies, respectively, and y_1 and y_2 are the probabilities (frequencies), with which player II chooses his pure strategies, respectively.

Since $x_1 + x_2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$ from the properties of probabilities, then $x_2 = 1 - x_1$. Similarly, $y_1 + y_2 = 1$, $y_1 \geq 0$, $y_2 \geq 0$, therefore $y_2 = 1 - y_1$.

Graphical method for solving matrix games in this case consists in constructing the straight line γ_1 through the pair of points $(0, a_{21})$, $(1, a_{11})$, and the straight line γ_2 through the points $(0, a_{22})$, $(1, a_{12})$. These straight lines are defined as follows

$$\gamma_1(x_1): (a_{11} - a_{21})x_1 + a_{21},$$

$$\gamma_2(x_1): (a_{12} - a_{22})x_1 + a_{22},$$

respectively.

The profit of player I is $\bar{\gamma} \stackrel{\text{def}}{=} \min\{\gamma_1, \gamma_2\}$ and the purpose of this player is to maximize this profit $\bar{\gamma}$.

The graph for player I is presented in Figure 1 below.

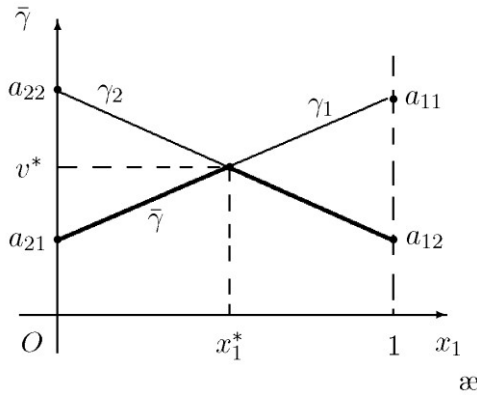


Figure 1
Graph for player I in Case 2×2

The maximum of $\bar{\gamma}$ is attained for $\gamma_1 = \gamma_2$, that is, for $(a_{11} - a_{21})x_1 + a_{21} = (a_{12} - a_{22})x_1 + a_{22}$. Hence,

$$x_1^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}}, x_2^* = 1 - x_1^*.$$

The values of y_1^* and y_2^* are calculated analogously, using the pair of points $(0, a_{12})$, $(1, a_{11})$, and the points $(0, a_{22})$, $(1, a_{21})$, respectively, defining the straight lines γ_I and γ_{II} , respectively:

$$\gamma_I(y_1): (a_{11} - a_{12})y_1 + a_{12},$$

$$\gamma_{II}(y_1): (a_{21} - a_{22})y_1 + a_{22}.$$

The loss of player II is $\bar{\gamma} \stackrel{\text{def}}{=} \max\{\gamma_I, \gamma_{II}\}$ (Figure 2 below) and the purpose of this player is to minimize this loss $\bar{\gamma}$.

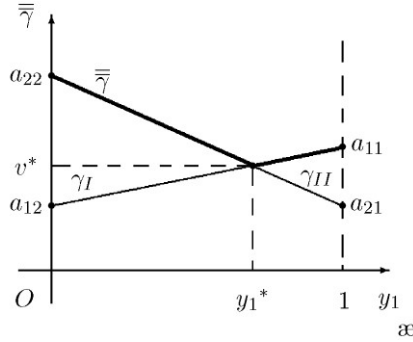


Figure 2
Graph for player II in Case 2×2

The minimum of $\bar{\gamma}$ is attained for $\gamma_I = \gamma_{II}$, that is, for $(a_{11} - a_{12})y_1 + a_{12} = (a_{21} - a_{22})y_1 + a_{22}$. Therefore

$$y_1^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}, y_2^* = 1 - y_1^*.$$

Example 1: Given the 2×2 matrix game defined by the following payoff matrix

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}.$$

Find the optimal strategies for both players.

Solution: Minimal elements of A per rows are -1 and -3 , respectively, and maximal elements of A per columns are 2 and 4 , respectively. Calculate

$$\alpha = \max_i \min_j a_{ij} = \max\{-1, -3\} = -1, \beta = \min_j \max_i a_{ij} = \min\{2, 4\} = 2.$$

Since $\alpha \neq \beta$, this game has no saddle point.

Construct the straight lines γ_1 and γ_2 :

$$\gamma_1(x_1): (2 - (-3))x_1 + (-3), \text{ that is, } \gamma_1(x_1): 5x_1 - 3,$$

$$\gamma_2(x_1): (-1 - 4)x_1 + 4, \text{ that is, } \gamma_2(x_1): -5x_1 + 4,$$

as well as the straight lines $\gamma_I(y_1)$ and $\gamma_{II}(y_1)$ (Figure 3 below):

$$\gamma_I(y_1): (2 - (-1))y_1 + (-1), \text{ that is, } \gamma_I(y_1): 3y_1 - 1,$$

$$\gamma_{II}(y_1): (-3 - 4)y_1 + 4, \text{ that is, } \gamma_{II}(y_1): -7y_1 + 4.$$

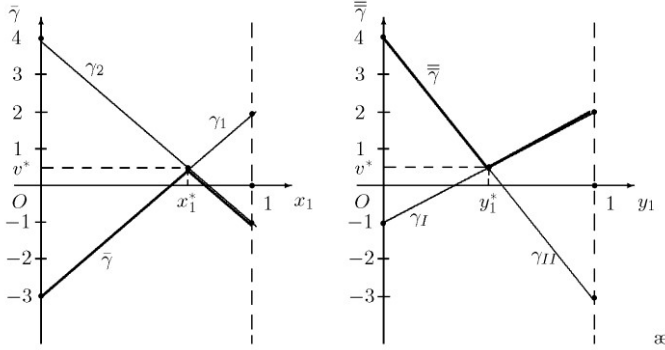


Figure 3
Graphs for players I and II in Example 1

By definition, $\bar{\gamma} = \min\{\gamma_1, \gamma_2\}$.

Obviously $\gamma_1 = \gamma_2$ for $(2 - (-3))x_1 + (-3) = (-1 - 4)x_1 + 4$, from where $x_1^* = 0,7$.

Therefore

$$v^* = \gamma_1(x_1^*) = 5,0,7 - 3 = 0,5 = \gamma_2(x_1^*) = -5,0,7 + 4,$$

that is, $\max \bar{\gamma}$ is attained for $x_1^* = 0,7$ and it is equal to $v^* = 0,5$.

By definition, $\bar{\bar{\gamma}} = \max\{\gamma_I, \gamma_{II}\}$.

Obviously $\gamma_I = \gamma_{II}$ for $(2 - (-1))y_1 + (-1) = (-3 - 4)y_1 + 4$, from where $y_1^* = 0,5$.

Hence

$$v^* = \gamma_I(y_1^*) = 3,0,5 - 1 = 0,5 = \gamma_{II}(y_1^*) = -7,0,5 + 4,$$

that is, $\min \bar{\bar{\gamma}}$ is attained for $y_1^* = 0,5$ and it is equal to $v^* = 0,5$.

Therefore the optimal strategies of players I and II are $S_I^* = \{0,7; 0,3\}$ and $S_{II}^* = \{0,5; 0,5\}$, respectively, and value of the game is $v^* = 0,5$.

The obtained optimal strategies can be interpreted as follows: player I must apply more often the first strategy because $x_1^* = 0,7 > x_2^* = 0,3$, whereas player II must apply both strategies with the same frequency because $y_1^* = y_2^* = 0,5$.

Case $2 \times n$. Consider matrix game with payoff matrix

player I \ player II	y_1	y_2	...	y_n
x_1	a_{11}	a_{12}	...	a_{1n}
$x_2 = 1 - x_1$	a_{21}	a_{22}	...	a_{2n}

In this case, player I has two strategies, hence $x_1 + x_2 = 1$, that is, $x_2 = 1 - x_1$, and $x_1 \geq 0$, $x_2 \geq 0$ from the properties of probabilities. Similarly, $y_1 + y_2 + \dots + y_n = 1$ and $y_1 \geq 0$, $y_2 \geq 0$, ..., $y_n \geq 0$.

Construct the straight lines

$$\gamma_1(x_1): (a_{11} - a_{21})x_1 + a_{21},$$

$$\gamma_2(x_1): (a_{12} - a_{22})x_1 + a_{22},$$

⋮

$$\gamma_n(x_1): (a_{1n} - a_{2n})x_1 + a_{2n}.$$

Similarly to the Case 2×2 , profit of player I is $\bar{\gamma} \stackrel{\text{def}}{=} \min\{\gamma_1, \dots, \gamma_n\}$, and the purpose of this player is to maximize this profit $\bar{\gamma}$.

Let $\max \bar{\gamma}$ be attained for $\gamma_k = \gamma_l$, where $1 \leq k, l \leq n$.

Value v of the game is calculated as in Case 2×2 .

The optimal strategy for player II has the form

$$S_{II} = \{0, \dots, 0, \lambda, 0, \dots, 0, 1 - \lambda, 0, \dots, 0\}, \lambda \in [0, 1],$$

where the values of λ and $1 - \lambda$ are on positions No. k and No. l in S_{II} , respectively.

In fact, components of S_{II} , are the values $y_1, y_2, \dots, y_n$ of the above table, and they obviously satisfy $y_1 + y_2 + \dots + y_n = 0 + \dots + 0 + \lambda + 0 + \dots + 0 + (1 - \lambda) + 0 + \dots + 0 = 1$ and $y_1 \geq 0, y_2 \geq 0, \dots, y_n \geq 0$ as probabilities.

The value of $\lambda \in [0, 1]$ is determined from the equation

$$\lambda \gamma_k(x_1) + (1 - \lambda) \gamma_l(x_1) = v^*, \lambda \in [0, 1],$$

where the left-hand side of this equation is the *convex combination* of $\gamma_k(x_1)$ and $\gamma_l(x_1)$.

Once the value of λ is determined, the optimal strategy S_{II} becomes known.

Example 2: Solve the 2×3 matrix game defined by the following payoff matrix

$$A = \begin{pmatrix} 7 & 9 & 8 \\ 10 & 6 & 9 \end{pmatrix}.$$

Solution: Minimal elements of A per rows are 7 and 6, respectively, and maximal elements of A per columns are 10, 9, and 9, respectively. Calculate

$$\alpha = \max_i \min_j a_{ij} = \max\{7, 6\} = 7, \quad \beta = \min_j \max_i a_{ij} = \min\{10, 9, 9\} = 9.$$

Since $\alpha \neq \beta$, this game has no saddle point.

Construct the straight lines

$$\gamma_1(x_1) : (7 - 10)x_1 + 10,$$

$$\gamma_2(x_1) : (9 - 6)x_1 + 6,$$

$$\gamma_3(x_1) : (8 - 9)x_1 + 9,$$

that is, the straight lines

$$\gamma_1(x_1) : -3x_1 + 10,$$

$$\gamma_2(x_1) : 3x_1 + 6,$$

$$\gamma_3(x_1) : -x_1 + 9.$$

The profit of player I is $\bar{\gamma} \stackrel{\text{def}}{=} \min\{\gamma_1, \gamma_2, \gamma_3\}$.

In Figure 4 below, $\bar{\gamma}$ is drawn by a bold broken line, and it is formed by segments of straight lines γ_1 and γ_2 .

It can be seen that $\max \bar{\gamma}$ is attained for $\gamma_1 = \gamma_2$, that is, for $-3x_1 + 10 = 3x_1 + 6$, hence $x_1^* = \frac{2}{3}$. Therefore $x_2^* = 1 - x_1^* = \frac{1}{3}$.

The value of this game is

$$v^* = \gamma_1(x_1^*) = -3 \cdot \frac{2}{3} + 10 = 8 = \gamma_2(x_1^*) = 3 \cdot \frac{2}{3} + 6,$$

and the optimal mixed strategy for player I is $S_I^* = \{\frac{2}{3}, \frac{1}{3}\}$.

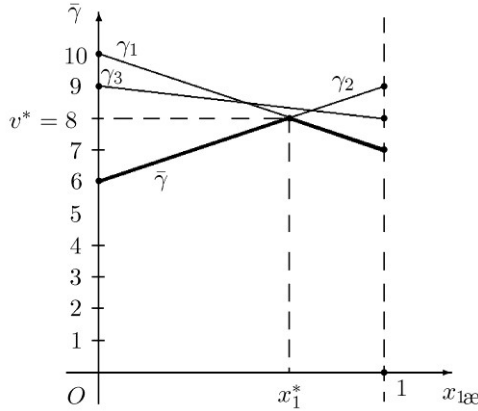


Figure 4
Graph for player I in Example 2

From Figure 4 it can be seen that player II has to use his/her first and second pure strategies, which correspond to the straight lines γ_1 and γ_2 , respectively, because $\max \bar{\gamma}$ is attained for $\gamma_1 = \gamma_2$. The optimal mixed strategy $S_{II}^* = \{\lambda, 1 - \lambda, 0\}, \lambda \in [0, 1]$ must be a combination between the first and second strategies of player I, with associated straight line through the point $(x_1^*, v^*) = (\frac{2}{3}, 8)$, which is parallel to the abscissa axis. As pointed out, we have to find a $\lambda \in [0, 1]$, such that the convex combination (with coefficients $\lambda \in [0, 1]$ and $1 - \lambda$) of the first and the second strategies of player II to be equal to $v^* = 8$, that is,

$$\lambda \gamma_1(x_1) + (1 - \lambda) \gamma_2(x_1) = v^*,$$

$$\lambda(-3x_1 + 10) + (1 - \lambda)(3x_1 + 6) = 8.$$

This equation is satisfied for $\lambda = \frac{1}{2} \in [0, 1]$. Therefore $1 - \lambda = \frac{1}{2}$, and the optimal mixed strategy of player II is $S_{II}^* = \{\frac{1}{2}, \frac{1}{2}, 0\}$.

Case $m \times 2$. Consider matrix game defined by the following payoff matrix

player I \ player II	y_1	$y_2 = 1 - y_1$
x_1	a_{11}	a_{12}
x_2	a_{21}	a_{22}
$\vdots$	$\vdots$	$\vdots$
x_m	a_{m1}	a_{m2}

Since player II has two strategies, then $y_1 + y_2 = 1$, that is, $y_2 = 1 - y_1$ and $y_1 \geq 0$, $y_2 \geq 0$ (properties of probabilities). Similarly, $x_1 + x_2 + \dots + x_m = 1$ and $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_m \geq 0$.

Construct the straight lines

$$\gamma_I(y_1): (a_{11} - a_{12})y_1 + a_{12},$$

$$\gamma_{II}(y_1): (a_{21} - a_{22})y_1 + a_{22},$$

... ..

$$\gamma_m(y_1): (a_{m1} - a_{m2})y_1 + a_{m2}.$$

The expected losses of player II are $\bar{\gamma} \stackrel{\text{def}}{=} \max\{\gamma_I, \gamma_{II}, \dots, \gamma_m\}$.

We have to find $\min \bar{\gamma}$, that is, minimum of the expected losses of player II.

Let $\min \bar{\gamma}$ be attained for $\gamma_p = \gamma_q$, where $1 \leq p, q \leq m$.

Value v of the game is calculated as in Case 2×2 and Case $2 \times n$.

The optimal strategy of player I is of the form

$$S_I = \{0, \dots, 0, \lambda, 0, \dots, 1 - \lambda, 0, \dots, 0\}, \lambda \in [0, 1],$$

where the values of λ and $1 - \lambda$ are on positions No. p and No. q in S_I , respectively.

Components of S_I are the values $x_1, x_2, \dots, x_m$ of the above table, and they obviously satisfy $x_1 + x_2 + \dots + x_m = 1$ and $x_1 \geq 0$, $x_2 \geq 0$, ..., $x_m \geq 0$ as probabilities.

The value of $\lambda \in [0, 1]$ is determined from the equation

$$\lambda \gamma_p(y_1) + (1 - \lambda) \gamma_q(y_1) = v^*, \lambda \in [0, 1],$$

where the left-hand side of this equation is the convex combination of $\gamma_p(y_1)$ and $\gamma_q(y_1)$.

Once the value of λ is determined, the optimal strategy S_I becomes known.

Example 3: Solve the 4×2 matrix game defined by the following payoff matrix

$$A = \begin{pmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 2 \\ -2 & 6 \end{pmatrix}.$$

Solution: Minimal elements of A per rows are 2, 2, 2, and -2 , respectively, and maximal elements of A per columns are 3 and 6, respectively. Calculate

$$\alpha = \max_i \min_j a_{ij} = \max\{2, 2, -2\} = 2,$$

$$\text{and } \beta = \min_j \max_i a_{ij} = \min\{3, 6\} = 3.$$

Since $\alpha \neq \beta$, this game has no saddle point.

Construct the straight lines

$$\gamma_1(y_1) : (2 - 4)y_1 + 4,$$

$$\gamma_2(y_1) : (2 - 3)y_1 + 3,$$

$$\gamma_3(y_1) : (3 - 2)y_1 + 2,$$

$$\gamma_4(y_1) : (-2 - 6)y_1 + 6,$$

that is, the straight lines (Figure 5 below):

$$\gamma_1(y_1) : -2y_1 + 4,$$

$$\gamma_2(y_1) : -y_1 + 3,$$

$$\gamma_3(y_1) : y_1 + 2,$$

$$\gamma_4(y_1) : -8y_1 + 6.$$

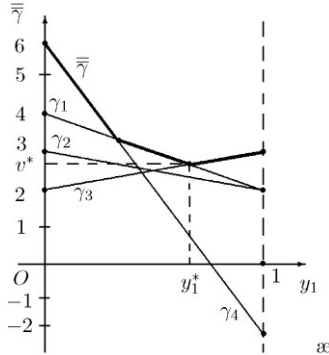


Figure 5

Graph for player II in Example 3

The expected losses of player II are $\bar{\bar{y}} \stackrel{\text{def}}{=} \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.

In Figure 5, $\bar{\bar{y}}$ is drawn by a bold broken line, and it is formed by segments of straight lines γ_1 , γ_3 and γ_4 .

It can be seen that $\min \bar{\bar{y}}$ is attained for $\gamma_1 = \gamma_3$ (Figure 5), that is, for $-2y_1 + 4 = y_1 + 2$, hence $y_1^* = \frac{2}{3}$. Therefore $y_2^* = 1 - y_1^* = \frac{1}{3}$.

The value of this game is

$$v^* = \gamma_1(y_1^*) = -2 \cdot \frac{2}{3} + 4 = \frac{8}{3} = \gamma_3(y_1^*) = \frac{2}{3} + 2,$$

and the optimal mixed strategy of player II is $S_{II}^* = \{\frac{2}{3}, \frac{1}{3}\}$.

From Figure 5 it can be seen that player I has to use his/her first and third pure strategies, which correspond to the straight lines γ_1 and γ_3 , respectively, because $\min \bar{\bar{y}}$ is attained for $\gamma_1 = \gamma_3$. The optimal mixed strategy $S_I^* =$

$\{\lambda, 0, 1 - \lambda, 0\}, \lambda \in [0, 1]$ must be a combination between the first and third strategies of player II. As described above, we have to find a $\lambda \in [0, 1]$, such that the convex combination (with coefficients $\lambda \in [0, 1]$ and $1 - \lambda$) of the first and the third strategies of player I to be equal to $v^* = \frac{8}{3}$, that is,

$$\begin{aligned}\lambda\gamma_1(y_1) + (1 - \lambda)\gamma_3(y_1) &= v^*, \\ \lambda(-2y_1 + 4) + (1 - \lambda)(y_1 + 2) &= \frac{8}{3}.\end{aligned}$$

This equation is satisfied for $\lambda = \frac{1}{3} \in [0, 1]$. Therefore $1 - \lambda = \frac{2}{3}$, and the optimal mixed strategy for player I is $S_1^* = \{\frac{1}{3}, 0, \frac{2}{3}, 0\}$.

3. Conclusion

In this paper, the graphical solution of 2×2 , $2 \times n$, $m \times 2$ matrix games is considered, and examples for applying this approach in the three cases are presented.

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