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On the fractional part of the sum and product of two p-adic β-integers

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Abstract

This paper aims to explore arithmetic properties in the set of p-adic β-integers. We show that the quantities $L_{\odot}$ and $L_{\oplus}$ are finite if β is an algebraic integer of odd degree. Here, $L_{\odot}$ (resp. $L_{\oplus}$) denotes the largest shift after the decimal point in the product (resp. sum) of two β-integers when this shift is finite.

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1. Introduction

β-expansions are an important mathematical concept that generalizes the idea of base expansions (like binary or decimal) to non-integer bases. They play a crucial role in various areas of mathematics and have significant theoretical and practical implications.

The generalization of β-expansions was first introduced by the Hungarian mathematician Alfréd Rényi in 1957. In his groundbreaking work, he extended the concept of base expansions to non-integer bases, laying the foundation for a new area of study in number theory and dynamical systems.

In [8], A. Rényi associated the expansion with a dynamical system defined by the class of the β -transformation $\{T_\beta, \beta > 1\}$. Let $\beta > 1$. The β-transformation is a linear piecewise function on $[0,1[$, defined as:

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$$T_\beta : x \mapsto \beta x - [\beta x],$$

such that x is a real number in $[0,1]$.

For all $i \geq 1$, we set

$$x_i = [\beta T_\beta^{i-1}(x)],$$

which leads to the representation

$$x = \sum_{i \geq 1} \frac{x_i}{\beta^i}.$$

The sequence $d_\beta(x) = 0 \bullet x_1 x_2 x_3 \cdots$ is called the β-expansion of x.

Now, $\forall\ x > 0$, there exists an integer $m > 0$ where $\beta^{-m-1} x \in [0,1[$. This allows us to express x in the form

$$x = x_{-m}\beta^m + \cdots + x_{-1}\beta + x_0 + \frac{x_1}{\beta} + \cdots.$$

Consequently, $d_\beta(x)$ is expressed as

$$x_{-m} \ldots x_{-1} x_0 \bullet x_1 x_2 x_3 \ldots.$$

The β-expansion is finite if the sequence $(x_i)_{i \geq 1}$ becomes 0 from a certain rank, and it's eventually periodic if there exist integers $l \geq 1$ and $s \geq 1$ such that for all $j \geq s$, we have the condition $x_j = x_{j+l}$. The set of real numbers in the interval $[0,1]$ with periodic β-expansions is denoted by $Per(\beta)$, while $Fin(\beta)$ is the set of numbers with finite β-expansions.

Any real $x > 1$ admits a unique expansion as follows:

$$x = \underbrace{\sum_{i=0}^{n} x_{-i}\beta^i}_{[x]_\beta} + \underbrace{\sum_{i>0} x_i \beta^{-i}}_{\{x\}_\beta}.$$

The part $[x]_\beta$ is referred to as the β-integer part of x. Conversely, the part $\{x\}_\beta = x - [x]_\beta$ is known the β-fractional part. This decomposition extends the conventional notion of integers. Denote by $\mathbb{Z}_\beta$ the set of β-integers.

In general, $Fin(\beta)$, $Per(\beta)$ and $\mathbb{Z}_\beta$ are not necessarily stable neither by addition nor by multiplication. Despite this, considering these operations in β-arithmetics is useful in computer science. Consequently, studying arithmetic properties of β-integers, such as computing their sum or product where fractional parts may arise, is very interesting.

To analyze this, we introduce the quantities $L_\oplus$ and $L_\odot$ as defined in [1].

Definition 1.1: Let $\beta > 1$. We define:

$$L_\oplus := \min\{m \in \mathbb{N} : \text{if } x, y \in \mathbb{Z}_\beta \text{ and } x + y \in \mathrm{Fin}(\beta), \text{then } \beta^m(x+y) \in \mathbb{Z}_\beta\}.$$

$$L_\odot := \min\{m \in \mathbb{N} : \text{if } x, y \in \mathbb{Z}_\beta \text{ and } xy \in \mathrm{Fin}(\beta), \text{then } \beta^m(xy) \in \mathbb{Z}_\beta\}.$$

In convention that, $+\infty$ otherwise.

Many researchers have been interested in examining the finiteness of these two quantities. For instance, in 1992, Frogny and Solmyak showed in [5] that $L_\oplus$ is finite if β is a Pisot number (an algebraic integer greater than 1, such that its

conjugates are all in the open unit disk). Later, M. Hbaib and I. Ben Salah strengthened this result through this theorem:

Theorem 1.2: [4] *Let* $\beta > 1$. *If either* β *or* $\frac{1}{\beta}$ *is an algebraic integer without archimedean conjugates of modulus* 1, *then* $L_{\oplus}$ *and* L_e *are finite.*

To date, no example of β is known for which $L_{\oplus}$ and $L_{\odot}$ are infinite. Additionally, several studies have aimed at determining explicit values of $L_{\oplus}$ and $L_{\odot}$ in various bases. Let's first consider the quadratic case and cite the following theorem:

Theorem 1.3: [3] *Let* $\beta > 1$. *Suppose* β *is a solution of either* $x^2 = kx - 1$, *where* $k \in \mathbb{N}$ *and* $k \geq 3$, *or* $x^2 = kx + 1$, *where* $k \in \mathbb{N}$. *In the first case, we get* $L_{\odot} = L_{\oplus} = 1$. *In the second case, we get* $L_{\odot} = L_{\oplus} = 2$.

The first cubic case was investigated by A. Messaoudi, who proved the following result for the Tribonacci number:

Theorem 1.4: [6] *If* β *is the Tribonacci number, then* $L_{\odot} \leq 9$.

Subsequently, this result has been refined as follows:

Theorem 1.5: [1] *If* β *is the Tribonacci number, therefore* $5 \leqslant L_{\oplus} \leqslant 6$ *and* $4 \leqslant L_{\odot} \leqslant 5$.

The exact value of $L_{\oplus}$ was later determined by J. Bernat.

Theorem 1.6: [2] *If* β *is the Tribonacci number, then* $L_{\oplus} = 5$.

For higher-degree values of β, determining the exact values of $L_{\oplus}$ and $L_{\odot}$ remains a challenging problem. Even establishing upper or lower bounds for these constants proves difficult. Many examples are analyzed in [1], where a new method was presented to give upper bounds for these two quantities for certain classes of Pisot numbers that satisfy specific properties.

Thus, the finiteness and explicit values of these two quantities constitute an important research topic that has attracted considerable attention. Motivated by this topic, we explore analogous concepts in the field $\mathbb{Q}_p$ and we introduce the algorithm of β-expansion in this setting, which is an emerging area of research.

It is worth noting that the quantities $L_{\oplus}$ and $L_{\odot}$ have not yet been studied in the context of p-adic numbers. The objective of this article is to investigate these quantities in this field and to establish that $L_{\odot}$ and $L_{\oplus}$ are finite for specific families of algebraic numbers β.

The organization of this paper is outlined below:

In Section 2, we cite fundamental definitions of p-adic numbers in the field $\mathbb{Q}_p$ and we define the β-expansion algorithm for p-adic numbers. In Section 3, we show that for any algebraic integer β or $\frac{1}{\beta}$ without archimedean conjugates of modulus 1, the quantities $L_{\odot}$ and $L_{\oplus}$ are finite.

2. β −expansion in the p-adic field

Let's take p is a prime number. In the field of p-adic numbers, β-expansions are a generalization of the usual expansions in powers of a prime p, but with a non-integer base β. They involve adapting the real β-expansion framework to the p-adic context, considering the unique arithmetic and topological properties of the p-adic numbers. This extension leads to new challenges in defining convergence and studying the ergodic properties of p-adic β-expansions, with potential applications in number theory, cryptography, and dynamical systems. So, before introducing the β-expansion in $\mathbb{Q}_p$, let us define first this field.

The p-adic valuation v_p is a function $v_p: \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ defined as follows:

- For any non-zero rational number y, we express y in the form $y = p^k \frac{a}{b}$, where a and b are integers not divisible by p. Then, $v_p(y) = k$.
- For $y = 0$, by convention, we have $v_p(0) = \infty$.

The p-adic valuation satisfies the following properties:

1. Non-negativity: For any non-zero $y \in \mathbb{Q}$, $v_p(y) \geq 0$.
2. Multiplicativity: For any $y_1, y_2 \in \mathbb{Q}$,
$$v_p(y_1 y_2) = v_p(y_1) + v_p(y_2).$$
3. Non-Archimedean Property: For any $y_1, y_2 \in \mathbb{Q}$,
$$v_p(y_1 + y_2) \geq \min\{v_p(y_1), v_p(y_2)\}.$$
Moreover, if $v_p(y_1) \neq v_p(y_2)$, then
$$v_p(y_1 + y_2) = \min\{v_p(y_1), v_p(y_2)\}.$$

So, we define the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ as:

- For $y \neq 0$: $|y|_p = p^{-v_p(y)}$.
- For $y = 0$: $|0|_p = 0$.

This p-adic absolute satisfies the non-Archimedean (Ultrametric) Property:
$$|y_1 + y_2|_p \leq \max\{|y_1|_p, |y_2|_p\} \text{ for all } y_1, y_2 \in \mathbb{Q}.$$
Additionally, if $|y_1|_p \neq |y_2|_p$, then $|y_1 + y_2|_p = \max\{|y_1|_p, |y_2|_p\}$.

Let $|.|_\infty$ denote the standard (Archimedean) absolute value on $\mathbb{Q}$. Then $|.|_p$ and $|.|_\infty$ verify the product formula

$$\prod_{p\in\mathbb{P}\cup\{\infty\}} |y|_p = 1, \text{for all } y \in \mathbb{Q}\backslash\{0\}.$$

Recall that the field $(\mathbb{Q}, |\cdot|_p)$ is not complete. Its completion with respects to $|\cdot|_p$ is the p-adic field $\mathbb{Q}_p$.

Now, we define

$$\mathbb{A}_p = \{kp^h | h, k \in \mathbb{Z}\} = \mathbb{Z}\left[\frac{1}{p}\right].$$

This set $\mathbb{A}_p$, included in $\mathbb{Q}$, forms a principal ring. The group of units in $\mathbb{A}_p$ is given by

$$\{\pm p^h | h \in \mathbb{Z}\},$$

and its field of fractions is $\mathbb{Q}$. Consequently, we have the following inclusions:

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

In particular, define

$$\mathbb{A}_p \cap [0,1[= \mathbb{A}_p,.$$

Moreover, the field $\mathbb{Q}_p$ can be defined as the field of fractions of $\mathbb{Z}_p$:

$$\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p),$$

where

$$\mathbb{Z}_p = \{\sum_{k=0}^{+\infty} a_k p^k; _k \in \{0, \dots, p-1\}, k \geq 0\}.$$

The valuation v_p, introduced earlier, extends naturally to $\mathbb{Q}_p$ in the following way:

Consider a non zero element $y \in \mathbb{Q}_p$ with

$$y = \sum_{k=k_0}^{\infty} y_k p^k,$$

where $k_0 \in \mathbb{Z}$, $y_{k_0} \neq 0$, and $y_k \in \{0, \dots, p-1\}$. Then the valuation $v_p(y)$ is defined as:

$$\text{If } y \neq 0, v_p(y) = k_0 \text{ and if } y = 0, v_p(y) = \infty.$$

We note that every nonzero element $y \in \mathbb{Q}_p$ has a unique p-adic expansion expressed as

$$y = \sum_{k=k_0}^{\infty} y_k p^k, (*)$$

with $k_0 \in \mathbb{Z}$, $y_{k_0} \neq 0$, and $y_k \in \{0, \dots, p-1\}$.

Using this expansion, we adopt the notation

$$y = \cdots p_2 p_1 p_0 . p_{-1} p_{-2} \dots p_{n_0}.$$

Definition 2.1: Every $y \in \mathbb{Q}_p$ with the expansion given in $(*)$ admits a unique decomposition, known as the Artin decomposition:

$$y = [y]_p + \{y\}_p,$$

with

$$[y]_p = \sum_{i\geq 0} y_i p^i \text{ and } \{y\}_p = \sum_{i<0} y_i p^i.$$

The term $[y]_p \in \mathbb{Z}_p$ is called the p-adic integer part of y, while $\{y\}_p \in [0,1) \cap \mathbb{A}_p$ is referred to as the p-adic fractional part of y.

Algebraic numbers in $\mathbb{Q}_p$ are powerful tools for studying local properties of numbers, with far-reaching implications for both local and global number theory. For this, our aim now is to introduce these numbers.

Definition 2.2: An element γ is said to be algebraic over $\mathbb{A}_p$ if there exists a polynomial

$$Q(x) = a_m x^m + \cdots + a_1 x + a_0 \in \mathbb{A}_p[x]$$

such that $a_m \neq 0$ and $Q(\gamma) = 0$. If $Q(x)$ is irreducible over $\mathbb{A}_p$, it is called the *minimal polynomial* of γ. In particular:

- If the leading coefficient satisfies $a_m = p^h$ for some $h \in \mathbb{Z}$, then γ is an *algebraic integer*. Since p^h is a unit in $\mathbb{A}_p$, we may assume without loss of generality that $a_m = 1$.
- If the constant term satisfies $a_0 = p^{h'}$ for some $h' \in \mathbb{Z}$, then γ is called an *algebraic unit*.

It is important to note that not all algebraic elements over $\mathbb{Q}$ belong to $\mathbb{Q}_p$. However, in our context, we only need the fact that the absolute value $|\cdot|_p$ uniquely extends from $\mathbb{Q}_p$ to all of its algebraic extensions. This stems from a basic theorem in the theory of non-Archimedean fields (see [7]).

Remark 2.3: Let γ be algebraic number of degree n belongs to $\mathbb{A}_p$. We denote by:

- $\gamma^{(1)}, \dots, \gamma^{(n)}$ the *non-archimedean* conjugates of γ.
- $\gamma^{(n+1)}, \dots, \gamma^{(2n)}$ the *archimedean* conjugates of γ.

Definition 2.4: Let $\gamma \in \mathbb{Q}_p$ such that $|\gamma|_p > 1$. γ is called a Pisot-Chabauty number (or PC number) if it satisfies these conditions:

- The element $\gamma^{(1)} = \gamma$ is an algebraic integer over $\mathbb{A}_p$.
- For all non-Archimedean conjugates $\gamma^{(i)}$ of γ, with $i \in \{2, \dots, n\}$, we have $|\gamma^{(i)}|_p \leq 1$.
- For all Archimedean conjugates $\gamma^{(i)}$ of γ, with $i \in \{n+1, \dots, 2n\}$, we have $|\gamma^{(i)}|_\infty < 1$.

Pisot-Chabauty numbers are a special class of algebraic numbers. They have deep connections to number theory, particularly in the fields of Diophantine approximation, sequences approaching integers, and certain dynamical systems. Their fascinating properties make them a rich subject of study in various mathematical disciplines.

Finally, we reach to study the β-expansions in $\mathbb{Q}_p$: Let β and y two p-adic numbers such that $|\beta|_p > 1$ and $|y|_p \leq 1$. A β-representation of y is a sequence $(a_i)_{i\geq 1}$, $a_i \in \mathbb{A}_p$, with

$$y = \sum_{i\geq 1} \frac{a_i}{\beta^i}.$$

In general , a β-representation is not unique.

Now, we introduce $d_\beta(y)$ the β-expansion of y by using T_β in $\mathbb{Z}_p$ which is defined as follows:

$$T_\beta(y) = \beta y - [\beta y]_p.$$

Thereby $d_\beta(y) = (a_i)_{i\geq 1}$ with values in $\mathbb{A}_{\beta,p} = \{y \in \mathbb{A}_p : |y|_p \leq |\beta|_p\} \cap [0,1)$ where $a_i = \{\beta T^{i-1}(y)\}_p$ for all $i \geq 1$.

We can define the sequence $(a_i)_{i\geq 1}$:

$$r_0 = y, a_i = \{\beta r_{k-1}\}_p \text{ where } r_i = \beta r_{i-1} - a_i \text{ for all } i \geq 1.$$

If $d_\beta(y) = (a_i)_{i\geq 1}$, we can write $y = 0.d_1 d_2 d_3$ ....

We note that $d_\beta(y)$ is finite ($a_j = 0$ for all but finitely many $j \geq 1$) if and only if there exists $h \geq 0$ such that $T_\beta^h(y) = 0$ and $d_\beta(y)$ is eventually periodic if and only if there exists $h \geq 0$ and $t \geq 1$ when $T_\beta^{h+t}(y) = T_\beta^h(y)$.

Let's define the following sets.

$$Fin(\beta) = \{y \in \mathbb{Z}_p : d_\beta(y) \text{ is finite}\} \text{ and}$$

$$Per(\beta) = \{y \in \mathbb{Z}_p : d_\beta(y) \text{ is periodic}\}.$$

We can generalize the β-expansion of p-adic number $y \in \mathbb{Q}_p$ such that $|y|_p > 1$; because in this case, there exists a unique $s \in \mathbb{N}$ with $|\beta|_p^s \leq |y|_p < |\beta|_p^{s+1}$.

So $|\frac{y}{\beta^{s+1}}|_p < 1$; hence we obtain the β-expansion of y by shifting $d_\beta(\frac{y}{\beta^{s+1}})$ by s digits.

If $d_\beta(y) = d_l d_{l-1} \dots d_0 . d_{-1} d_{-2}$ ..., let $[y]_\beta = d_l\beta^l + d_{l-1}\beta^{l-1} + \cdots + d_0$ and $\{y\}_\beta = y - [y]_\beta$.

If $d_\beta(y)$ is finite with $y = \sum_l^m a_i \beta^{-i}$ where $m, l \in \mathbb{Z}$, then we put $\mathrm{ord}_\beta(y) = -m$ and $\mathrm{ord}_\beta(y) = -\infty$ otherwise.

We also define $(\mathbb{Z}_\beta)_p$ the set of β-integers as follows:

$$(\mathbb{Z}_\beta)_p = \{y \in \mathbb{Q}_p : y = [y]_\beta\}.$$

Remark 2.5: The set $(\mathbb{Z}_\beta)_p$ is not stable either by addition or by multiplication.

Recently, the beta-expansion in $\mathbb{Q}_p$ has become a source of interest for several researchers, as it presents many applications in various areas of mathematics. In 2016, K. Schecher, V. Sirvent, and P. Sureir succeeded in proving interesting results concerning this concept for instance the following result $KVFPS$:

Theorem 2.6: *Let $\beta \in \mathbb{Q}_p$ be a PC number. Hence*

$$Per(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p.$$

The sets $\mathrm{Fin}(\beta)$ and $(\mathbb{Z}_\beta)_p$ are not stable by addition and multiplication. Therefore it is important to study the two quantities $L_\oplus$ and L_{e}.

Definition 2.7: Let $|\beta| > 1$. We denote

$$L_\odot = \begin{cases} \min S_\beta & \text{if } S_\beta \neq \phi \\ \infty & \text{if } S_\beta = \phi \end{cases} \quad \text{and} \quad L_\odot = \begin{cases} \min S'_\beta & \text{if } S'_\beta \neq \phi \\ \infty & \text{if } S'_\beta = \phi \end{cases}$$

With

$$S_\beta = \{m \in \mathbb{N}: \text{if } x, y \in (\mathbb{Z}_\beta)_p \text{ and } xy \in Fin(\beta), \text{then } \beta^m(xy) \in (\mathbb{Z}_\beta)_p\}.$$

and

$$S'_\beta = \{m \in \mathbb{N}: \text{if } x, y \in (\mathbb{Z}_\beta)_p \text{ and } x + y \in Fin(\beta), \text{then } \beta^m(x + y) \in (\mathbb{Z}_\beta)_p\}.$$

3. Results

Our goal now is to establish the finiteness of two quantities L_{e} and $L_\oplus$ in the context of an algebraic integer basis. To achieve this, let's first introduce some fundamental notions.

Consider an algebraic number β whose minimal polynomial is

$$P(X) = a_0 + \cdots + a_{d-1}X^{d-1} + X^d,$$

with $a_i \in \mathbb{A}_p$, for all $i \in \{0, \dots, d-1\}$. Let $\beta^{(2)}, \dots, \beta^{(d)}$ denote the non-Archimedean conjugates of β. We define the vector of these conjugates as

$$\overline{\beta} = \begin{pmatrix} \beta^{(2)} \\ \vdots \\ \beta^{(d)} \end{pmatrix}.$$

For any element $x \in \mathbb{Q}(\beta)$ of the form

$$x = r_0 + r_1\beta + \cdots + r_{d-1}\beta^{d-1},$$

where $r_i \in \mathbb{Q}$, the j-th non-Archimedean conjugate of x is given by

$$x^{(j)} = r_0 + r_1\beta^{(j)} + \cdots + r_{d-1}(\beta^{(j)})^{d-1}.$$

Definition 3.1 Define the following vector norms On $\mathbb{A}_p^n$:

- The p-adic norm: $||(a_1, \dots, a_n)||_p = \max_{1 \le i \le n} |a_i|_p$.
- The infinity norm: $||(a_1, \dots, a_n)\mathrm{P}_\infty = \max_{1 \le i \le n} |a_i|_\infty$.

Before establishing the finiteness of $L_\odot$ and $L_\oplus$, we first present this lemma.

Lemma 3.2: [9] *Assume $A \subset \mathbb{A}_p$. If A is bounded to both the p-adic norm $|\cdot|_p$ and the Archimedean norm $|\cdot|_\infty$, meaning that*

$$\sup_{a\in A}|a|_p < \infty \text{ and } \sup_{a\in A}|a|_\infty < \infty,$$

then A is necessarily finite.

Theorem 3.3: *Let $\beta \in \mathbb{Q}_p$ such that $|\beta|_p > 1$. If β is an algebraic integer of odd degree, then $L_\oplus$ and $L_\odot$ are finite.*

Proof:

Let x, $y \in (\mathbb{Z}_\beta)_p$. Suppose that $d_\beta(xy) \in Fin(\beta)$, so $d_\beta(xy) = c_n \dots c_1 c_0 \cdot c_{-1} \dots c_{-m}$, where

$$x = a_s\beta^s + a_{s-1}\beta^{s-1} + \cdots + a_0 \text{ with } |a_i|_p < |\beta|_p$$

and

$$y = b_k\beta^k + b_{k-1}\beta^{k-1} + \cdots + b_0 \text{ with } |b_i|_p < |\beta|_p.$$

Hence

$$xy = c_n\beta^n + \cdots + c_0 + \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m}.$$

Then,

$$\{xy\}_\beta = xy - c_n\beta^n - \cdots - c_0 = \frac{c_{-1}}{\beta} + \frac{c_{-2}}{\beta^2} + \cdots + \frac{c_{-m}}{\beta^m}.$$

Let now $(\beta^{(j)})_{2\le j\le d}$ denote the non-Archimedean conjugates of β. To show this theorem, we first distinguish these two cases:

Case 1: If $|\beta^{(j)}|_p > 1$.

In this case, we have

$$\{xy\}_\beta^{(j)} = \frac{c_{-1}}{\beta^{(j)}} + \frac{c_{-2}}{(\beta^{(j)})^2} + \cdots + \frac{c_{-m}}{(\beta^{(j)})^m}.$$

Since

$$|c_i|_p \le |\beta|_p \text{ for all } i \in \{-1, \dots, -m\},$$

we get

$$|\{xy\}_\beta^{(j)}|_p < |\beta|_p.$$

Case 2: If $|\beta^{(j)}|_p \le 1$.

In this case, we have

$$\begin{aligned}|\{xy\}_\beta^{(j)}|_p &= |(a_s(\beta^{(j)})^s + \cdots + a_0)(b_k(\beta^{(j)})^k + \cdots + b_0)\\ &\quad -c_n(\beta^{(j)})^n - \cdots - c_0|_p\\ &= |\textstyle\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l - c_n(\beta^{(j)})^n - \cdots - c_0|_p\\ &\le max\{|\textstyle\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l|_p; |c_n(\beta^{(j)})^n - \cdots - c_0|_p\}.\end{aligned}$$

As $|\beta^{(j)}|_p \le 1$, we get

$$|\textstyle\sum_{l=0}^{s+k}(\sum_{p=0}^{l} b_p a_{l-p})(\beta^{(j)})^l|_p \le |\beta|_p^2 \; and \; |c_n(\beta^{(j)})^n - \cdots - c_0|_p \le |\beta|_p.$$

Consequently,

$$|\{xy\}_\beta^{(j)}|_p \leq |\beta|_p^2.$$

Therefore the p-adic norm of $\begin{pmatrix} \{xy\}_\beta \\ \{xy\}_\beta^{(2)} \\ \vdots \\ \{xy\}_\beta^{(d)} \end{pmatrix}$ is less than $|\beta|_p^2$.

Now, since $\{xy\}_\beta = \sum_{l=0}^{s+k} (\sum_{p=0}^{l} b_p a_{l-p})\beta^l - c_n\beta^n - \cdots - c_0$ and β is an algebraic integer of degree d, we easily deduce that $\{xy\}_\beta = A_0 + A_1\beta + \cdots + A_{d-1}\beta^{d-1}$ with $A_i \in \mathbb{A}_p$.

Hence for all $j \in \{2, \dots, d\}$, we have

$$\{xy\}_\beta^{(j)} = A_0 + A_1(\beta^{(j)}) + \cdots + A_{d-1}(\beta^{(j)})^{d-1}.$$

Thus,

$$\begin{pmatrix} \{xy\}_\beta \\ \{xy\}_\beta^{(2)} \\ \vdots \\ \vdots \\ \{xy\}_\beta^{(d)} \end{pmatrix} = M \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_{d-1} \end{pmatrix}, \text{ where } M = \begin{pmatrix} 1 & \beta & \dots & \dots & \beta^{d-1} \\ 1 & \beta^{(2)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (\beta^{(d-1)})^{d-1} \\ 1 & \beta^{(d)} & \dots & \dots & (\beta^{(d)})^{d-1} \end{pmatrix}.$$

We have $\det M = \prod_{i<j} (\beta^{(i)} - \beta^{(j)}) \neq 0$ which implies that M is invertible, therefore it transforms all bounded vector in a bounded vector. From the two cases and since $|\{xy\}_\beta|_p < 1$, we get $\begin{pmatrix} \{xy\}_\beta \\ \{xy\}_\beta^{(2)} \\ \vdots \\ \{xy\}_\beta^{(d)} \end{pmatrix}$ is bounded, so $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ is also bounded in $\mathbb{Q}_p$ and moreover belongs to $\mathbb{A}_p^d$.

Furthermore, β has d archimedean conjugates $\beta^{(j)}$, with $d+1 \leq j \leq 2d$.

Recall that β has not a conjugate of modulus 1 because if one of the conjugates has modulus 1, then by the symmetry of Galois conjugates, its inverse (with the same modulus) must also be a conjugate.

However, if β has an odd degree, this symmetry would imply that at least one root (either β or one of its conjugates) would not be paired symmetrically, leading to difficulties in maintaining the modulus 1 condition. For this, we must distinguish these two cases:

Case 1: If $|\beta^{(j)}|_\infty < 1$.

In this situation, we have

$$|\{xy\}_{\beta^{(j)}}|_\infty = |(a_s(\beta^{(j)})^s + \cdots + a_0)(b_k(\beta^{(j)})^k + \cdots + b_0) - c_n(\beta^{(j)})^n - \cdots - c_0|_\infty.$$

Which implies that

$$|\{xy\}_{\beta^{(j)}}|_\infty \leq \frac{1}{(1-|\beta^{(j)}|_\infty)^2} + \frac{1}{1-|\beta^{(j)}|_\infty}.$$

Case 2: If $|\beta^{(j)}|_\infty > 1$.

In this situation, we have

$$\begin{aligned} |\{xy\}_\beta^{(j)}|_\infty &= |xy - [xy]_{\beta^{(j)}}|_\infty. \\ &= |\frac{c_{-1}}{\beta^{(j)}} + \frac{c_{-2}}{(\beta^{(j)})^2} + \cdots + \frac{c_{-m}}{(\beta^{(j)})^m}|_\infty. \end{aligned}$$

Consequently, we get

$$|\{xy\}_\beta^{(j)}|_\infty \leq \frac{1}{1-\frac{1}{|\beta^{(j)}|_\infty}}.$$

As $|\{xy\}_\beta^{(j)}|_\infty < 1$ and through the two cases, we deduce that the infinity norm of the vector $\begin{pmatrix} \{xy\}_\beta^{(d+1)} \\ \{xy\}_\beta^{(d+2)} \\ \vdots \\ \{xy\}_\beta^{(2d)} \end{pmatrix}$ is less than the quantity

$$\max_{d+1\leq j\leq 2d}\left(\max\left(\frac{1}{(1-|\beta^{(j)}|_\infty)^2} + \frac{1}{1-|\beta^{(j)}|_\infty}, \left|\beta^{(j)}|_\infty < 1\right), \max\left(\frac{1}{1-\frac{1}{|\beta^{(j)}|_\infty}}, \left|\beta^{(j)}|_\infty > 1\right)\right)\right).$$

Recall that $\begin{pmatrix} \{xy\}_\beta^{(d+1)} \\ \{xy\}_\beta^{(d+2)} \\ \vdots \\ \vdots \\ \{xy\}_\beta^{(2d)} \end{pmatrix}$=M' $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_{d-1} \end{pmatrix}$, where

$$M' = \begin{pmatrix} 1 & \beta^{(d+1)} & \ldots & \ldots & (\beta^{(d+1)})^{d-1} \\ 1 & \beta^{(d+2)} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & (\beta^{(2d-1)})^{d-1} \\ 1 & \beta^{(2d)} & \ldots & \ldots & (\beta^{(2d)})^{d-1} \end{pmatrix}.$$

Since M' is invertible, so the vector $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ is also bounded.

Finally, by Lemma 3.2, we conclude that $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ takes a finite values. Thus we obtain that L_{e} is finite.

We proceed in the same manner as in the case of finiteness of L_{e} and we prove that the p-adic norm of $\begin{pmatrix} \{x+y\}_\beta \\ \{x+y\}_\beta^{(2)} \\ \vdots \\ \{x+y\}_\beta^{(d)} \end{pmatrix}$ is less than $|\beta|_p$.

and the infinity norm of $\begin{pmatrix} \{x+y\}_\beta^{(d+1)} \\ \{x+y\}_\beta^{(d+2)} \\ \vdots \\ \{x+y\}_\beta^{(2d)} \end{pmatrix}$ is less than the quantity

$$\max_{d+1\le j\le 2d}\left(\max\left(\frac{3}{1-|\beta^{(j)}|_\infty},\left|\beta^{(j)}|_\infty<1\right.\right),\max\left(\frac{1}{1-\frac{1}{|\beta^{(j)}|_\infty}},\left|\beta^{(j)}|_\infty>1\right.\right)\right).$$

Since M and M' are invertible and through Lemma 3.2, we conclude that $\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{d-1} \end{pmatrix}$ takes a finite values. Therefore, we obtain that $L_{\oplus}$ is finite.

Now, we suppose that $\frac{1}{\beta}$ is an algebraic integer of degree d whose minimal polynomial is $P(X)=b_0+\cdots+b_{d-1}X^{d-1}+X^d$, where $b_i\in\mathbb{A}_p$ for $i\in\{0,\dots,d-1\}$.

we can easily check that $\{xy\}_{\frac{1}{\beta}}=B_0+B_1\frac{1}{\beta}+\cdots+B_{d-1}(\frac{1}{\beta})^{d-1}$ with $B_i\in\mathbb{A}_p$.

Hence, for all $j\in\{2,\dots,d,d+1,\dots,2d\}$, we get

$$\{xy\}_{\frac{1}{\beta}}^{(j)}=B_0+B_1(\tfrac{1}{\beta})^{(j)}+\cdots+B_{d-1}((\tfrac{1}{\beta})^{(j)})^{d-1}.$$

Moreover, we have

$$\{x+y\}_{\frac{1}{\beta}}=B_0+B_1\frac{1}{\beta}+\cdots+B_{d-1}(\tfrac{1}{\beta})^{d-1}\ \text{with}\ B_i\in\mathbb{A}_p.$$

Thereby, for all $j\in\{2,\dots,d,d+1,\dots,2d\}$, we get

$$\{x+y\}_{\frac{1}{\beta}}^{(j)}=B_0+B_1(\tfrac{1}{\beta})^{(j)}+\cdots+B_{d-1}((\tfrac{1}{\beta})^{(j)})^{d-1}.$$

We continue as in the previous way, we obtain that quantities $L_{\oplus}$ and L_{e} are both finite.

Corollary 3.4: *Let $\beta \in \mathbb{Q}_p$ such that $|\beta|_p > 1$. If β is a PC number of odd degree, so $L_{\oplus}$ and $L_{\odot}$ are finite.*

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