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On the extreme points of the sets

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Abstract

In this paper, extreme points of the convex sets are considered. Some properties of the extreme points are stated and proved.

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1. Introduction

Definition 1: Let $X \subseteq \mathbf{R}^n$ be a convex closed set. The point $x^0 \in X$ is said to be the *extreme point* of the set X if there do not exist $x_1,\ x_2 \in X,\ x_1 \neq x_2$, such that

$$x^0 = \lambda x_1 + (1-\lambda)x_2, \text{ where } \lambda \in (0,1),$$

that is, if there does not exist a line segment with endpoints of the set X, which contains x^0 as an interior point.

An equivalent definition of extreme point is as follows.

Definition 2: Let $X \subseteq \mathbf{R}^n$ be a convex closed set. The point $x^0 \in X$ is said to be the *extreme point* of the set X if $X \backslash \{x^0\}$ is a convex set.

The totality of all extreme points of a set X is denoted by one of the following ways:

$$\hat{X} = \text{ext}\, X = \text{extr}\, X.$$

Example 1:

1.1. Let X be a closed line segment, whose endpoints are x_1 and x_2, that is, $X = [x_1, x_2]$. Then $\hat{X} = \{x_1, x_2\}$.

1.2. Let X be a triangle with vertices $x_1,\ x_2,\ x_3$. Then $\hat{X} = \{x_1, x_2, x_3\}$.

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1.3. Let X be a polygon or polyhedron with vertices x_1, x_2, ..., x_n. Then $\hat{X} = \{\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n\}$.

The topics, considered in this paper, and related to them, are studied in reference titles [1]–[19], etc.

The rest of the paper is organized as follows. In Section 2, some basic properties of the extreme points are stated and proved. In Section 3, an open problem for research is formulated.

2. Extreme points

In this section, some properties of extreme points of the convex sets are stated and proved.

Theorem 1: *Let X be a closed convex set in $\mathbf{R}^n$, H be a supporting hyperplane for X, and $X_1 \stackrel{\text{def}}{=} X \cap H$. Then $\hat{X}_1 \subseteq \hat{X}$.*

Proof: Let $\hat{\mathrm{x}} \in \hat{X}_1$, $\langle \mathrm{a}, \mathrm{x} \rangle = \alpha$ be the equation defining the supporting hyperplane H and let $\langle \mathrm{a}, \mathrm{x} \rangle \leq \alpha$ for $\mathrm{x} \in X$. Assume that $\hat{x}$ is not an extreme point for the set X. Then according to Definition 1 there exist two distinct points x' and x'' in X, so that $\hat{\mathrm{x}} = \lambda \mathrm{x}' + (1-\lambda)\mathrm{x}''$ with some constant $\lambda \in (0,1)$. However, from

$$\alpha = \langle a, \hat{\mathrm{x}} \rangle = \langle \mathrm{a}, \lambda \mathrm{x}' + (1-\lambda)\mathrm{x}'' \rangle = \lambda \langle \mathrm{a}, \mathrm{x}' \rangle + (1-\lambda)\langle \mathrm{a}, \mathrm{x}'' \rangle \leq \lambda\alpha + (1-\lambda)\alpha \equiv \alpha$$

it follows that $\lambda \langle \mathrm{a}, \mathrm{x}' \rangle = \alpha$ and $\langle \mathrm{a}, \mathrm{x}'' \rangle = \alpha$, i.e., x', $\mathrm{x}'' \in H$ or x', $\mathrm{x}'' \in X_1$, which is impossible because $\hat{\mathrm{x}}$ by assumption is an extreme point for X_1. Therefore the assumption was wrong, hence $\hat{\mathrm{x}}$ is an extreme point for X. Since $\hat{\mathrm{x}} \in \hat{X}_1$ was arbitrarily chosen, then $\hat{X}_1 \subseteq \hat{X}$.

Theorem 2: *A nonempty closed convex set X in $\mathbf{R}^n$ has an extreme point if and only if X does not contain a straight line.*

Proof: *Necessity.* Assume that $\hat{\mathrm{x}} \in X$ is an extreme point for X, that is, let $\hat{\mathrm{x}} \in \hat{X}$. If we assume that X contains the straight line $\{\mathrm{x}_0 + t\mathrm{b} : t \in \mathbf{R}\}$, then X would also contain the straight line $\{\hat{\mathrm{x}} + t\mathrm{b} : t \in \mathbf{R}\}$ according to a known theorem, which is a contradiction with the assumption that $\hat{\mathrm{x}} \in \hat{X}$ because $\hat{\mathrm{x}}$ can be expressed as follows

$$\hat{\mathrm{x}} = \frac{1}{2}(\mathrm{x}' + \mathrm{x}''), \mathrm{x}' = \hat{\mathrm{x}} + \mathrm{b} \in X, \mathrm{x}'' = \hat{\mathrm{x}} - \mathrm{b} \in X.$$

Sufficiency. Let the closed convex set $X \neq \emptyset$ do not contain a straight line. The statement will be proved by induction on the dimension of the set X.

The one-dimensional convex sets, which do not contain straight lines, are the line segment and the ray, and both sets have extreme points, i.e., the statement of Theorem 2 holds true for the one-dimensional sets.

Induction hypothesis. Let statement be true for closed convex sets with dimension, less than k.

We will prove that it is also true for the closed convex sets with dimension k.

Let $\dim X = k$. Through an arbitrary relative boundary point x of X (such a point exists under the assumptions of Theorem 2) construct a supporting hyperplane H which does not contain entirely the set X (Theorem 7, [19]). The set $X_1 \stackrel{\text{def}}{=} X \cap H$ is closed and convex, it does not contain a straight line, and $\dim X_1 < k$. By the induction hypothesis, X_1 has an extreme point, that is, $\hat{X}_1 \neq \emptyset$. This conclusion and Theorem 1 imply that $\hat{X} \neq \emptyset$.

Then the Mathematical induction principle implies that the statement is true for each positive integer k.

Corollary 1: *Each nonempty convex compact set has at least one extreme point.*

Proof: Since by assumption the set is compact, in particular, it is bounded, therefore it does not contain a straight line. Then sufficiency of Theorem 2 implies the statement of Corollary 1.

Corollary 2: *If $X \subseteq \mathbf{R}^n$ is a closed convex set, $\hat{X} \neq \emptyset$, H is a supporting hyperplane for X, and $Y \stackrel{\text{def}}{=} X \cap H$, then $\hat{Y} \neq \emptyset$.*

Proof: The assumption that $\hat{X} \neq \emptyset$ and the necessity of Theorem 2 imply that X does not contain a straight line, therefore the set $Y \stackrel{\text{def}}{=} X \cap H \neq \emptyset$ also does not contain a straight line. Then sufficiency of Theorem 2 implies that $\hat{Y} \neq \emptyset$.

Theorem 3*: Each relative interior point of the convex compact set X lies on a line segment whose endpoints are relative boundary points of X, that is, each relative interior point of the convex compact set can be presented as convex combination of two relative boundary points.*

Proof: Let $\mathrm{x} \in \operatorname{ri} X$, where $\operatorname{ri} X$ denotes the relative interior of X (see, e.g., [19]). Through the point x construct the straight line $p: \{\mathrm{x} + t\mathrm{b}, t \in \mathbf{R}\}$ such that $p \subset \operatorname{aff} X$.

Let

$$t_1 = \max\{t: \mathrm{x} + t\mathrm{b} \in X\}, \quad t_2 = \max\{t: \mathrm{x} - t\mathrm{b} \in X\}.$$

By the assumption, X is a compact set, therefore the above maxima are attained and they are finite, that is, $0 \leq t_1, t_2 < \infty$.

Define the points

$$\mathrm{x}_1 = \mathrm{x} + t_1\mathrm{b} \in X, \quad \mathrm{x}_2 = \mathrm{x} - t_2\mathrm{b} \in X.$$

From these definitions we have $\mathrm{x}_1, \mathrm{x}_2 \in r\,\partial X$.

By assumption, the set X is compact, hence it is closed.

Present the point x as follows

$$\mathrm{x} = \frac{t_2}{t_1 + t_2}\mathrm{x}_1 + \frac{t_1}{t_1 + t_2}\mathrm{x}_2.$$

This presentation is correct because

$$\frac{t_2}{t_1+t_2}\mathrm{x}_1+\frac{t_1}{t_1+t_2}\mathrm{x}_2=\frac{t_2}{t_1+t_2}(\mathrm{x}+t_1\mathrm{b})+\frac{t_1}{t_1+t_2}(\mathrm{x}-t_2\mathrm{b})$$
$$=\frac{1}{t_1+t_2}(t_2\mathrm{x}+t_1t_2\mathrm{b}+t_1\mathrm{x}-t_1t_2\mathrm{b})=\frac{1}{t_1+t_2}(t_1+t_2)\mathrm{x}\equiv\mathrm{x}.$$

Moreover

$$\frac{t_2}{t_1+t_2}\geq 0,\frac{t_1}{t_1+t_2}\geq 0,\frac{t_2}{t_1+t_2}+\frac{t_1}{t_1+t_2}=1,$$

that is, x is presented as convex combination of two relative boundary points x_1 and x_2.

The next Theorem 4 is called the Minkowski-Krein-Milman theorem.

Theorem 4: *Let X be a nonempty convex compact set in $\mathbf{R}^n$. Then $X=co\,\hat{X}$.*

Proof: The statement of Theorem 4 will be proved by two-way inclusion of the sets X and $co\,\hat{X}$.

I. From definition of a convex set, definition of a convex hull, and a known theorem it follows that $co\,\hat{X}\subseteq X$.

II. The converse inclusion $X\subseteq co\,\hat{X}$ will be proved by induction on $dim\,X$.

When $\dim X=1$, the statement is true because each point of a line segment, being a convex set by definition, is presented as convex combination of the endpoints (that is, of the extreme points) of this line segment, that is, $X\subseteq co\,\hat{X}$.

Induction hypothesis. Let the statement that $X\subseteq co\,\hat{X}$ be true for $dim\,X<k$.

Verify the statement for $dim\,X=k$. Let $\mathrm{x}\in X$. There are two possible cases for $\mathrm{x}\in X$:

Case 1: Let $\mathrm{x}\in r\,\partial X$.

Then a supporting hyperplane H passes through the relative boundary point x with $X\not\subseteq H$ and $\mathrm{x}\in H$ according to Theorem 7 ([19]).

Consider the set $X_1\overset{\mathrm{def}}{=}X\cap H$ which is a convex compact set and its dimension is $dim\,X_1<k$.

According to the induction hypothesis we have

$$\mathrm{x}=\sum_{i=1}^{s}\alpha_i\hat{\mathrm{x}}_i,\quad \hat{\mathrm{x}}_i\in\hat{X}_1,\quad \alpha_i\geq 0,i=1,\dots,s,\sum_{i=1}^{s}\alpha_i=1. \qquad (1)$$

However, $\hat{X}_1\subseteq\hat{X}$ in accordance with Theorem 1, that is, $\hat{\mathrm{x}}_i\in\hat{X}$ for each $i=1,\dots,s$.

This conclusion and (1) imply that $\mathrm{x}\in co\,\hat{X}$, that is, $\mathrm{x}\in r\,\partial X$ implies $\mathrm{x}\in co\,\hat{X}$.

Case 2: Let $x \in ri\, X$.

From Theorem 3 it follows that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad x_1, x_2 \in r\,\partial X, \quad \lambda \in [0,1], \tag{2}$$

and from Case 1. we obtain that $x_1, x_2 \in co\,\hat{X}$. Therefore

$$x_1 = \sum_{i=1}^{s} \alpha_i x'_i, x_2 = \sum_{i=1}^{s} \beta_i x''_i,$$

where $x'_i, x''_i \in \hat{X}$ for all $i = 1, \dots, s$, and

$$\alpha_i \geq 0, i = 1, \dots, s, \quad \sum_{i=1}^{s} \alpha_i = 1, \quad \beta_i \geq 0, i = 1, \dots, s, \quad \sum_{i=1}^{s} \beta_i = 1.$$

After substituting the expressions of x_1 and x_2 in (2), we get

$$x = \lambda x_1 + (1-\lambda)x_2 = \sum_{i=1}^{s} \lambda\alpha_i x'_i + \sum_{i=1}^{s} (1-\lambda)\beta_i x''_i. \tag{3}$$

About coefficients of expression (3) we have

$$\lambda\alpha_i \geq 0, (1-\lambda)\beta_i \geq 0, i = 1, \dots, s,$$

$$\sum_{i=1}^{s} \lambda\alpha_i + \sum_{i=1}^{s} (1-\lambda)\beta_i = \lambda \sum_{i=1}^{s} \alpha_i + (1-\lambda) \sum_{i=1}^{s} \beta_i = \lambda.1 + (1-\lambda).1 = 1,$$

that is, $x \in ri\, X$ is presented in (3) as a convex combination of $x'_i, x''_i \in \hat{X}$, $i = 1, \dots, s$. Therefore $x \in co\,\hat{X}$.

Case 1. and Case 2. include all possible cases because $r\,\partial X \cup ri\, X = \overline{X}$.

We have proved that $x \in X$ for an arbitrary x implies $x \in co\,\hat{X}$. Therefore $X \subseteq co\,\hat{X}$.

From both opposite inclusions, obtained in I) and II), it follows that $X = co\,\hat{X}$.

Theorem 4 characterizes the convex compact sets, whose number of extreme points is finite.

For example, if X is a polygon or polyhedron with vertices x_1, ..., x_n, then $\hat{X} = \{x_1, \dots, x_n\}$, and $X = co\,\hat{X} = co\,\{x_1, \dots, x_n\}$ according to Theorem 4.

However, the statement of Theorem 4 is not true if some of its assumptions are not satisfied, for example, if the requirement that the set X be bounded (and, therefore, be compact) is not satisfied. In such cases, $co\,\hat{X}$ possibly characterizes only the compact part of the set X, if there is such a part.

3. Conclusion

In this paper, extreme points of the convex sets are considered. Some properties of the extreme points are stated and proved.

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