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On the affine cohomology class for filiform Lie algebra

Midoune Noureddine †
Department of Mathematics
Faculty of Mathematics and Computer
Laboratory of Pure and Applied Mathematics
M'sila University
M'sila
Algeria

Rakdi Mohamed Anouar *
Department of Mathematics
Higher Normal School of Kouba
Laboratory of Pure and Applied Mathematics
M'sila University
Algeria

and

Laboratory of Fixed Point Theorem and Applications
Higher Normal School of Kouba
Algiers
Algeria

Abstract

We investigate symplectic structures on 2-step nilpotent Lie algebras as well as affine structures on filiform Lie algebras, and we establish a necessary condition for the existence of such affine structures.

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† *E-mail:* **noureddine.midoune@univ-msila.dz**
* *E-mail:* **mohamedanouar.rakdi@g.ens-kouba.dz (Corresponding Author)**

1. Introduction

Let $\mathfrak{g}$ denote a nilpotent Lie algebra over a field $\mathbb{K}$. The descending sequence of ideals known as the lower central series, written $\{\mathfrak{g}^k\}$, is constructed by setting $\mathfrak{g}^0 = \mathfrak{g}$ and defining each subsequent term as $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ for all $k \geq 1$. The smallest integer p satisfying $\mathfrak{g}^p = 0$ while $\mathfrak{g}^{p-1} \neq 0$ is called the nilindex of $\mathfrak{g}$; in such a case $\mathfrak{g}$ is said to be p-step nilpotent. Two important subclasses are obtained for specific values of p: when $p = 2$ the algebra is called 2-step nilpotent, and when $p = n - 1$ (where $n = \dim\mathfrak{g}$) it is referred to as a filiform Lie algebra.

These two families—2-step and filiform algebras–are central to the study of geometric structures on Lie algebras, particularly symplectic and affine structures. Indeed, the search for symplectic forms on a Lie group can be reformulated in terms of its tangent Lie algebra: a symplectic form on $\mathfrak{g}$ induces a left-invariant symplectic structure on the corresponding simply connected Lie group G (see [2]). This connection motivates the investigation of which Lie algebras admit symplectic forms.

Although a symplectic Lie algebra must have even dimension, no general necessary or sufficient conditions for the existence of a symplectic structure are known. The class of 2-step nilpotent Lie algebras, owing to its relatively simple structure, provides an ideal setting in which to search for such criteria. In the present work we examine symplectic structures on 2-step nilpotent algebras and demonstrate that they are, in fact, symplectic only in exceptional cases.

Symplectic Lie algebras constitute a special case within the broader class of Lie algebras that admit affine structures. An affine structure on a Lie algebra corresponds infinitesimally to a left-invariant affine structure on the associated Lie group. Affine concepts appear in a wide range of mathematical disciplines, from differential geometry to optimization and cryptography. For example, affine mappings have been employed in combinatorial optimization and graph theory [3], and affine transformations are used in the design of cryptographic systems [5]. These diverse applications underscore the richness of affine ideas, which also play a fundamental role in the structure theory of Lie algebras.

The problem of deciding whether a given Lie algebra admits an affine structure remains open in general, despite considerable effort (see [1]). In this paper we focus on filiform Lie algebras and establish a link between the vanishing of a certain subspace $\Delta_2(\mathfrak{g})$ and the existence of a nontrivial affine cohomology class in $H^2(\mathfrak{g}, \mathbb{K})$. Specifically, we prove that if $\Delta_2(\mathfrak{g}) = \{0\}$, then such a class exists, thereby providing a sufficient condition for the existence of an affine structure on a filiform Lie algebra.

2. 2-step nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. Denote by $D(\mathfrak{g})$ its derived ideal, i.e., the vector subspace spanned by all brackets $[x, y]$ with $x, y \in \mathfrak{g}$. The center of $\mathfrak{g}$ is written $z(\mathfrak{g})$ and consists of all elements $x \in \mathfrak{g}$ satisfying $[x, \cdot] = 0$.

Given a bilinear alternating form ω on $\mathfrak{g}$, the set

$$\mathrm{Rad}(\omega) = \{x \in \mathfrak{g} | \omega(x, \cdot) = 0\}$$

is called the radical of ω. If $Rad(\omega) = \{0\}$, then ω is said to be nondegenerate.

Definition 1: A finite-dimensional Lie algebra $\mathfrak{g}$ is called **2-step nilpotent** (or **metabelian**) if its derived ideal is contained in its center; that is, $D(\mathfrak{g}) \subset z(\mathfrak{g})$. Equivalently, $[[x, y], z] = 0$ for every $x, y, z \in \mathfrak{g}$.

Examples 1:
1. Every abelian Lie algebra is trivially 2-step, since all brackets vanish.
2. The Heisenberg algebra H_k of dimension $2k + 1$ is a standard example. With respect to a basis $\{e_1, e_2, \dots, e_{2k+1}\}$ its non-zero brackets are

$$[e_1, e_2] = [e_3, e_4] = \cdots = [e_{2k-1}, e_{2k}] = e_{2k+1},$$

and all other brackets are zero. One readily checks that H_k is 2-step nilpotent.

Propositon 1: ([6]). Any 2-step nilpotent Lie algebra $\mathfrak{g}$ decomposes as a direct product

$$\mathfrak{g} \cong \mathfrak{g}_1 \times \mathfrak{g}_2,$$

where $\mathfrak{g}_1$ is abelian and $\mathfrak{g}_2$ is a 2-step nilpotent Lie algebra satisfying $D(\mathfrak{g}_2) = z(\mathfrak{g}_2)$.

3. Symplectic 2-step Lie algebras

This section provides an adaptation of results contained in [4].

Definition 2: Let $\mathfrak{g}$ be a Lie algebra over a field K. A bilinear alternating map $\omega: \mathfrak{g} \times \mathfrak{g} \to K$ is called a **2-cocycle** if it satisfies

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \qquad (2.1)$$

for every triple $x, y, z \in \mathfrak{g}$. The vector space of all 2-cocycles is denoted by $Z^2(\mathfrak{g}, K)$.

A 2-cocycle ω is said to be a **2-coboundary** when there exists a linear functional $f \in Hom(\mathfrak{g}, \mathbb{K})$ such that

$$\omega(x, y) = f([x, y]) \text{ for all } x, y \in \mathfrak{g}.$$

The subspace of 2-coboundaries is written $B^2(\mathfrak{g}, K)$.

The **second cohomology space** of $\mathfrak{g}$ (with coefficients in K) is the quotient

$$H^2(\mathfrak{g}, K) = Z^2(\mathfrak{g}, K) \,/\, B^2(\mathfrak{g}, K).$$

Its dimension, $b_2(\mathfrak{g}) = \dim H^2(\mathfrak{g}, K)$, is called the **second Betti number** of $\mathfrak{g}$.

Definition 3: A Lie algebra $\mathfrak{g}$ is called **symplectic** if there exists a nondegenerate 2-cocycle $\omega \in Z^2(\mathfrak{g}, \mathbb{K})$.

Example 1: Let $\mathfrak{g} = H_k \times \mathbb{K}^{2j+1}$ be the direct product of the Heisenberg algebra H_k (of dimension $2k + 1$) with an abelian algebra of odd dimension. Then $\mathfrak{g}$ is symplectic exactly when $k = 1$.

For $k = 1$ write $\mathfrak{g} = H_1 \times \mathbb{K}^{2j+1}$. Choose a basis $\{g_1, \dots, g_{2j+1}\}$ of $\mathbb{K}^{2j+1}$ and a basis $\{e_1, e_2, f\}$ of H_1 with $[e_1, e_2] = f$. Evaluating the cocycle condition for all triples $x, y, z \in \{e_1, e_2, f, g_1, \dots, g_{2j+1}\}$ gives $\omega(f, f) = 0$ and $\omega(f, g_i) = 0$ for every $i = 1, \dots, 2j + 1$. One may then construct the nondegenerate alternating form

$$\omega = e_1^* \wedge f^* + e_2^* \wedge g_1^* + \sum_{i=1}^{j} g_{2i}^* \wedge g_{2i+1}^*,$$

which shows that $\mathfrak{g}$ is indeed symplectic.

Now let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra and x a nonzero element of $\mathfrak{g}$. The centralizer $z(x) = \ker adx$ contains both $z(\mathfrak{g})$ and $D(\mathfrak{g})$; consequently it is an ideal of $\mathfrak{g}$. Denote by D_x the derived ideal of $z(x)$, i.e.

$$D_x = [\ker\ \mathrm{ad}\ x, \qquad \ker\ \mathrm{ad}\ x].$$

For an arbitrary nilpotent Lie algebra $\mathfrak{g}$ we introduce the following subspaces:

$$\Delta_1(\mathfrak{g}) = \cap_{x \in \mathfrak{g}} D_x, \Delta_2(\mathfrak{g}) = \cap_{\omega \in Z^2(\mathfrak{g}, \mathbb{K})} Rad\ \omega.$$

Clearly $\Delta_2(\mathfrak{g})$ is a vector subspace of $\mathfrak{g}$.

Propositon 2: We have the chain of inclusions $\Delta_1(\mathfrak{g}) \subset \Delta_2(\mathfrak{g}) \subset z(\mathfrak{g})$.

Proof: Note that $z(\mathfrak{g}) = \cap_{\omega \in B^2(\mathfrak{g}, \mathbb{K})} \mathrm{Rad}\ \omega$. If $x \notin z(\mathfrak{g})$, there exists $y \in \mathfrak{g}$ with $[x, y] \neq 0$ and a linear functional $f \in \mathfrak{g}^*$ such that $f([x, y]) \neq 0$; hence the coboundary d_f satisfies $d_f(x, y) \neq 0$, so $x \notin \mathrm{Rad}(d_f)$.

Now take $y \in \Delta_1(\mathfrak{g})$ and $z \in \mathfrak{g}$. Because $y \in D_x$, we can write $y = \sum_i [u_i, v_i]$ with $u_i, v_i \in \ker\ \mathrm{ad}\, x$. For any $\omega \in Z^2(\mathfrak{g}, \mathbb{K})$,

$$\omega(y, z) = \sum_i \omega([u_i, v_i], z) = -\sum_i (\omega(u_i, [v_i, x]) + \omega(v_i, [x, u_i])) = 0,$$

since $[v_i, x] = [x, u_i] = 0$. Hence $\omega(y, z) = 0$, showing that $\Delta_2(\mathfrak{g}) \supset \Delta_1(\mathfrak{g})$.

Corollary 1: *If $\Delta_1(\mathfrak{g}) \neq \{0\}$, then $\mathfrak{g}$ cannot be symplectic.*

Lemma 1: *Let H_k be the $(2k + 1)$-dimensional Heisenberg algebra. Then $\Delta_1(H_k) = z(H_k)$ for every $k \geq 2$.*

Proof: A straightforward computation gives $D_{e_i} = z(H_k)$ for all $i = 1, \dots, 2k + 1$, which immediately yields $\Delta_1(H_k) = z(H_k)$.

Consequently, for $k \geq 2$ the algebra $\mathfrak{g} = H_k \times \mathbb{K}^{2j+1}$ of Example 1 is never symplectic.

Lemma 2: *A necessary condition for $\mathfrak{g}$ to be symplectic is that*

$$D(\mathfrak{g}) \perp_\omega z(\mathfrak{g}) \text{ for every } \omega \in Z^2(\mathfrak{g}, \mathbb{K}).$$

Proof: Indeed, for $z \in z(\mathfrak{g})$ the cocycle relation gives

$$\omega([x,y],z) = -\omega([y,z],x) - \omega([z,x],y) = 0.$$

Thus $D(\mathfrak{g})$ is totally isotropic with respect to any cocycle, and we obtain the dimension estimate

$$\dim\ D(\mathfrak{g}) + \dim z(\mathfrak{g}) \leq \dim \mathfrak{g}.$$

Hence the following lemma.

Lemma 3: *If* $\mathfrak{g}$ *is a 2-step nilpotent Lie algebra (so* $\mathfrak{g} \supset z(\mathfrak{g}) \supset D(\mathfrak{g})$*) satisfying* $dimD(\mathfrak{g}) > \frac{1}{2}\dim\ \mathfrak{g}$*, then* $\mathfrak{g}$ *cannot be symplectic.*

We now illustrate with several examples of 2-step nilpotent Lie algebras whose derived ideal has small dimension.

Examples 2:

1. Let $\mathfrak{g} = H_1 \times H_1$, a 6-dimensional Lie algebra (the generic example of this type). It is symplectic. Take a basis $\{e_1, e_2, e_3, e_4, f_1, f_2\}$ with $[e_1, e_2] = f_1$ and $[e_3, e_4] = f_2$. One checks that $D_{e_i} = \mathbb{K}f_2$ for $i = 1,2$ and $D_{e_i} = \mathbb{K}f_1$ for $i = 3, 4$; hence $\Delta_1(\mathfrak{g}) = \{0\}$. A direct computation shows that
$$\omega(f_1, e_3) = \omega(f_1, e_4) = \omega(f_2, e_1) = \omega(f_2, e_2) = 0,$$
and the alternating form
$$\omega = e_1^* \wedge e_3^* + e_2^* \wedge f_1^* + e_4^* \wedge f_2^*$$
is nondegenerate.
2. Let $\mathfrak{g} = H_2 \times H_1$, an 8-dimensional algebra. With basis $\{e_1, e_2, e_3, e_4, e_5, e_6, f_1, f_2\}$ and brackets $[e_1, e_2] = [e_3, e_4] = f_1$, $[e_5, e_6] = f_2$, we have $D_{e_i} = D(\mathfrak{g})$ for $i \leq 4$ and $D_{e_i} = \mathbb{K}f_1$ for $i = 5,6$. Thus $\Delta_1(\mathfrak{g}) = Kf_1 \neq \{0\}$, so $\mathfrak{g}$ is not symplectic.
The same conclusion holds for $\mathfrak{g} = H_k \times H_1$ with $k \geq 3$. In that case $\dim\mathfrak{g} = 2k + 4$ and the brackets are
$$[e_1, e_2] = [e_3, e_4] = \cdots = [e_{2k-1}, e_{2k}] = f_1, [e_{2k+1}, e_{2k+2}] = f_2,$$
which again gives $\Delta_1(\mathfrak{g}) = Kf_1 \neq \{0\}$.

Propositon 3: The 2-step nilpotent Lie algebras $\mathfrak{g}$ defined by the brackets

$$\begin{gathered}[e_1, e_2] = f_1,\\ [e_3, e_4] = f_1 + f_2,\\ [e_5, e_6] = f_1 + \lambda_4 f_2,\\ \vdots\\ [e_{2n-3}, e_2] = f_1 + \lambda_n f_2,\\ = f_2,\end{gathered}$$

where $\{e_1, \dots, e_{2n}, f_1, f_2\}$ is a basis, $n \geq 3$, and $\lambda_4, \dots, \lambda_n \notin \{0,1\}$, are never symplectic.

Proof: For such an algebra one computes

$$z(e_{2i-1}) = (\bigoplus_{j\neq 2i} Ke_j) \oplus D(\mathfrak{g}), z(e_{2i}) = (\bigoplus_{j\neq 2i-1} Ke_j) \oplus D(\mathfrak{g}),$$

hence $D_{e_{2i}} = D_{e_{2i-1}} = D(\mathfrak{g})$ for every $i = 1, \dots, n$. Consequently $\Delta_1(\mathfrak{g}) = D(\mathfrak{g}) \neq \{0\}$, which proves the claim and also shows why the condition $n \geq 3$ is necessary.

We now outline a general construction that leads to the following question. Let $\mathcal{L}_{\mathbb{K}}$ be an $(n+r)$-dimensional 2-step nilpotent Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero, with basis $\{e_i, f_k\}$ $(1 \leq i \leq n,\ 1 \leq k \leq r)$ and brackets

$$[e_i, e_j] = \textstyle\sum_{k=1}^{r} c_{ij}^k f_k, \qquad [e_i, f_k] = [f_k, f_\ell] = 0.$$

The center $z(\mathcal{L}_{\mathbb{K}})$ has basis $\{f_1, \dots, f_r\}$ and dimension r. Assume now that $\binom{n-r-2}{2} > r$. In $\mathcal{L}_{\mathbb{K}}$ one has $D_{e_i} = [\ker ade_i, \ker ade_i] = z(\mathcal{L}_{\mathbb{K}}) = D(\mathcal{L}_{\mathbb{K}})$ for all $i = 1, \dots, n$.

Let $\omega \in Z^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K})$. Any element $f \in D(\mathcal{L}_{\mathbb{K}})$ can be expressed as $f = \sum_\alpha \lambda_{\alpha,i}[u_{\alpha,i}, v_{\alpha,i}]$ with $u_{\alpha,i}, v_{\alpha,i} \in \ker ade_i$. Then

$$\omega(e_i, f) = \textstyle\sum_\alpha \lambda_{\alpha,i}\, \omega(e_i, [u_{\alpha,i}, v_{\alpha,i}]) = 0,$$

because

$$\omega(e_i, [u_{\alpha,i}, v_{\alpha,i}]) = -\omega(u_{\alpha,i}, [v_{\alpha,i}, e_i]) - \omega(v_{\alpha,i}, [e_i, u_{\alpha,i}]) = 0.$$

Moreover, for $f_i \in z(\mathcal{L}_{\mathbb{K}})$,

$$\omega(f_i, f) = \omega([e_1, e'_{j+1}], f) = -\omega([e'_{j+1}, f], e_1) - \omega([f, e_1], e'_{j+1}) = 0.$$

Thus $\mathrm{Rad}\ \omega \supset D(\mathcal{L}_{\mathbb{K}})$. Conversely, any alternating bilinear form ω on $\mathcal{L}_{\mathbb{K}}$ whose radical contains $D(\mathcal{L}_{\mathbb{K}})$ is a 2-cocycle, since then $\omega([x,y],z) = \omega([y,z],x) = \omega([z,x],y) = 0$. We therefore obtain the following characterization.

Theorem 1: *An alternating bilinear form ω on $\mathcal{L}_{\mathbb{K}}$ belongs to $Z^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K})$ if and only if $Rad\ \omega \supset D(\mathcal{L}_{\mathbb{K}})$. Consequently,*

$$Z^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K}); (\wedge^2\ (\mathcal{L}_{\mathbb{K}}/D(\mathcal{L}_{\mathbb{K}})))^*.$$

Since $B^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K}); D(\mathcal{L}_{\mathbb{K}})^*$, we obtain a description of the cohomology group:

$$H^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K}); Z^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K})/B^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K}),\ \ b_2(\mathcal{L}_{\mathbb{K}}) = \dim H^2(\mathcal{L}_{\mathbb{K}}, \mathbb{K}) = \binom{n}{2} - r.$$

Corollary 2: *If $n \geq r + \sqrt{2r} + 2$, then the 2-step nilpotent Lie algebra $\mathcal{L}_{\mathbb{K}}$ cannot be symplectic.*

Example 2: All 8-dimensional 2-step nilpotent Lie algebras (with $n = 6,\ r = 2$) are non-symplectic. Representative examples are:

$$\begin{aligned}
\mathfrak{g}_1 &: [e_1,e_3]=[e_2,e_4]=f_1, [e_1,e_5]=[e_2,e_6]=f_2,\\
\mathfrak{g}_2 &: [e_1,e_4]=[e_2,e_5]=[e_3,e_6]=f_1, [e_1,e_2]=f_2,\\
\mathfrak{g}_3 &: [e_1,e_4]=[e_2,e_5]=[e_3,e_6]=f_1, [e_1,e_2]=[e_3,e_4]=f_2,\\
\mathfrak{g}_4 &: [e_1,e_2]=[e_3,e_4]=f_1, [e_1,e_3]=[e_5,e_6]=f_2,\\
\mathfrak{g}_5 &: [e_1,e_2]=f_1, [e_3,e_4]=[e_5,e_6]=f_2,\\
\mathfrak{g}_6 &: [e_1,e_2]=f_1+f_2, [e_5,e_6]=f_1, [e_3,e_4]=f_2,
\end{aligned}$$

where $\{e_1,\dots,e_6,f_1,f_2\}$ is a basis of each $\mathfrak{g}_i$ $(i=1,\dots,6)$.

4. Filiform nilpotent Lie algebras

Definition 4: *A nilpotent Lie algebra $\mathfrak{g}$ of dimension n is called **filiform** if its lower central series satisfies*

$$\dim \mathfrak{g}^k = n-k-1 \quad \text{for } 1\le k\le n-1.$$

An immediate corollary is that the center $z(\mathfrak{g})$ of a filiform Lie algebra satisfies $\dim z(\mathfrak{g})=1$.

Examples 3: Two standard constructions of filiform Lie algebras are:

(i) **The Lie algebra L_n:** For $n\ge 3$, take a basis $\{e_1,\dots,e_n\}$ and set $[e_1,e_i]=e_{i+1}$ for $i=2,\dots,n-1$; all other brackets are zero. This is the simplest filiform Lie algebra of dimension n.

(ii) **The Lie algebra Q_n (odd dimension):** Let $n=2k+1$. On the basis $\{e_0,\dots,e_n\}$ define $[e_0,e_i]=e_{i+1}$ ($1\le i\le n-1$) and $[e_i,e_{n-i}]=(-1)^i e_n$ ($1\le i\le k$). The algebra Q_n is $(n+1)$-dimensional and filiform. Under the basis change $f_0=e_0+e_1$, $f_i=e_i$, the brackets take the form $[f_0,f_i]=f_{i+1}$ $(1\le i\le n-2)$ and $[f_i,f_{n-i}]=(-1)^i f_n$.

Every $(n+1)$-dimensional filiform Lie algebra over a field of characteristic zero arises as a linear deformation of L_n.

The collection of n-dimensional filiform Lie algebras over a field $\mathbb{K}$ ($\operatorname{char}\mathbb{K}=0$) is written $\mathfrak{F}_n(\mathbb{K})$. A natural generalization of the symplectic condition is the existence of an affine structure on the Lie algebra.

5. Affine cohomology classes

Let $\mathfrak{g}\in\mathfrak{F}_n(\mathbb{K})$ be an n-dimensional filiform Lie algebra over a field $\mathbb{K}$ of characteristic zero. We recall several definitions and facts concerning filiform Lie algebras from [1]; our main goal is to prove that an affine cohomology class exists in $H^2(\mathfrak{g},\mathbb{K})$ whenever $\Delta_2(\mathfrak{g})=\{0\}$.

Definition 5: For $\mathfrak{g}\in\mathfrak{F}_n(\mathbb{K})$, a 2-cocycle $\omega\in Z^2(\mathfrak{g},\mathbb{K})$ is called **affine** if

$$\omega(x,z)\neq 0 \ \text{ for some } x\in\mathfrak{g} \text{ and } z\in z(\mathfrak{g}).$$

Lemma 4: [1] *Let* $\mathrm{g} \in \mathfrak{F}_n(\mathbb{K})$ *and let* $\omega \in Z^2(\mathrm{g}, \mathbb{K})$ *be an affine* 2*-cocycle. Then its cohomology class* $[\omega] \in H^2(\mathrm{g}, \mathbb{K})$ *is affine and nonzero.*

Propositon 4: [1] Let $\mathrm{g} \in \mathfrak{F}_n(\mathbb{K})$ and suppose there exists an affine cohomology class $[\omega] \in H^2(\mathrm{g}, \mathbb{K})$. Then g admits an affine structure.

Lemma 5: [1] *An alternating bilinear form* ω *is an affine* 2*-cocycle if and only if either* $\omega(e_1, e_n)$ *or* $\omega(e_2, e_n)$ *is nonzero.*

Examples 4:

1. Consider L_6, the 6-dimensional algebra with basis $\{e_1, \ldots, e_6\}$ and brackets $[e_1, e_i] = e_{i+1}$ for $i = 1, \ldots, 5$. For any 2-cocycle ω, we have $\omega(e_1, e_6) \neq 0$ while the center $z(L_6) = \langle e_6 \rangle$; consequently, L_6 admits an affine structure.
2. All symplectic Lie algebras are affine.
3. Every 2-step nilpotent Lie algebra admits an affine structure.
4. Any nilpotent Lie algebra of dimension $n \leq 6$ admits an affine structure.

Propositon 5: If $\Delta_2(\mathrm{g}) = \{0\}$, then g admits an affine structure.

Proof: Now let $\mathrm{g} \in \mathfrak{F}_n(\mathbb{K})$. There always exists an element $x \in \mathrm{g}$ such that $\dim\ker adx = 2$; for such an element we have

$$\ker ad\, x = \mathbb{K}x \oplus z(\mathrm{g}) \quad \text{(abelian)}, \quad D_x = \{0\},$$

and consequently

$$\Delta_1(\mathrm{g}) = \cap_{x \in \mathrm{g}} D_x = \{0\}.$$

Since $\Delta_1(\mathrm{g}) = \{0\} \subset \Delta_2(\mathrm{g}) \subset z(\mathrm{g})$ and $\dim z(\mathrm{g}) = 1$, only two possibilities can occur:

- $\Delta_2(\mathrm{g}) = z(\mathrm{g})$; then every $\omega \in Z^2(\mathrm{g}, \mathbb{K})$ satisfies $\omega(\cdot, z) = 0$ for $z \in z(\mathrm{g})$.
- $\Delta_2(\mathrm{g}) = \{0\}$; then there exists $\omega \in Z^2(\mathrm{g}, \mathbb{K})$ with $\omega(x, z) \neq 0$ for some $x \in \mathrm{g}$.

In the second case ($\Delta_2(\mathrm{g}) = \{0\}$) such an ω is an affine 2-cocycle, and its cohomology class $[\omega] \in H^2(\mathrm{g}, \mathbb{K})$ is affine and nonzero. Thus we obtain a sufficient condition for a filiform Lie algebra to carry an affine structure.

Examples 5:

(1) Let g be the 6-dimensional filiform Lie algebra with basis $\{e_1, \ldots, e_6\}$ and brackets

$$[e_1, e_i] = e_{i+1}\ (i = 1, \ldots, 5), [e_2, e_3] = \beta e_5\ (\beta \in \mathbb{K}).$$

It is easy to verify that $\Delta_2(\mathrm{g}) = \{0\}$: one can exhibit a 2-cocycle ω with $\omega(e_1, e_6) \neq 0$. Hence g possesses an affine cohomology class $[\omega]$ and therefore an affine structure.

(2) Let $\mathfrak{g} \in \mathfrak{F}_6(\mathbb{K})$ be defined by

$$[e_1, e_i] = e_{i+1}\ (i = 1, \dots, 5), \qquad [e_2, e_3] = \beta e_5\ (\beta \in \mathbb{K}),$$
$$[e_3, e_4] = \alpha e_6\ (\alpha \neq 0).$$

For every $\omega \in Z^2(\mathfrak{g}, \mathbb{K})$ we have $\omega(\,\cdot\,, e_6) = 0$, so $\Delta_2(\mathfrak{g}) = z(\mathfrak{g}) = \langle e_6 \rangle$. Consequently no affine cohomology class exists in $H^2(\mathfrak{g}, \mathbb{K})$.

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