

On the (2+1)-dimensional time-fractional Sakovich equation : Lie symmetry analysis, exact solutions, and conservation laws

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Abstract

Herein, we extend the framework of Lie symmetry group theory and conservation laws to analyze the $(2+1)$ -dimensional time-fractional Sakovich equation $((2+1)D$ -TFSE). The distinctive feature of this equation, setting it apart from conventional time-fractional partial differential equations (TFPDEs), lies in its unique combination of the Riemann-Liouville time-fractional derivative and the integer-order derivative with respect to x . Therefore, we first transform the original equation into the form of a system of equations containing only fractional order derivatives in time, and then apply Lie symmetry methods to this system of equations. After finding these symmetries, the system of equations is transformed into a lower dimensional form and its power series solution is obtained by means of Erdélyi-Kober (E-K) fractional order differential operators. Finally, a new Noether's theorem is used to establish a conservation law for the system of equations.

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1. Introduction

The investigation of fractional-order differential equations (FPDEs) holds paramount significance due to their superior descriptive capabilities compared to traditional integer-order counterparts. These equations provide a more precise portrayal of complex dynamic behaviors encountered in natural and artificial systems, capturing essential nonlocality and long-term memory effects [5, 14, 19, 20]. Numerous researchers have delved into the theoretical underpinnings and practical implications of FPDEs, leading to significant advancements. Benmehidi and Dahmani explored sequential problems with Riemann-Liouville (R-L) integral conditions, paving the way for more accurate modeling of systems with historical dependencies [1]. Bezziou and Dahmani further enriched the theoretical framework by establishing random inequalities via fractional integration, which broadens the application scope of these equations in stochastic scenarios [2]. Additionally, Houas investigated the solvability and stability of fractional differential equations involving dual R-L fractional orders, enhancing our understanding of their adaptability in representing diverse dynamic behaviors [9]. These studies collectively drive the continuous evolution of FPDE research, bridging the gap between theory and real-world applications. In contrast to integer-order equations, fractional-order formulations offer enhanced flexibility and richness in dynamic behaviors, accommodating nonlinear, oscillatory, and chaotic phenomena. Furthermore, their wide-ranging applications in signal processing, image enhancement, control systems, and optimization underscore their significance in providing efficient algorithms and strategies, particularly in scenarios necessitating consideration of non-local effects and long-term dependencies [1, 2, 5, 9, 14, 15, 19–21, 23]. The study of FPDEs not only deepens our understanding of complex systems but also drives innovation in scientific and engineering applications, enhancing predictive accuracy and analytical frameworks. As FPDEs emerge as powerful tools for modeling intricate

phenomena, researchers are extending methodologies from integer-order PDEs to explore their fractional counterparts. Prominent techniques include the Adomian decomposition approach [4], numerical discretization via finite differences [16], perturbative solution based on homotopy [17], sub-equation-based strategy [27], variational iteration procedure [18], symmetry reduction through Lie group analysis [6], and invariant subspace-based solution construction [7]. Among these, symmetry group analysis proves especially versatile, effectively handling both integer and fractional derivative equations.

The notion of investigating the invariance properties of fractional diffusion equations under Lie group transformations was first proposed by Buckwar and Luchko in their work [3]. Building upon this, Gazizov et al. [6, 8] extended this concept to address fractional differential equations. Consequently, numerous nonlinear models of fractional order, grounded in physical principles, have been investigated. However, the majority of FPDEs examined using Lie symmetry technique predominantly involve purely time-fractional, purely space-fractional, or both forms [8, 10, 13, 24-26, 28]. Conversely, there exists a scarcity of Lie symmetry findings pertaining to PDEs involving hybrid fractional-integer order derivatives.

In reference [22], the authors investigated the three-dimensional second-order nonlinear wave equation as follows:

$$u_{xt} + u_{yy} + 2uu_{xy} + 6u^2u_{xx} + 2u_{xx}^2 = 0. \quad (1.1)$$

This equation has recently emerged as an intriguing special case in the Painlevé analysis of a class of second-order equations quadratic in u_{xx} . The authors obtained multi-soliton solutions reminiscent of the Korteweg-de Vries (KdV) type.

This paper concentrates on the exploration of Lie symmetries and conservation laws in the (2+1)D-TFSE

$$D_t^\alpha u_x + u_{yy} + 2uu_{xy} + 6u^2u_{xx} + 2u_{xx}^2 = 0, \quad (1.2)$$

with $0 < \alpha < 1$. For this type of equations, we use a different approach than that in the literature [29]. The core principle of our approach involves transforming FPDE (1.2) into an equivalent system through strategic variable substitution:

$$\begin{cases} D_t^\alpha v + u_{yy} + 2uv_y + 6u^2v_x + 2v_x^2 = 0, \\ v = u_x. \end{cases} \quad (1.3)$$

This is followed by the application of Lie symmetry analysis (LSA) to the system. As commonly acknowledged, fractional derivatives encompass various definitions, including Riemann-Liouville, Caputo, Weyl types, among others. In this study, we adopt the R-L fractional derivative formulation specified by

$$\begin{aligned} {}_b D_t^r f(t, x) &= D_t^m {}_b I_t^{m-r} f(t, x) = \\ &\begin{cases} \frac{1}{\Gamma(m-r)} \frac{\partial^m}{\partial t^m} \int_b^t \frac{f(s, x)}{(t-s)^{r-m+1}} ds, & m-1 < r < m, m \in \mathbf{N} \\ D_t^m f(t, x), & r = m \in \mathbf{N} \end{cases} \end{aligned}$$

with $t > b$. Throughout this paper, we denote the operator ${}_0D_t^\alpha$ as D_t^α . Additionally, we use $D_t^{-\alpha} = I_t^\alpha$ to represent the R-L fractional integral.

This study systematically derives the complete Lie symmetry algebra of Eqs. (1.3) within the Lie symmetry analysis framework. The symmetries are then exploited via invariant transformations to reduce the equations to either (1+1)D fractional-in-time PDEs or ODEs. Moreover, closed-form solutions of the simplified system are constructed, and establish symmetry-induced conservation laws via a recently proposed theorem combined with fractional Noether operators.

The layout of the present work is organized sequentially: Section 2 conducts a comprehensive symmetry analysis of Eqs. (1.3) to identify their invariance properties. Section 3 derives reduced forms and characterizes exact solutions invariant under these symmetries. Section 4 establishes conservation vector fields for each symmetry, linking local invariance to global conservation. The final section synthesizes these results.

2. Systematic Lie symmetry analysis of the model (1.3)

Given that the (2+1)D-TFSE (1.3) exhibits symmetry under the continuous transformations of an ε -parameterized Lie group, whose elements are local point transformations

$$\begin{aligned}\bar{t} &= t + \varepsilon \xi_1(t, x, y, u, v) + o(\varepsilon), \\ \bar{x} &= x + \varepsilon \xi_2(t, x, y, u, v) + o(\varepsilon), \\ \bar{y} &= y + \varepsilon \xi_3(t, x, y, u, v) + o(\varepsilon), \\ \bar{u} &= u + \varepsilon \eta_1(t, x, y, u, v) + o(\varepsilon), \\ \bar{v} &= v + \varepsilon \eta_2(t, x, y, u, v) + o(\varepsilon),\end{aligned}\tag{2.1}$$

where ξ_1, ξ_2, ξ_3 represent the symmetry components in time and space, while η_1, η_2 correspond to the field-dependent generators.

This group action has the infinitesimal generator

$$V = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \xi_3 \frac{\partial}{\partial y} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v}.\tag{2.2}$$

As a result, this extended generator V corresponding to this symmetry is expressed as

$$prV = V + \eta_2^{\alpha,t} \frac{\partial}{\partial t^\alpha v} + \eta_1^x \frac{\partial}{\partial u_x} + \eta_2^x \frac{\partial}{\partial v_x} + \eta_2^y \frac{\partial}{\partial v_y} + \eta_1^{yy} \frac{\partial}{\partial u_{yy}},\tag{2.3}$$

where

$$\eta_1^x = D_x(\eta_1) - u_t D_x(\xi_1) - u_x D_x(\xi_2) - u_y D_x(\xi_3),\tag{2.4}$$

$$\eta_1^y = D_y(\eta_1) - u_t D_y(\xi_1) - u_x D_y(\xi_2) - u_y D_y(\xi_3),\tag{2.5}$$

$$\eta_2^x = D_x(\eta_2) - v_t D_x(\xi_1) - v_x D_x(\xi_2) - v_y D_x(\xi_3),\tag{2.6}$$

$$\eta_2^y = D_y(\eta_2) - v_t D_y(\xi_1) - v_x D_y(\xi_2) - v_y D_y(\xi_3),\tag{2.7}$$

$$\eta_1^{yy} = D_y(\eta_1^y) - u_{yt} D_y(\xi_1) - u_{xy} D_y(\xi_2) - u_{yy} D_x(\xi_3),\tag{2.8}$$

and

$$\begin{aligned}
\eta_2^{\alpha,t} &= \frac{\partial^\alpha \eta_2}{\partial t^\alpha} + ((\eta_2)_v - \alpha D_t(\tau)) \frac{\partial^\alpha v}{\partial t^\alpha} - u \frac{\partial^\alpha (\eta_2)_u}{\partial t^\alpha} + ((\eta_2)_u \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha (\eta_2)_u}{\partial t^\alpha}) \\
&\quad + \mu_1 + \mu_2 + \sum_{m=1}^{\infty} \left[\binom{\alpha}{m} \frac{\partial^m (\eta_2)_v}{\partial t^m} - \binom{\alpha}{m+1} D_t^{m+1}(\xi_1) \right] D_t^{\alpha-m}(v) \\
&\quad + \sum_{m=1}^{\infty} \binom{\alpha}{m} \frac{\partial^m (\eta_2)_u}{\partial t^m} D_t^{\alpha-m}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{m} D_t^m(\xi_2) D_t^{\alpha-m}(v_x) \\
&\quad - \sum_{m=1}^{\infty} \binom{\alpha}{m} D_t^m(\xi_3) D_t^{\alpha-m}(v_y),
\end{aligned} \tag{2.9}$$

with

$$\begin{aligned}
\mu_1 &= \sum_{r_4=2}^{\infty} \sum_{r_3=2}^{r_4} \sum_{r_2=2}^{r_3} \sum_{r_1=0}^{r_2-1} \binom{\alpha}{r_4} \binom{r_4}{r_3} \binom{r_2}{r_1} \frac{t^{r_4-\alpha} (-u)^{r_1}}{r_2! \Gamma(r_4+1-\alpha)} \frac{\partial^{r_3} u^{r_2-r_1}}{\partial t^{r_3}} \frac{\partial^{r_4-r_3+r_2} \eta_2}{\partial t^{r_4-r_3} \partial u^{r_2}}, \\
\mu_2 &= \sum_{r_4=2}^{\infty} \sum_{r_3=2}^{r_4} \sum_{r_2=2}^{r_3} \sum_{r_1=0}^{r_2-1} \binom{\alpha}{r_4} \binom{r_4}{r_3} \binom{r_2}{r_1} \frac{t^{r_4-\alpha} (-v)^{r_1}}{r_2! \Gamma(r_4+1-\alpha)} \frac{\partial^{r_3} v^{r_2-r_1}}{\partial t^{r_3}} \frac{\partial^{r_4-r_3+r_2} \eta_2}{\partial t^{r_4-r_3} \partial v^{r_2}},
\end{aligned}$$

where D_t , D_x , D_y are the total derivatives for t , x , y .

Remark 1: Preserving the structural integrity of the R-L fractional derivative operator in (2.1) requires that the manifold $t = 0$, which defines the integral's lower limit, remains invariant under the group transformations. This condition leads to the conclusion that

$$\xi_1(t, x, y, u, v)|_{t=0} = 0. \tag{2.10}$$

Remark 2: For $\eta_1 = a_1 u + b_1 v$ and $\eta_2 = a_2 u + b_2 v$, the invariants μ_1 and μ_2 satisfy $\mu_1 = \mu_2 = 0$. Thus,

$$\frac{\partial^2 \eta_2}{\partial u^2} = \frac{\partial^2 \eta_2}{\partial v^2} = 0. \tag{2.11}$$

Under the group transformations framework defined by (2.1), Eqs. (1.3) remain invariant if and only if the following condition holds

$$\begin{cases} prV(D_t^\alpha v + u_{yy} + 2uv_y + 6u^2 v_x + 2v_x^2)|_{(1.3)} = 0, \\ prV(v - u_x)|_{(1.3)} = 0, \end{cases} \tag{2.12}$$

which can be rewritten as

$$\begin{cases} (\eta_2^{\alpha,t} + \eta_1^{yy} + 2u\eta_2^y + 6u^2\eta_2^x + 4v_x\eta_2^x + 2v_y\eta_1 + 12uv_x\eta_1)|_{(1.3)} = 0, \\ (\eta_2 - \eta_1^x)|_{(1.3)} = 0. \end{cases} \tag{2.13}$$

Upon substituting the symmetry components $\eta_1^{\alpha,t}$, η_1^x , η_2^x , η_1^y , and η_1^{yy} into the determining equation (2.13), the invariance condition mandates that all derivative coefficients of u and v must annihilate, yielding the following infinitesimal expressions

$$\xi_1 = c_1 t, \xi_2 = \frac{\alpha}{5} c_1 x + c_3, \xi_3 = \frac{3\alpha}{5} c_1 y + c_3, \eta_1 = -\frac{2\alpha}{5} c_1 u, \eta_2 = -\frac{3\alpha}{5} c_1 v, \tag{2.14}$$

where c_1 , c_2 , c_3 be free parameters.. Therefore, Eqs. (1.3) yields a 3D Lie algebra with generators

$$V_1 = t \frac{\partial}{\partial t} + \frac{\alpha}{5} x \frac{\partial}{\partial x} + \frac{3\alpha}{5} y \frac{\partial}{\partial y} - \frac{2\alpha}{5} u \frac{\partial}{\partial u} - \frac{3\alpha}{5} v \frac{\partial}{\partial v}, V_2 = \frac{\partial}{\partial x}, V_3 = \frac{\partial}{\partial y}. \tag{2.15}$$

3. Reducing Eqs. (1.3) via Similarity Transformations: Invariant Solutions Constructed

This section focuses on reducing the target system (1.3) to two classes of FPDEs: (1+1)-D models incorporating left-sided E-K fractional derivatives and analytically tractable FPDEs involving R-L fractional operators. The reduction aims to simplify the complexity of the original equations for subsequent analytical treatment.

Case 1: $V_1 = t \frac{\partial}{\partial t} + \frac{\alpha}{5} x \frac{\partial}{\partial x} + \frac{3\alpha}{5} y \frac{\partial}{\partial y} - \frac{2\alpha}{5} u \frac{\partial}{\partial u} - \frac{3\alpha}{5} v \frac{\partial}{\partial v}$

Given the group generator V_1 , its characteristic equation is expressed as

$$\frac{dt}{t} = \frac{dx}{\frac{\alpha}{5}x} = \frac{dy}{\frac{3\alpha}{5}y} = \frac{du}{-\frac{2\alpha}{5}u} = \frac{dv}{-\frac{3\alpha}{5}v}. \quad (3.1)$$

Building on this foundation, we have determined the similarity variables $xt^{-\frac{\alpha}{5}}$, $yt^{-\frac{3\alpha}{5}}$, $ut^{-\frac{2\alpha}{5}}$, and $vt^{-\frac{3\alpha}{5}}$. As a result, the invariant solutions for Eqs. (1.3) are derived accordingly

$$u = t^{-\frac{2\alpha}{5}} f(\omega_1, \omega_2), v = t^{-\frac{3\alpha}{5}} g(\omega_1, \omega_2), \tau_1 = xt^{-\frac{\alpha}{5}}, \tau_2 = yt^{-\frac{3\alpha}{5}}. \quad (3.2)$$

Theorem 3.1: *To reduce the dimensionality of the original system (1.3), we seek solutions of the self-similar form:*

$$u = t^{-\frac{2\alpha}{5}} f(\omega_1, \omega_2), v = t^{-\frac{3\alpha}{5}} g(\omega_1, \omega_2),$$

where $\omega_1 = xt^{-\frac{\alpha}{5}}$, $\omega_2 = yt^{-\frac{3\alpha}{5}}$. This ansatz systematically reduces the system to a set of (l+1)-D FPDEs

$$\begin{cases} (\mathcal{P}_{\frac{5}{5}}^{1-\frac{8\alpha}{5}\alpha} g)(\omega_1, \omega_2) + f_{\omega_1, \omega_2} + 2fg_{\omega_2} + 6f^2g_{\omega_1} + 2g_{\omega_1}^2 = 0, \\ g - f_{\omega_1} = 0, \end{cases} \quad (3.3)$$

where $(\mathcal{P}_{\delta_1, \delta_2}^{\iota, k})$ represents the left-sided E-K fractional differential operator, which is defined by

$$(\mathcal{P}_{\delta_1, \delta_2}^{\iota, k} \phi)(\omega_1, \omega_2) := \prod_{j=0}^{s-1} \left(\iota + j - \frac{1}{\delta_1} \omega_1 \frac{d}{d\omega_1} - \frac{1}{\delta_2} \omega_2 \frac{d}{d\omega_2} \right) (\mathcal{K}_{\delta_1, \delta_2}^{\iota+k, s-k} \phi) (\omega_1, \omega_2), k > 0,$$

$$s = \begin{cases} [k] + 1, & \text{if } k \notin \mathbf{N}, \\ k, & \text{if } k \in \mathbf{N}, \end{cases} \quad (3.4)$$

with

$$(\mathcal{K}_{\delta_1, \delta_2}^{\iota, k} \phi)(\omega_1, \omega_2) := \begin{cases} \frac{1}{\Gamma(k)} \int_1^\infty (\tau - 1)^{k-1} \tau^{-(\iota+k)} \phi(\omega_1 \tau^{\frac{1}{\delta_1}}, \omega_2 \tau^{\frac{1}{\delta_2}}) d\tau, & k > 0, \\ \phi(\omega_1, \omega_2), & k = 0, \end{cases} \quad (3.5)$$

denotes the left-sided E-K fractional integral operator.

Proof: With α restricted to the open interval $(0,1)$, the fractional differentiation of $u(t, x)$ in the R-L sense proceeds as specified hereafter

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} (t^{-\frac{3\alpha}{5}} g(\omega_1, \omega_2)) = \frac{\partial}{\partial t} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{-\frac{3\alpha}{5}} g\left(xs^{-\frac{\alpha}{5}}, ys^{-\frac{3\alpha}{5}}\right) ds \right].$$

Assuming $r = \frac{t}{s}$, we have

$$\begin{aligned} \frac{\partial^\alpha v}{\partial t^\alpha} &= \frac{\partial}{\partial t} \left[\frac{t^{1-\frac{8\alpha}{5}}}{\Gamma(1-\alpha)} \int_1^\infty (r-1)^{-\alpha} r^{\frac{8\alpha}{5}-2} g(\omega_1 r^{\frac{\alpha}{5}}, \omega_2 r^{\frac{3\alpha}{5}}) dr \right] = \\ &= \frac{\partial}{\partial t} \left[t^{1-\frac{8\alpha}{5}} (\mathcal{K}_{\frac{5}{\alpha^3 \alpha}}^{1-\frac{3\alpha}{5}, 1-\alpha} g)(\omega_1, \omega_2) \right]. \end{aligned}$$

Because of $\omega_1 = xt^{-\frac{\alpha}{5}}$ and $\omega_2 = yt^{-\frac{3\alpha}{5}}$, the following relationship is valid

$$\begin{aligned} t \frac{\partial}{\partial t} g(\omega_1, \omega_2) &= t(-\frac{\alpha}{5} xt^{-\frac{\alpha}{5}-1} g_{\omega_1} - \frac{3\alpha}{5} yt^{-\frac{3\alpha}{5}-1} g_{\omega_2}) = \\ &= -\frac{\alpha}{5} \omega_1 \frac{d}{d\omega_1} g - \frac{3\alpha}{5} \omega_2 \frac{d}{d\omega_2} g. \end{aligned}$$

Hence, we arrive at

$$\begin{aligned} \frac{\partial^\alpha v}{\partial t^\alpha} &= t^{-\frac{8\alpha}{5}} \left[\left(1 - \frac{8\alpha}{5} - \frac{\alpha}{5} \omega_1 \frac{d}{d\omega_1} - \frac{3\alpha}{5} \omega_2 \frac{d}{d\omega_2}\right) (\mathcal{K}_{\frac{5}{\alpha^3 \alpha}}^{1-\frac{3\alpha}{5}, 1-\alpha} g)(\omega_1, \omega_2) \right] = \\ &= t^{-\frac{8\alpha}{5}} (\mathcal{P}_{\frac{5}{\alpha^3 \alpha}}^{1-\frac{8\alpha}{5}, \alpha} g)(\omega_1, \omega_2). \end{aligned}$$

Meanwhile,

$$\begin{aligned} u_{yy} + 2uv_y + 6u^2 v_x + 2v_x^2 &= t^{-\frac{8\alpha}{5}} (f_{\omega_2, \omega_2} + 2f g_{\omega_2} + 6f^2 g_{\omega_1} + 2g_{\omega_1}^2) = 0, \\ u_x - v &= t^{-\frac{3\alpha}{5}} (f_{\omega_1} - g) = 0. \end{aligned}$$

The proof is now finished.

The power series solutions for system (3.3) are next derived through application of the power series method. Let us make the following assumption

$$\begin{aligned} f(\omega_1, \omega_2) &= \sum_{m_1, m_2=0}^\infty a_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \quad g(\omega_1, \omega_2) = \\ &= \sum_{m_1, m_2=0}^\infty b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \end{aligned} \quad (3.6)$$

then

$$\begin{aligned} \frac{\partial f}{\partial \omega_1} &= \sum_{m_1, m_2=0}^\infty (m_1 + 1) a_{m_1+1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \quad \frac{\partial f}{\partial \omega_2} \\ &= \sum_{m_1, m_2=0}^\infty (m_2 + 1) a_{m_1, m_2+1} \omega_1^{m_1} \omega_2^{m_2}, \\ \frac{\partial g}{\partial \omega_1} &= \sum_{m_1, m_2=0}^\infty (m_1 + 1) b_{m_1+1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \\ \frac{\partial g}{\partial \omega_2} &= \sum_{m_1, m_2=0}^\infty (m_2 + 1) b_{m_1, m_2+1} \omega_1^{m_1} \omega_2^{m_2}, \\ \frac{\partial^2 f}{\partial \omega_2^2} &= \sum_{m_1, m_2=0}^\infty (m_2 + 2)(m_2 + 1) a_{m_1, m_2+2} \omega_1^{m_1} \omega_2^{m_2} \end{aligned} \quad (3.7)$$

and

$$\begin{aligned}
& (\mathcal{P}_{\frac{5}{\alpha^2 3\alpha}}^{1-\frac{8\alpha}{5}, \alpha} g)(\omega_1, \omega_2) \\
&= (1 - \frac{8\alpha}{5} - \frac{\alpha}{5} \omega_1 \frac{d}{d\omega_1} - \frac{3\alpha}{5} \omega_2 \frac{d}{d\omega_2}) \\
&\times (\frac{1}{\Gamma(1-\alpha)} \int_1^\infty (s-1)^{-\alpha} s^{\frac{8\alpha}{5}-2} g(\omega_1 s^{\frac{\alpha}{5}}, \omega_2 s^{\frac{3\alpha}{5}}) ds) \\
&= (1 - \frac{8\alpha}{5} - \frac{\alpha}{5} \omega_1 \frac{d}{d\omega_1} - \frac{3\alpha}{5} \omega_2 \frac{d}{d\omega_2}) \\
&\times (\frac{1}{\Gamma(1-\alpha)} \int_1^\infty (s-1)^{-\alpha} s^{\frac{(m_1+3m_2+8)\alpha}{5}-2} \sum_{m_1, m_2=0}^\infty (b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}) ds) \\
&= (1 - \frac{8\alpha}{5} - \frac{\alpha}{5} \omega_1 \frac{d}{d\omega_1} - \frac{3\alpha}{5} \omega_2 \frac{d}{d\omega_2}) \\
&\times (\sum_{m_1, m_2=0}^\infty \frac{\Gamma(1-\frac{3\alpha}{5} - \frac{k_1\alpha}{5} - \frac{3m_2\alpha}{5})}{\Gamma(2-\frac{8\alpha}{5} - \frac{m_1\alpha}{5} - \frac{3m_2\alpha}{5})} b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}) \\
&= \sum_{m_1, m_2=0}^\infty \frac{\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}.
\end{aligned} \tag{3.8}$$

Upon substituting the expressions given in Eqs. (3.6)-(3.8) into the simplified system (3.3) and conducting coefficient comparison for different ω_1 and ω_2 power combinations, the forms of a_{m_1, m_2} and b_{m_1, m_2} are established.

$$\begin{aligned}
& \sum_{m_1, m_2=0}^\infty \frac{\Gamma(1-\frac{(3+k_1+3k_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2} \\
&+ \sum_{m_1, m_2=0}^\infty (m_2+2)(m_2+1) a_{m_1, m_2+2} \omega_1^{m_1} \omega_2^{m_2} \\
&+ 2 \sum_{m_1, m_2=0}^\infty \sum_{j=0}^{m_1} \sum_{j=0}^{m_2} (j+1) a_{m_1-m, m_2-j} b_{m, j+1} \omega_1^{m_1} \omega_2^{m_2} \\
&+ 6 \sum_{m_1, m_2=0}^\infty \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{r=0}^m (n+ \\
&\quad 1) b_{n+1, r} a_{m-n, j-r} a_{m_1-m, m_2-j} \omega_1^{m_1} \omega_2^{m_2}, \\
&+ 2 \sum_{m_1, m_2=0}^\infty \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (m+1)(m_1-m+ \\
&\quad 1) b_{m+1, j} b_{m_1-m+1, m_2-j} \omega_1^{m_1} \omega_2^{m_2} = 0, \\
&\sum_{m_1, m_2=0}^\infty b_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2} - \sum_{m_1, m_2=0}^\infty (m_1+1) a_{m_1+1, m_2} \omega_1^{m_1} \omega_2^{m_2} = 0. \tag{3.9}
\end{aligned}$$

For $m_1, m_2 \geq 0$, we have

$$\begin{aligned}
& b_{m_1, m_2} = (m_1+1) a_{m_1+1, m_2}, \\
& a_{m_1, m_2+2} = -\frac{1}{(m_2+2)(m_1+1)} \left[\frac{\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} b_{m_1, m_2} \right. \\
&+ 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (j+1) a_{m_1-m, m_2-j} b_{m, j+1} \\
&+ 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (m+1)(m_1-m+1) b_{m+1, j} b_{m_1-m+1, m_2-j} \\
&\left. + 6 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j (n+1) b_{n+1, r} a_{m-n, j-r} a_{m_1-m, m_2-j} \right], \tag{3.10}
\end{aligned}$$

where the constants a_{m_1, m_2} are characterized by $a_{m_1, m_2} = \frac{\partial^{m_1}}{\partial \omega_1^{m_1}} \frac{\partial^{m_2}}{\partial \omega_2^{m_2}} f(0, 0)$ ($m_2 = 0, 1, m_1 = 0, 1, 2, \dots$). The solution to Eqs. (3.3) follows directly in the form of a power series

$$\begin{aligned}
 f(\omega_1, \omega_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^1 a_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2} \\
 &\quad - \sum_{m_1, m_2=0}^{\infty} \frac{\omega_1^{m_1} \omega_2^{m_2+2}}{(m_2+2)(m_1+1)} \left[\frac{\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} b_{m_1, m_2} \right. \\
 &\quad \left. + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (j+1) a_{m_1-m, m_2-j} b_{m, j+1} \right. \\
 &\quad \left. + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (m+1)(m_1-m+1) b_{m+1, j} b_{m_1-m+1, m_2-j} \right. \\
 &\quad \left. + 6 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j (n+1) b_{n+1, r} a_{m-n, j-r} a_{m_1-m, m_2-j} \right], \\
 g(\omega_1, \omega_2) &= \sum_{m_1, m_2=0}^{\infty} (m_1+1) a_{m_1+1, m_2} \omega_1^{m_1} \omega_2^{m_2}. \tag{3.11}
 \end{aligned}$$

Consequently, the power series solution corresponding to Eqs. (1.3) is as follows

$$\begin{aligned}
 u &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^1 a_{m_1, m_2} x^{m_1} y^{m_2} t^{\frac{-(m_1+3m_2+2)\alpha}{5}} \\
 &\quad - \sum_{m_1, m_2=0}^{\infty} \frac{x^{m_1} y^{m_2+2} t^{\frac{-(m_1+3m_2+4)\alpha}{5}}}{(m_2+2)(m_1+1)} \left[\frac{\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} b_{m_1, m_2} \right. \\
 &\quad \left. + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (j+1) a_{m_1-m, m_2-j} b_{m, j+1} \right. \\
 &\quad \left. + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (m+1)(m_1-m+1) b_{m+1, j} b_{m_1-m+1, m_2-j} \right. \\
 &\quad \left. + 6 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j (n+1) b_{n+1, r} a_{m-n, j-r} a_{m_1-m, m_2-j} \right], \\
 v &= \sum_{m_1, m_2=0}^{\infty} (m_1+1) a_{m_1+1, m_2} x^{m_1} y^{m_2} t^{\frac{-(m_1+3m_2+3)\alpha}{5}}. \tag{3.12}
 \end{aligned}$$

Theorem 3.2: *Convergence of the series (3.12) is guaranteed within a vicinity centered at $(0, |a_{0,0}|, M|a_{1,0}|)$.*

Proof: Derived from (3.10), we can acquire

$$\begin{aligned}
 |b_{m_1, m_2}| &\leq |m_1+1| |a_{m_1, m_2}|, \\
 |a_{m_1, m_2+2}| &\leq \frac{1}{(m_2+2)(m_1+1)} \left| \frac{\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})}{\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})} \right| |b_{m_1, m_2}| \\
 &\quad + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (j+1) |a_{m_1-m, m_2-j}| |b_{m, j+1}| \\
 &\quad + 2 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (m+1)(m_1-m+1) |b_{m+1, j}| |b_{m_1-m+1, m_2-j}| \\
 &\quad + 6 \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j (n+1) |b_{n+1, r}| |a_{m-n, j-r}| |a_{m_1-m, m_2-j}|. \tag{3.13}
 \end{aligned}$$

Using properties of the Gamma function, we readily establish that

$$\frac{|\Gamma(1-\frac{(3+m_1+3m_2)\alpha}{5})|}{|\Gamma(1-\frac{(8+m_1+3m_2)\alpha}{5})|} \leq 1 \text{ for any natural numbers } m_1 \text{ and } m_2. \text{ Consequently, Eqs. (3.13) transforms to}$$

$$\begin{aligned}
|b_{k+1}| &\leq M|a_{m_1, m_2}|, \\
|a_{m_1, m_2+2}| &\leq M[|b_{m_1, m_2}| + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} (j+1)|a_{m_1-m, m_2-j}||b_{m, j+1}| \\
&\quad + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} |b_{m+1, j}||b_{m_1-m+1, m_2-j}| \\
&\quad + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j |b_{n+1, r}||a_{m-n, j-r}||a_{m_1-m, m_2-j}|], \quad (3.14)
\end{aligned}$$

where

$$M = \max\{m_1 + 1, \frac{1}{(m_2+2)(m_2+1)}, \frac{2(j+1)}{(m_2+2)(m_2+1)}, \frac{2(m+1)}{(m_2+2)(m_2+1)}, \frac{6(n+1)}{(m_2+2)(m_2+1)}\}.$$

Consider another power series

$$P(\omega_1, \omega_2) = \sum_{m_1, m_2=0}^{\infty} p_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \quad Q(\omega_1, \omega_2) = \sum_{k=0}^{\infty} q_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2}, \quad (3.15)$$

where $p_{m_1, m_2} = |a_{m_1, m_2}| (m_2 = 0, 1, m_1 = 0, 1, 2, \dots)$, and

$$\begin{aligned}
q_{m_1, m_2} &= M p_{m_1+1, m_2}, \quad m_1, m_2 \geq 0, \\
p_{m_1, m_2+2} &= M[q_{m_1, m_2} + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} p_{m_1-m, m_2-j} q_{m, j+1} \\
&\quad + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} q_{m+1, j} q_{m_1-m+1, m_2-j} \\
&\quad + \sum_{m=0}^{m_1} \sum_{j=0}^{m_2} \sum_{n=0}^m \sum_{r=0}^j q_{n+1, r} p_{m-n, j-r} p_{m_1-m, m_2-j}], \quad m_1, m_2 \geq 0. \quad (3.16)
\end{aligned}$$

Thus, we observe that $|a_{m_1, m_2}| \leq p_{m_1, m_2}$ and $|b_{m_1, m_2}| \leq q_{m_1, m_2}$ for $m_1, m_2 = 0, 1, 2, \dots$. This implies that the series (3.15) serves as the dominant series over (3.6). We proceed to demonstrate the convergence of the series (3.15). Through straightforward computation, we obtain

$$\begin{aligned}
P &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^1 p_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2} + M(Q\omega_2^2 + P(Q \\
&\quad - \sum_{m_1=0}^{\infty} q_{m_1, 0} \omega_1^{m_1})\omega_2 + (Q - \sum_{m_2=0}^{\infty} q_{0, m_2} \omega_2^{m_2})^2 \omega_1^{-2} \omega_2^2 \\
&\quad + P^2(Q - \sum_{m_2=0}^{\infty} q_{0, m_2} \omega_2^{m_2})\omega_1^{-1} \omega_2^2), \\
Q &= M(P - \sum_{m_2=0}^{\infty} p_{0, m_2} \omega_2^{m_2})\omega_1^{-1}. \quad (3.17)
\end{aligned}$$

Consider the implicit function involving the independent variables ω_1 and ω_2 :

$$\begin{aligned}
F(\omega_1, \omega_2, P, Q) &= P - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^1 p_{m_1, m_2} \omega_1^{m_1} \omega_2^{m_2} \\
&\quad - M(Q\omega_2^2 + P(Q - \sum_{m_1=0}^{\infty} q_{m_1, 0} \omega_1^{m_1})\omega_2 \\
&\quad + (Q - \sum_{m_2=0}^{\infty} q_{0, m_2} \omega_2^{m_2})^2 \omega_1^{-2} \omega_2^2), \\
G(\omega_1, \omega_2, P, Q) &= Q - M(P - \sum_{m_2=0}^{\infty} p_{0, m_2} \omega_2^{m_2})\omega_1^{-1}. \quad (3.18)
\end{aligned}$$

By virtue of the analyticity of F, G near $(0, 0, p_{0,0}, M p_{1,0})$ and their vanishing at this point, together with the non-vanishing Jacobian condition $\frac{\partial(F, G)}{\partial(P, Q)}|_{(0,0,p_{0,0}, M p_{1,0})} = 1 \neq 0$, the implicit function theorem ensures the local analyticity of (3.15). This directly implies the convergence of solution (3.6) in a neighborhood of $(0, 0, |a_{0,0}|, M|a_{1,0}|)$, completing the argument.

Case 2: $V_2 = \frac{\partial}{\partial x}$

For the group generator V_2 , its corresponding characteristic equation takes the form

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dy}{0} = \frac{du}{0} = \frac{dv}{0}. \quad (3.19)$$

Consequently, the similarity variables t , y , u , v , which characterize invariant solutions to (1.3), follow directly from Eq. (3.19). Their specific forms are

$$u = f_1(t, y), v = f_2(t, y). \quad (3.20)$$

Performing the transformation (3.20) in (1.3) yields the corresponding simplified system

$$\begin{cases} D_t^\alpha f_2 + (f_1)_{yy} + 2f_1(f_2)_y = 0, \\ f_2 = 0. \end{cases} \quad (3.21)$$

Based on this, we can obtain the following straightforward solution

$$u = F_1(t)y + F_2(t), v = 0, \quad (3.22)$$

where $F_1(t)$ and $F_2(t)$ are taken as completely arbitrary functions of t .

Case 3: $V_3 = \frac{\partial}{\partial y}$

For the group generator V_3 , its corresponding characteristic equation takes the form

$$\frac{dy}{1} = \frac{dx}{0} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}. \quad (3.23)$$

Consequently, the similarity variables t , y , u , v , which characterize invariant solutions to (1.3), follow directly from Eq. (3.23). Their specific forms are

$$u = u(t, x), v = v(t, x), \quad (3.24)$$

along with the simplified equations

$$\begin{cases} D_t^\alpha v + 6u^2 v_x + 2v_x^2 = 0, \\ v = u_x, \end{cases} \quad (3.25)$$

which is (1+1)-D time fractional PDEs. Applying the LSA to Eqs. (3.25) once again results in the following generator

$$\Lambda_1 = t \frac{\partial}{\partial t} + \frac{\alpha}{5} x \frac{\partial}{\partial x} - \frac{2\alpha}{5} u \frac{\partial}{\partial u} - \frac{3\alpha}{5} v \frac{\partial}{\partial v}, \Lambda_2 = \frac{\partial}{\partial x}. \quad (3.26)$$

(1) For Λ_1 , the equation defining its characteristics is

$$\frac{dt}{t} = \frac{dx}{\frac{\alpha}{5}x} = \frac{du}{-\frac{2\alpha}{5}u} = \frac{dv}{-\frac{3\alpha}{5}v}. \quad (3.27)$$

Thus, we derive the similarity variables $xt^{-\frac{\alpha}{5}}$, $ut^{\frac{2\alpha}{5}}$, and $vt^{\frac{3\alpha}{5}}$. As a result, the solutions that are invariant are obtained

$$u = t^{-\frac{2\alpha}{5}} \Phi_1(z), v = t^{-\frac{3\alpha}{5}} \Phi_2(z), z = xt^{-\frac{\alpha}{5}}. \quad (3.28)$$

Theorem 3.3: To reduce Eqs. (3.25), we apply the scaling transformation:

$$u = t^{-\frac{2\alpha}{5}}\Phi_1(z), v = t^{-\frac{3\alpha}{5}}\Phi_2(z),$$

which yields the following FODEs

$$\begin{cases} (\mathcal{P}_{\frac{5}{\alpha}}^{1-\frac{8\alpha}{5},\alpha}\Phi_2)(z) + 6\Phi_1^2\Phi_{2'} + 2(\Phi_{2'})^2 = 0, \\ \Phi_{1'} - \Phi_2 = 0, \end{cases} \quad (3.29)$$

where $z = xt^{-\frac{\alpha}{5}}$.

Proof: With α restricted to the open interval $(0,1)$, the fractional differentiation of $v(t, x)$ in the R-L sense proceeds as specified hereafter

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} (t^{-\frac{3\alpha}{5}}\Phi_2(z)) = \frac{\partial}{\partial t} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{-\frac{3\alpha}{5}} \Phi_2(xs^{-\frac{\alpha}{5}}) ds \right].$$

Assuming $r = \frac{t}{s}$, we have

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial}{\partial t} \left[\frac{t^{1-\frac{8\alpha}{5}}}{\Gamma(1-\alpha)} \int_1^\infty (r-1)^{-\alpha} r^{\frac{8\alpha}{5}-2} \Phi_2(z) dr \right] = \frac{\partial}{\partial t} \left[t^{1-\frac{8\alpha}{5}} (\mathcal{K}_{\frac{5}{\alpha}}^{1-\frac{3\alpha}{5},1-\alpha} \Phi_2)(z) \right].$$

Because of $z = xt^{-\frac{\alpha}{5}}$, the following relationship is valid:

$$t \frac{\partial}{\partial t} \Phi_2(z) = t \left(-\frac{\alpha}{5} xt^{-\frac{\alpha}{5}-1} (\Phi_2)_z \right) = -\frac{\alpha}{5} z \frac{d}{dz} \Phi_2.$$

Hence, we arrive at

$$\frac{\partial^\alpha v}{\partial t^\alpha} = t^{-\frac{8\alpha}{5}} \left[\left(1 - \frac{8\alpha}{5} - \frac{\alpha}{5} z \frac{d}{dz} \right) (\mathcal{K}_{\frac{5}{\alpha}}^{1-\frac{3\alpha}{5},1-\alpha} \Phi_2)(z) \right] = t^{-\frac{8\alpha}{5}} (\mathcal{P}_{\frac{5}{\alpha}}^{1-\frac{8\alpha}{5},\alpha} \Phi_2)(z).$$

Meanwhile,

$$\begin{aligned} 6u^2 v_x + 2v_x^2 &= 6t^{-\frac{8\alpha}{5}} \Phi_1^2 \Phi_{2'} + 2t^{-\frac{8\alpha}{5}} (\Phi_{2'})^2, \\ u_x - v &= t^{-\frac{3\alpha}{5}} (\Phi_{1'} - \Phi_2) \end{aligned}$$

The proof is now concluded.

We will employ the power series approach to derive solutions for the reduced equations (3.29). Assuming

$$\Phi_1(z) = \sum_{h=0}^\infty a_h z^h, \Phi_2(z) = \sum_{h=0}^\infty b_h z^h \quad (3.30)$$

then

$$\frac{\partial \Phi_1}{\partial z} = \sum_{h=0}^\infty (h+1) a_{h+1} z^h, \frac{\partial \Phi_2}{\partial z} = \sum_{h=0}^\infty (h+1) b_{h+1} z^h, \quad (3.31)$$

and

$$\begin{aligned} &(\mathcal{P}_{\frac{5}{\alpha}}^{1-\frac{8\alpha}{5},\alpha} \Phi_2)(z) \\ &= \left(1 - \frac{8\alpha}{5} - \frac{\alpha}{5} z \frac{d}{dz} \right) \left(\frac{1}{\Gamma(1-\alpha)} \int_1^\infty (s-1)^{-\alpha} s^{\frac{8\alpha}{5}-2} \Phi_2(zs^{\frac{\alpha}{5}}) ds \right) \end{aligned}$$

$$\begin{aligned}
&= \left(1 - \frac{8\alpha}{5} - \frac{\alpha}{5} z \frac{d}{dz}\right) \left(\frac{1}{\Gamma(1-\alpha)} \int_1^\infty (s-1)^{-\alpha} s^{\frac{(h+8)\alpha}{5}-2} \sum_{h=0}^\infty (b_h z^h) ds\right) \\
&= \left(1 - \frac{8\alpha}{5} - \frac{\alpha}{5} z \frac{d}{dz}\right) \left(\sum_{h=0}^\infty \frac{\Gamma(1-\frac{3\alpha}{5}-\frac{h\alpha}{5})}{\Gamma(2-\frac{8\alpha}{5}-\frac{h\alpha}{5})} b_h z^h\right) \\
&= \sum_{h=0}^\infty \frac{\Gamma(1-\frac{(3+h)\alpha}{5})}{\Gamma(1-\frac{(8+h)\alpha}{5})} b_h z^h. \tag{3.32}
\end{aligned}$$

Upon substituting the expressions given in Eqs. (3.30)-(3.32) into the simplified system (3.29) and conducting coefficient comparison for different z power combinations, the forms of a_h and b_h are established

$$\begin{aligned}
&\sum_{h=0}^\infty \frac{\Gamma(1-\frac{(3+h)\alpha}{5})}{\Gamma(1-\frac{(8+h)\alpha}{5})} b_h z^h + 6 \sum_{h=0}^\infty \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} (h_1+1) a_{h_2-h_1} a_{h-h_1} b_{h_1+1} z^h \\
&+ 2 \sum_{h=0}^\infty \sum_{h_2=0}^h (h_2+1)(h-h_2+1) b_{h_2+1} b_{h-h_2+1} z^h = 0, \\
&\sum_{h=0}^\infty b_h z^h - \sum_{h=0}^\infty (h+1) a_{h+1} z^h = 0. \tag{3.33}
\end{aligned}$$

When $h \geq 0$, we have

$$\begin{aligned}
a_{h+1} &= \frac{1}{h+1} b_h, \\
b_h &= -\frac{\Gamma(1-\frac{(8+h)\alpha}{5})}{\Gamma(1-\frac{(3+h)\alpha}{5})} (2 \sum_{h_2=0}^h (h_2+1)(h-h_2+1) b_{h_2+1} b_{h-h_2+1} \\
&+ 6 \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} (h_1+1) a_{h_2-h_1} a_{h-h_1} b_{h_1+1}), \tag{3.34}
\end{aligned}$$

where $a_0 = \Phi_1(0)$ and $a_1 = \Phi_{1'}(0)$ represent free constants. Therefore, Eqs. (1.3) admit a power series solution of the form

$$u = \sum_{h=0}^\infty a_h x^h t^{\frac{-(h+2)\alpha}{5}}, v = \sum_{h=0}^\infty b_h x^h t^{\frac{-(h+3)\alpha}{5}}, \tag{3.35}$$

with a_h and b_h denoting the coefficients defined through the equations in (3.34).

(2) The equation that characterizes Λ_2 is expressed as

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0} = \frac{dv}{0}. \tag{3.36}$$

Based on this, the invariant solutions follow from the similarity variables t , u , v and are expressed as

$$u = \Psi_1(t), v = \Psi_2(t). \tag{3.37}$$

Substituting the expression from (3.37) into Eqs. (3.25) yields the vanishing solution $\Psi_2(t) = 0$. This implies that $u = \Psi_1(t)$, where $\Psi_1(t)$ is any function, serves as a trivial solution to Eq. (1.2).

4. Conservation laws of Eqs. (1.3)

In this section, we aim to derive the conservation laws for Eqs. (1.3) by utilizing generalized Noether symmetries and the corresponding invariance theorem [11, 12].

Eqs. (1.3) can be formulated as follows

$$\begin{cases} F_1 = D_t^\alpha v + u_{yy} + 2uv_y + 6u^2 v_x + 2v_x^2 = 0, \\ F_2 = v - u_x = 0, \end{cases} \quad (4.1)$$

with the corresponding formal Lagrangian being expressed as

$$\begin{aligned} \mathfrak{L} &= H_1(t, x, y)F_1 + H_2(t, x, y)F_2 \\ &= H_1(t, x, y)(D_t^\alpha v + u_{yy} + 2uv_y + 6u^2 v_x + 2v_x^2) + H_2(t, x, y)(v - u_x), \end{aligned} \quad (4.2)$$

here $H_1(t, x, y)$ and $H_2(t, x, y)$ denote the newly introduced dependent variables, with the Euler-Lagrange (E-L) operators given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y^2 \frac{\partial}{\partial u_{yy}}, \quad (4.3)$$

$$\frac{\delta}{\delta v} = (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha v)} + \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_y \frac{\partial}{\partial v_y}, \quad (4.4)$$

where the adjoint operator associated with D_t^α , denoted by $(D_t^\alpha)^*$, is mathematically characterized through the right-sided Caputo fractional derivative. Specifically, this operator is defined as

$$(D_t^\alpha)^* f(t, x) \equiv {}_t D_t^\alpha f(t, x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_t^\tau \frac{\partial^n f(s, x)}{(t-s)^{\alpha-n+1}} ds, & n-1 < \alpha < n, n \in \mathbf{N} \\ D_t^n f(t, x), & \alpha = n \in \mathbf{N}. \end{cases}$$

The adjoint equations corresponding to equation (4.1) are formulated as

$$\begin{cases} F_1^* = \frac{\delta \mathfrak{L}}{\delta u} = (H_1)_{yy} + 2H_1 v_y + 12H_1 u v_x + (H_2)_x = 0, \\ F_2^* = \frac{\delta \mathfrak{L}}{\delta v} = (D_t^\alpha)^*(H_1) - 2(H_1)_y u - 2H_1 u_y - 6(H_1)_x u^2 - \\ \quad 12H_1 u u_x - 4(H_1)_x v_x - 4H_1 v_{xx} + H_2 = 0. \end{cases} \quad (4.5)$$

Subsequently, we shall utilize the aforementioned adjoint equations in combination with the newly derived conservation theorem to establish conservation laws for the system given by Eqs. (1.3). In accordance with the classical definition, a conserved vector is a triplet $T = (T^t, T^x, T^y)$ such that $[D_t T^t + D_x T^x + D_y T^y]_{F_1, F_2=0} = 0$. Using generalized Noether operators leads to the determination of the vector components.

Initially, deriving from the fundamental operator identity, namely

$$\begin{aligned} prV + D_t \xi_1 \cdot \mathfrak{F} + D_x \xi_2 \cdot \mathfrak{F} + D_y \xi_3 \cdot \mathfrak{F} &= \Omega^u \cdot \frac{\delta}{\delta u} + \Omega^v \cdot \frac{\delta}{\delta v} + D_t \mathcal{N}^t + \\ &\quad D_x \mathcal{N}^x + D_y \mathcal{N}^y. \end{aligned} \quad (4.6)$$

Using the prolonged vector field prV given in (2.3), we introduce the characteristic functions Ω^u and Ω^v associated with the symmetry generator V :

$$\Omega^u = \eta_1 - \xi_1 u_t - \xi_2 u_x - \xi_3 u_y, \Omega^v = \eta_2 - \xi_1 v_t - \xi_2 v_x - \xi_3 v_y.$$

These definitions, along with the identity operator $\mathfrak{F}$, lead to the Noether operators:

$$\mathcal{N}^t = \xi_1 \mathfrak{F} + \sum_{l=0}^{n-1} (-1)^l D_t^{\alpha-1-l} (\Omega^v) D_t^l \frac{\partial}{\partial (D_t^\alpha v)} - (-1)^n J(\Omega^v, D_t^n \frac{\partial}{\partial (D_t^\alpha v)}), \quad (4.7)$$

$$\mathcal{N}^x = \xi_2 \mathfrak{I} + \Omega^u \frac{\partial}{\partial u_x} + \Omega^v \frac{\partial}{\partial v_x}, \quad (4.8)$$

$$\mathcal{N}^y = \xi_3 \mathfrak{I} + D_y(\Omega^u) \frac{\partial}{\partial u_{yy}} + \Omega^v \frac{\partial}{\partial v_y}, \quad (4.9)$$

with $n = [\alpha] + 1$ and where J represents the following operator

$$J(F, G) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{F(\tau, x, y) G(\theta, x, y)}{(\theta - \tau)^{\alpha+1-n}} d\theta d\tau. \quad (4.10)$$

The conserved vector's components can be expressed as

$$T^t = \mathcal{N}^t \mathfrak{Q}, T^x = \mathcal{N}^x \mathfrak{Q}, T^y = \mathcal{N}^y \mathfrak{Q}. \quad (4.11)$$

Case 1: $V_1 = t \frac{\partial}{\partial t} + \frac{\alpha}{5} x \frac{\partial}{\partial x} + \frac{3\alpha}{5} y \frac{\partial}{\partial y} - \frac{2\alpha}{5} u \frac{\partial}{\partial u} - \frac{3\alpha}{5} v \frac{\partial}{\partial v}$

For the symmetry generator V_1 , the corresponding characteristics take the form

$$\Omega^u = -\frac{2\alpha}{5} u - t u_t - \frac{\alpha}{5} x u_x - \frac{3\alpha}{5} y u_y, \Omega^v = -\frac{3\alpha}{5} v - t v_t - \frac{\alpha}{5} x v_x - \frac{3\alpha}{5} y v_y. \quad (4.12)$$

As a result

$$\begin{aligned} T^t &= -H_1 D_t^{\alpha-1}(\Omega^v) + J(\Omega^v, (H_1)_t) \\ &= -H_1 D_t^{\alpha-1}(\frac{3\alpha}{5} v + t v_t + \frac{\alpha}{5} x v_x + \frac{3\alpha}{5} y v_y) \\ &\quad - J(\frac{3\alpha}{5} v + t v_t + \frac{\alpha}{5} x v_x + \frac{3\alpha}{5} y v_y, (H_1)_t), \\ T^x &= H_2(\frac{2\alpha}{5} u + t u_t + \frac{\alpha}{5} x u_x + \frac{3\alpha}{5} y u_y) \\ &\quad - (\frac{3\alpha}{5} v + t v_t + \frac{\alpha}{5} x v_x + \frac{3\alpha}{5} y v_y)(6H_1 u^2 + 4H_1 v_x), \\ T^y &= -H_1(\frac{2\alpha}{5} u_y + t u_{ty} + \frac{\alpha}{5} x u_{xy} + \frac{3\alpha}{5} u_y + \frac{3\alpha}{5} y u_{yy}) \\ &\quad - 2H_1 u(\frac{3\alpha}{5} v + t v_t + \frac{\alpha}{5} x v_x + \frac{3\alpha}{5} y v_y). \end{aligned} \quad (4.13)$$

Case 2: $V_2 = \frac{\partial}{\partial x}$

For the symmetry generator V_2 , the corresponding characteristics take the form

$$\Omega^u = -u_x, \Omega^v = -v_x. \quad (4.14)$$

As a result

$$\begin{aligned} T^t &= H_1 D_t^{\alpha-1}(\Omega^v) + J(\Omega^v, p_t) = -H_1 D_t^{\alpha-1} v_x - J(v_x, (H_1)_t), \\ T^x &= H_2 u_x - (6H_1 u^2 + 4H_1 v_x) v_x, \\ T^y &= -H_1 u_{xy} - 2H_1 u v_x. \end{aligned} \quad (4.15)$$

Case 3: $V_3 = \frac{\partial}{\partial y}$

For the symmetry generator V_3 , the corresponding characteristics take the form

$$\Omega^u = -u_y, \Omega^v = -v_y. \quad (4.16)$$

As a result

$$\begin{aligned} T^t &= H_1 D_t^{\alpha-1}(\Omega^v) + J(\Omega^v, (H_1)_t) = -H_1 D_t^{\alpha-1}v_y - J(v_y, (H_1)_t), \\ T^x &= H_2 u_y - (6H_1 u^2 + 4H_1 v_x)v_y, \\ T^y &= -H_1 u_{yy} - 2H_1 uv_y. \end{aligned} \quad (4.17)$$

5. Conclusion

When the space derivative and the fractional time derivative are both of great significance, the mixed derivative might carry a different physical connotation. Here, we present a method to compute the symmetry of a fractional order differential equation containing this mixed derivative. The method doesn't necessitate extensive theoretical groundwork; however, the process of identifying such reduced equations is straightforward, requiring less computational complexity. In addition, a power series solution and a conservation law for this differential equation are obtained by us.

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