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On solutions of the nonlocal generalized coupled systems beam-type and travelling waves

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Abstract

This paper is structured in two parts, both focusing on the analysis of nonlinear coupled systems using fractional calculus. In the first part, we study a beam deflection-type system with sequential Caputo derivatives. We prove an existence and uniqueness result under nonlocal boundary conditions by applying Banach's fixed point theorem. Additionally, we discuss the Ulam-Hyers stability of the system, supported by illustrative examples. In the second part, we apply the Tanh method to obtain travelling wave solutions for a coupled evolution system related to the sequential model.

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1. Introduction

The field of Fractional Calculus (FC) is witnessing remarkable development and growing significance. The power of this field lies in its broad and successful applications across various scientific and technological disciplines [19], [21], [25], [28]. It provides unique mathematical tools for modeling complex phenomena in dynamical systems, continuum mechanics, signal processing, quantum mechanics, biomedical engineering, and even in financial modeling and the study of environmental pollution and disease spread [1], [7], [9].

Within this framework, Fractional Differential Equations (FDEs) have emerged as a central research tool, offering a more accurate description of many physical and engineering phenomena compared to their integer-order counterparts. The use of derivatives of real order, such as the Caputo derivative [3], [15], demonstrates higher efficiency in representing reality, as in cases of anomalous diffusion or in the precise modeling of lithium-ion batteries or Zika virus dynamics [11].

Research efforts are not limited to developing models alone but extend to studying the theoretical foundations of these equations. This includes proving the existence and uniqueness of solutions using fixed-point theorems [30] and analyzing stability in the Ulam-Hyers sense [14], [16], [17], [22], [26], which ensures obtaining stable and highly reliable approximate solutions for practical

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applications. Thus, fractional calculus remains a field rich with potential and opportunities for researchers in mathematics and applied sciences [2].

Beam equations are fundamental in structural and civil engineering for analyzing the deflection, stress, and vibration of load-bearing elements like bridges and building frames [4], [24], [27]. An elastic beam's deformation under an external force g, sustained at both ends, is described, for instance, by the linear boundary-value problem:

$$\begin{cases} \frac{d^4\psi}{d\varsigma^4}(\varsigma) = F(\varsigma), & \varsigma \in (0,1), \\ \psi(0) = \psi(1) = \frac{d^2\psi}{d\varsigma^2}(0) = \frac{d^2\psi}{d\varsigma^2}(1) = 0. \end{cases}$$

Physically, the boundary conditions are derived from the assumption of zero bending moments at the beam's endpoints, a standard mechanical consideration referenced in [13]. Within this research direction, prior contributions such as [6] have examined solution properties for the following fractional problem:

$$\begin{cases} \mathcal{D}^{\flat}(\mathcal{D}^{\natural}\omega)(\varsigma) + G(\varsigma, \eta(\varsigma)) = 0, & \varsigma \in (0,1), \\ \eta(0) = \mathcal{D}^{\natural}\omega(0) = (\mathcal{D}^{\natural}\omega)'(0) = \mathcal{D}^{\natural}\eta(1) = 0, \end{cases}$$

For $\natural \in (0,1]$ and $\flat \in (2,3]$, consider a continuous mapping $G: [0,1] \times \mathbb{R}^2 \to \mathbb{R}$. Under certain additional conditions, the following result holds.

The research presented in [5] is dedicated to establishing the existence and uniqueness of solutions for a coupled system incorporating sequential Caputo fractional derivatives, formulated as:

$$\begin{cases} \partial^{\flat_1}\partial^{\natural_2}\partial^{\flat_3}\partial^{\natural_4}u(\varsigma) = F_1(\varsigma, u(\varsigma), v(\varsigma)) + c_1\phi_1(u(\varsigma)) + d_1\psi_1(\partial^{\flat_1}\partial^{\natural_2}u(\varsigma)), \varsigma \in I = [0,1], \\ \partial^{\natural_1}\partial^{\flat_2}\partial^{\natural_3}\partial^{\flat_4}v(\varsigma) = F_2(\varsigma, u(\varsigma), v(\varsigma)) + c_2\phi_2(v(\varsigma)) + d_2\psi_2(\partial^{\natural_1}\partial^{\flat_2}v(\varsigma)), \varsigma \in I = [0,1], \\ u(0) = u(1) = \partial^{\flat_1}\partial^{\natural_2}u(1) = \partial^{\natural_4}u(0) = 0, \\ v(0) = v(1) = \partial^{\natural_1}\partial^{\flat_2}v(1) = \partial^{\natural_4}v(0) = 0, \end{cases}$$

Here, the derivatives $\partial^{\flat i}$ and $\partial^{\natural i}$ are taken in the Caputo sense, with orders satisfying $0 < \flat_i \leq 1$ and $0 < \natural_i \leq 1$ for $i = 1, \ldots, 4$. Because the derivatives are applied one after another, the total order of each equation is between 3 and 4. We also impose the conditions $\flat_1 + \natural_2 < \natural_4$ and $\natural_1 + \natural_2 < \natural_4$.

The nonlinear terms are given by the continuous functions

$$\phi_j, \psi_j: \mathbb{R} \to \mathbb{R} \text{ and } F_j: [0,1] \times \mathbb{R}^2 \to \mathbb{R}, j = 1,2.$$

An important property of the system is that none of these functions vanish at the origin, that is,

$$F_i(\varsigma, 0,0) \neq 0, \quad \phi_i(0) \neq 0, \quad \psi_i(0) \neq 0, \quad i = 1,2.$$

In reference [8], Bensassa and co-authors examined a fractional order system describing a beam structure, formulated with Caputo derivatives. Using Banach's contraction principle and Schaeferâ€™s fixed point theorem, they derived conditions guaranteeing the existence and uniqueness of solutions for the following coupled problem:

$$\begin{cases} \mathcal{D}^{\flat_1}\mathcal{D}^{\natural_2}y(\varsigma) = \Phi_1(\varsigma, y(\varsigma), z(\varsigma)) + c_1\,\Psi_1(\varsigma, y(\varsigma)) + d_1\,\Theta_1(\varsigma, \mathcal{D}^{\sigma}y(\varsigma)), \\ \mathcal{D}^{\natural_1}\mathcal{D}^{\natural_2}z(\varsigma) = \Phi_2(\varsigma, y(\varsigma), z(\varsigma)) + c_2\,\Psi_2(\varsigma, y(\varsigma)) + d_2\,\Theta_2(\varsigma, \mathcal{D}^{\sigma}z(\varsigma)), \end{cases}$$

The boundary conditions are given by:

$$y(0) = y(1) = \flat, y'(0) = y'(1) = 0, z(0) = z(1) = \natural, z'(0) = z'(1) = 0.$$

where, for $i = 1, 2$, the constants c_i, d_i are real numbers, and $\mathcal{D}^{\flat i}, \mathcal{D}^{\natural i}, \mathcal{D}^{\sigma}$ denote Caputo fractional derivatives with $0 < \sigma \leq 1$. The non-linearities are continuous: $\Phi_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Psi_i, \Theta_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$. An important feature of the system stems from the orders of differentiation, which satisfy $0 < \flat_1, \natural_1 \leq 1$ and $2 < \natural_2, \natural_2 \leq 3$. This choice breaks the semi group and commutativity properties usually enjoyed by fractional differential operators, adding a distinctive technical difficulty to the analysis.

In this work, a new beam-type fractional differential system with non-local conditions and stated in terms of Caputo derivatives is studied. The following equations represent the system that we are analyzing:

$$\begin{aligned} \mathfrak{D}^{\flat_1}\mathfrak{D}^{\natural_1}\mathfrak{D}^{\natural_1}v_1(\varsigma) &= \mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big) + \mathcal{R}_1(\varsigma, v_1(\varsigma), I^{p}v_1(\varsigma), D^{\delta}v_1(\varsigma)), \\ \mathfrak{D}^{\flat_2}\mathfrak{D}^{\natural_2}\mathfrak{D}^{\natural_2}v_2(\varsigma) &= \mathcal{S}_2\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_2(\varsigma)\big) + \mathcal{R}_2(\varsigma, v_2(\varsigma), I^{p}v_2(\varsigma), D^{\delta}v_2(\varsigma)). \end{aligned} \tag{1}$$

subject to the boundary conditions:

$$\begin{cases} v_1(0) = \xi_1, v_2(0) = \xi_2, \\ \mathfrak{D}^{\natural_1}v_1(0) = 0, \mathfrak{D}^{\natural_2}v_2(0) = 0, \\ v_1(1) = \int_0^{\mu_1} k(\varsigma)v_1(\varsigma)\mathrm{d}\varsigma, v_2(1) = \int_0^{\mu_2} k(\varsigma)v_2(\varsigma)\mathrm{d}\varsigma, \\ \mathfrak{D}^{\natural_1}v_1(1) = \vartheta_1, \mathfrak{D}^{\natural_2}v_2(1) = \vartheta_2. \end{cases} \tag{2}$$

In this framework, $\mathfrak{D}^{\flat i}$, $\mathfrak{D}^{\natural i}$, $\mathfrak{D}^{\natural i}$ are Caputo fractional derivatives, The operator I^{p} stands for the Riemann--Liouville integral. We assume the orders satisfy $0 < \flat_i, \delta, \natural_i \leq 1$, $1 < \natural_i \leq 2$, and $\vartheta_i, \xi_i > 0$. Moreover, $\mathcal{S}_i, \mathcal{R}_i \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$ and $k \in C([0,1], \mathbb{R})$, for $i = 1, 2$.

This problem's structure offers a considerable extension beyond conventional fractional beam equations. A further distinctive feature lies in its nonlinear terms, which create a hybrid dependence on the system's fractional velocities alongside a Riemann-Liouville integral component ($I^{p}v_i(\varsigma)$). These unique dynamical characteristics are complemented by a full spectrum of non-local boundary restrictions. Together, these elements yield an advanced and versatile mathematical construct for modeling sophisticated mechanical behaviors.

The analysis relies on two main tools Banach's contraction principle and Schauder's fixed point theorem To examine the system. We construct two principal results concerning solution existence and uniqueness. The first theorem is validated through a numerical example, while Ulam-Hyers stability is demonstrated with two additional examples.

In this division, we present the tanh method and how it is used to derive novel traveling wave solutions (Figure 1) for the beam system formulated with Khalil's fractional derivative. [29]:

$$\begin{cases} T_{\varsigma}^{2\flat}u^1(\varsigma, x) + T_x^{4\natural}u^1(\varsigma, x) + G_1(u^1, u^2, T_x^{\natural}(u^1, u^2)(\varsigma, x), T_x^{2\natural}(u^1, u^2)(\varsigma, x)) = 0, \\ T_{\varsigma}^{2\flat}u^2(\varsigma, x) + T_x^{4\natural}u^2(\varsigma, x) + G_2(u^1, u^2, T_x^{\natural}(u^1, u^2)(\varsigma, x), T_x^{2\natural}(u^1, u^2)(\varsigma, x)) = 0. \end{cases} \tag{3}$$

where G_1, G_2 are two functions and $0 < \natural, \natural \leq 1$, $T_\varsigma^\flat T_x^\natural$ represent fractional derivatives as defined by Khalil's conformable derivative framework.

We draw the attention of the reader to the fact that this new evolutionary coupled system and the system (1), previously introduced are related in the sense that both of them are instances of a coupled system of fourth order beam type equation.

The structure of the paper is as follows: Section 2 establishes the foundational concepts for our investigation. In Section 3, we present and prove our main existence and uniqueness theorem; we then employ the same assumptions to analyze different types of Ulam stability, supported by illustrative examples. Finally, in Section 4, we use Khalil's conformable derivative framework to generalize the analysis to a system of beam-type equations. We systematically derive new progressive traveling wave solutions using the Tanh method.

2. Preliminaries

This section presents the necessary preliminary lemma and definitions used throughout this paper (see, [21], [25]).

Definition 1: For $\natural > 0$ and $v \in C([0,1])$, the R-L fractional integral of order $\natural$ is given by:

$$\mathfrak{J}^\natural v(\varsigma) = \int_0^s \frac{(s-\varsigma)^{\natural-1}}{\Gamma(\natural)} v(\varsigma) d\varsigma,$$

and for $\natural = 0$, as

$$\mathfrak{J}^0 v(\varsigma) = v(\varsigma).$$

Definition 2: Let $\natural > 0$. For $v \in C^\varrho([0,1])$, the Caputo derivative of order $\natural$ is defined by:

$$\mathfrak{D}^\natural v(\varsigma) = \frac{1}{\Gamma(\varrho - \#)} \int_0^\varsigma (\varsigma - \Delta)^{\varrho-\natural-1} v^{(\varrho)}(\Delta) d\Delta.$$

where, $\varrho = [\natural] + 1$ indicates the integer part of $\flat$ and $v^{(\varrho)}$ denotes the ϱ-th derivative of v.

Lemma 3: *Let $\natural > 0$. The fractional differential equation is*

$$\mathfrak{D}^\natural v(\varsigma) = 0,$$

with Caputo derivative $\mathfrak{D}^\natural$, admits as general solution on $[0,1]$:

$$v(\varsigma) = \sum_{k=0}^{n-1} a_k \varsigma^k,$$

where $a_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n-1$.

Lemma 4: Let $\natural > 0$. The differential equation is

$$\mathfrak{D}^\natural v(\varsigma) = 0, \tag{4}$$

has a general solution on $[0,1]$ given by

$$v(\varsigma) = \Delta_0 + \Delta_1 \varsigma + \Delta_2 \varsigma^2 + \cdots + \Delta_{\varrho-1} \varsigma^{\varrho-1},$$

where, $\Delta_i \in \mathbb{R}$*;* $i = 0, 1, 2, \dots, \varrho - 1$.

Lemma 5: *Let* $\natural > 0$ *and let* $b > 0$*, then the equation*

$$\mathfrak{I}^{\natural}\mathfrak{D}^{\natural}v(\varsigma) = v(\varsigma) + \Delta_0 + \Delta_1\varsigma + \Delta_2\varsigma^2 + \cdots + \Delta_{\varrho-1}\varsigma^{\varrho-1}, \tag{5}$$

where $\Delta_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, \varrho - 1$.

Lemma 6: *The following characteristics hold true for any* $\varsigma \in [0,1]$ *:*

- $\mathfrak{I}^{\natural}\mathfrak{I}^{\natural}v(\varsigma) = \mathfrak{I}^{\natural+\natural}v(\varsigma)$, for all $\natural > 0$ and $\natural > 0$.
- $\mathfrak{D}^{\natural}\mathfrak{I}^{\natural}v(\varsigma) = v(\varsigma)$, for all $\natural > 0$.
- $\mathfrak{D}^{\natural}\mathfrak{I}^{\natural}v(\varsigma) = \mathfrak{I}^{\natural-\natural}v(\varsigma)$, for all $\natural > \natural > 0$.

Lemma 7: (Banach's contraction principle). *Let* F *be a real Banach space. If a map* $f: F \to F$ *is a contraction, that is there exists a constant* $0 < k < 1$ *such that for all* $x, y \in F$

$$\|f(x) - f(y)\|_F \leq k\|x - y\|_F.$$

Then f *admits a unique fixed point in* F.

Lemma 8: *Let* $\mathcal{C}_1$ *,*$\mathcal{C}_2 \in C([0,1], \mathbb{R})$*. The linear problem*

$$\begin{cases} \mathfrak{D}^{\flat_1}\mathfrak{D}^{\natural_1}\mathfrak{D}^{\natural_1}v_1(\varsigma) = \mathcal{C}_1(\varsigma), \\ \mathfrak{D}^{\flat_2}\mathfrak{D}^{\natural_2}\mathfrak{D}^{\natural_2}v_2(\varsigma) = \mathcal{C}_2(\varsigma). \end{cases} \tag{6}$$

With:

$$\begin{cases} v_1(0) = \xi_1, v_2(0) = \xi_2, \\ \mathfrak{D}^{\natural_1}v_1(0) = 0\ \mathfrak{D}^{\natural_2}v_2(0) = 0, \\ v_1(1) = \int_0^{\mu_1} k(\varsigma)v_1(\varsigma)\,\mathrm{d}\varsigma, v_2(1) = \int_0^{\mu_2} k(\varsigma)v_2(\varsigma)\,\mathrm{d}\varsigma, \\ \mathfrak{D}^{\natural_1}v_1(1) = \vartheta_1, D^{\natural_2}v_2(1) = \vartheta_2, \end{cases} \tag{7}$$

Has a unique linear solution as follows for $i = 1, 2$

$$\begin{cases} v_i(\varsigma) = & \mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{C}_i(\varsigma) + \Gamma(\natural_i + 1)\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{C}_i(\varsigma)\big)\frac{\varsigma^{\natural_i+\natural_i}}{\Gamma(\natural_i+\natural_i+1)} + \xi_i \\ & +\big(1 - \int_0^{\mu_1} k(s)\,\mathrm{d}s\big)^{-1} \times [\int_0^{\mu_1} k(s)\mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{C}_i(s)\,\mathrm{d}s - \mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{C}_i(1) \\ & +\Gamma(\natural_i + 1)\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{C}_i(\varsigma)\big)\left(\int_0^{\mu_1} \frac{k(s)}{\Gamma(\natural_i+\natural_i+1)}\,\mathrm{d}s - \frac{1}{\Gamma(\natural_1+\natural_1+1)}\right) \\ & + \int_0^{\mu_1} \xi_i k(s)\,\mathrm{d}s - \xi_i]\varsigma. \end{cases} \tag{8}$$

Proof: We apply the fractional integral $\mathfrak{I}^{\flat_i}$ for $i = 1,2$ and use Lemma (5) to obtain:

$$\begin{cases} \mathfrak{D}^{\natural_1}\mathfrak{D}^{\natural_1}v_1(\varsigma) = \mathfrak{I}^{\flat_1}\mathcal{C}_1(\varsigma) + k_0, \\ \mathfrak{D}^{\natural_2}\mathfrak{D}^{\natural_2}v_2(\varsigma) = \mathfrak{I}^{\flat_2}\mathcal{C}_2(\varsigma) + k_1. \end{cases} \tag{9}$$

Applying the fractional integrals $\mathfrak{I}^{\natural_i}$ and $\mathfrak{I}^{\natural_i}$ successively for $i = 1,2$, and using Lemma (6), we derive:

$$\begin{cases} v_1(\varsigma) & = I^{\flat_1+\natural_1+\natural_1} C_1(\varsigma) + \frac{k_0 \varsigma^{\natural_1+\natural_1}}{\Gamma(\natural_1+\natural_1+1)} + \frac{k_1 \varsigma^{\natural_1}}{\Gamma(\natural_1+1)} + k_2 + k_3 \varsigma, \\ v_2(\varsigma) & = I^{\flat_2+\natural_2+\natural_2} C_2(\varsigma) + \frac{m_0 \varsigma^{\natural_2+\natural_2}}{\Gamma(\natural_2+\natural_2+1)} + \frac{m_1 \varsigma^{\natural_2}}{\Gamma(\natural_2+1)} + m_2 + m_3 \varsigma. \end{cases}$$

Using the initial conditions $v_i(0) = \xi_i$, we find $k_2 = \xi_1$ and $m_2 = \xi_2$. Furthermore, we have:

$$\begin{cases} D^{\natural_1} v_1(\varsigma) & = I^{\flat_1+\natural_1} C_1(\varsigma) + \frac{k_0 \varsigma^{\natural_1}}{\Gamma(\natural_1+1)} + k_1, \\ D^{\natural_2} v_2(\varsigma) & = I^{\flat_2+\natural_2} C_2(\varsigma) + \frac{m_0 \varsigma^{\natural_2}}{\Gamma(\natural_2+1)} + m_1. \end{cases} \tag{10}$$

These equations are obtained by differentiating (8). Evaluating at $\varsigma = 0$ yields $k_1 = m_1 = 0$. Substituting these values gives:

$$\begin{cases} v_1(\varsigma) & = I^{\flat_1+\natural_1+\natural_1} C_1(\varsigma) + \frac{k_0 \varsigma^{\natural_1+\natural_1}}{\Gamma(\natural_1+\natural_1+1)} + \xi_1 + k_3 \varsigma, \\ v_2(\varsigma) & = I^{\flat_2+\natural_2+\natural_2} C_2(\varsigma) + \frac{m_0 \varsigma^{\natural_2+\natural_2}}{\Gamma(\natural_2+\natural_2+1)} + \xi_2 + m_3 \varsigma. \end{cases} \tag{11}$$

Evaluating at $\varsigma = 1$ in equations (10) and (11), we determine the rest of the constants given by :

$$\begin{cases} k_0 & = \Gamma(\natural_1 + 1)\left[\vartheta_1 - I^{\flat_1+\natural_1} C_1(1)\right], \\ k_3 & = \left(1 - \int_0^{\mu_1} k(s) ds\right)^{-1} \times \left[\int_0^{\mu_1} k(s) I^{\flat_1+\natural_1+\natural_1} C_1(s) ds - I^{\flat_1+\natural_1+\natural_1} C_1(1)\right. \\ & \quad + \Gamma(\natural_1 + 1)\left(\vartheta_1 - I^{\flat_1+\natural_1} C_1(1)\right)\left(\int_0^{\mu_1} k(s) \frac{s^{\natural_1+\natural_1}}{\Gamma(\natural_1+\natural_1+1)} ds - \frac{1}{\Gamma(\natural_1+\natural_1+1)}\right) \\ & \quad \left. + \int_0^{\mu_1} \xi_1 k(s) ds - \xi_1\right]. \end{cases}$$

$$\begin{cases} m_0 & = \Gamma(\natural_2 + 1)\left[\vartheta_2 - I^{\flat_2+\natural_2} C_2(1)\right], \\ m_3 & = \left(1 - \int_0^{\mu_2} sk(s) ds\right)^{-1} \times \left[\int_0^{\mu_2} k(s) I^{\flat_2+\natural_2+\natural_2} C_2(s) ds - I^{\flat_2+\natural_2+\natural_2} C_2(1)\right. \\ & \quad + \Gamma(\natural_2 + 1)\left(\vartheta_2 - I^{\flat_2+\natural_2} C_2(1)\right)\left(\int_0^{\mu_2} k(s) \frac{s^{\natural_2+\natural_2}}{\Gamma(\natural_2+\natural_2+1)} ds - \frac{1}{\Gamma(\natural_2+\natural_2+1)}\right) \\ & \quad \left. + \int_0^{\mu_2} \xi_2 k(s) ds - \xi_2\right]. \end{cases}$$

Finally, substituting k_i and m_i for $i = 0, 3$ into (9) yields (8).

The integral problem (8) is equivalent to the original problem by Definition 2. The converse follows from Lemma 6 and the fact that $\mathfrak{D}^{\flat} k = 0$ for any constant k.

3. Analytical results

The functional framework for our subsequent study is the Banach space

$$F = \left(v : v \in C([0,1]),\ D^{\delta} v \in C([0,1])\right),$$

and also, consider the norm

$$\|v\|_F = \|v\|_\infty + \left\|D^{\delta} v\right\|_\infty$$

, such that

$$\|v\|_\infty = \max_{\varsigma \in [0,1]} |v(\varsigma)|, \left\|D^{\delta} v\right\|_\infty = \max_{\varsigma \in [0,1]} |D^{\delta} v(\varsigma)|.$$

Additionally, we present the product space $F \times F$, whose norm is provided by:

$$\|(v,w)\|_{F\times F} = \|v\|_F + \|w\|_F.$$

Let us consider the following sufficient conditions:

- The functions $\mathcal{S}_i, \mathcal{R}_i : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous for $i = 1,2$.
- There exist positive constants $l_{ij} > 0,\ i = 1,2,\ j = 1,2,3$, such that for all $\varsigma \in [0,1]$ and $(y,z,x), (y',z',x') \in \mathbb{R}^3$, we have
  $|\mathcal{S}_i(\varsigma,y,z,x) - \mathcal{S}_i(\varsigma,y',z',x')| \le l_{i1}|y-y'| + l_{i2}|z-z'| + l_{i3}|x-x'|,$
  where $L_i = \max\{l_{i1}, l_{i2}, l_{i3}\}$ for $i = 1,2$.
- There exist positive constants $h_{ij} > 0,\ i = 1,2,\ j = 1,2,3$, such that for all $\varsigma \in [0,1]$ and $(y,z,x), (y',z',x') \in \mathbb{R}^3$, we have
  $|\mathcal{R}_i(\varsigma,y,z,x) - \mathcal{R}_i(\varsigma,y',z',x')| \le h_{i1}|y-y'| + h_{i2}|z-z'| + h_{i3}|x-x'|,$
  where $H_i = \max\{h_{i1}, h_{i2}, h_{i3}\}$ for $i = 1,2$.

We define the following constants:

$$\Gamma' = \int_0^{\mu_1} \frac{k(s)}{\Gamma(\natural_i+\natural_i+1)}\,ds - \frac{1}{\Gamma(\natural_i+\natural_i+1)}, \mathcal{K} = \left(1 - \int_0^{\mu_1} k(s)ds\right)^{-1},$$

$$K_m = \max_{\varsigma \in [0,1]} |K(\varsigma)|,$$

we shall take :

$$\begin{aligned}
\tau_1 &= \left(\frac{|\mathcal{K}|+|\mathcal{K}||K_m|+1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \frac{|\mathcal{K}|\Gamma(\natural_1+1)\Gamma\prime}{\Gamma(\flat_1+\natural_1+1)} + \frac{\Gamma(\natural_1+1)}{(\Gamma(\flat_1+\natural_1+1))^2}\right)\cdot H_1,\\
\tau_2 &= \left(\frac{1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)} + \frac{2}{2-\delta}\cdot\frac{|\mathcal{K}||K_m|+|\mathcal{K}|}{\Gamma(\flat_1+\natural_1+\natural_1+1)}\right.\\
&\quad \left. + \frac{\Gamma(\natural_1+1)}{\Gamma(\natural_1+\natural_1+1)}\cdot\frac{\Gamma(\natural_1+\natural_1+1)}{\Gamma(\natural_1+\natural_1+1-\delta)} + \frac{|\mathcal{K}|\Gamma(\natural_1+1)\Gamma\prime}{\Gamma(\flat_1+\natural_1+1)}\cdot\frac{\Gamma(2)}{\Gamma(2-\delta)}\right)\cdot H_1,\\
\tau_3 &= \left(\frac{|\mathcal{K}|+|\mathcal{K}||K_m|+1}{\Gamma(\flat_2+\natural_2+\natural_2+1)} + \frac{|\mathcal{K}|\Gamma(\natural_2+1)\Gamma\prime}{\Gamma(\flat_2+\natural_2+1)} + \frac{\Gamma(\natural_2+1)}{(\Gamma(\flat_2+\natural_2+1))^2}\right)\cdot H_2,\\
\tau_4 &= \left(\frac{1}{\Gamma(\flat_2+\natural_2+\natural_2+1-\delta)} + \frac{2}{2-\delta}\cdot\frac{|\mathcal{K}||K_m|+|\mathcal{K}|}{\Gamma(\flat_2+\natural_2+\natural_2+1)}\right.\\
&\quad \left. + \frac{\Gamma(\natural_2+1)}{\Gamma(\natural_2+\natural_2+1)}\cdot\frac{\Gamma(\natural_2+\natural_2+1)}{\Gamma(\natural_2+\natural_2+1-\delta)} + \frac{|\mathcal{K}|\Gamma(\natural_2+1)\Gamma\prime}{\Gamma(\flat_2+\natural_2+1)}\cdot\frac{\Gamma(2)}{\Gamma(2-\delta)}\right)\cdot H_2,\\
\Theta &= \tau_1 + \tau_2 + \tau_3 + \tau_4.
\end{aligned}$$

3.1 *Exactly One Solution to system (1)-(2)*

This section starts by defining the operator. $T: F \times F \to F \times F$ as follows

$$T(v_1(\varsigma), v_2(\varsigma)) = (T_1(v_1(\varsigma), v_2(\varsigma)), T_2(w_1(\varsigma), w_2(\varsigma))),$$

where for $i = 1, 2$, we have:

$$\begin{aligned}
&T_i(v_1(\varsigma), v_2(\varsigma)) \ = \mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big),\\
&+\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\\
&+\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\big)\tfrac{\Gamma(\natural_i+1)}{\Gamma(\natural_i+\natural_i+1)}\varsigma^{\natural_i+\natural_i}\\
&+[\textstyle\int_0^{\mu_i} k(s)\mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\\
&-\mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(1, v_1(1), v_2(1), D^{\delta}v_i(1)\big), +\mathcal{R}_i(\varsigma, v_i(1), I^{p}v_i(1), D^{\delta}v_i(1))\\
&+\Gamma(\natural_i+1)\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\big)\\
&\times\left(\textstyle\int_0^{\mu_1}\frac{k(s)}{\Gamma(\natural_i+\natural_i+1)}\,\mathrm{d}s - \frac{1}{\Gamma(\natural_i+\natural_i+1)}\right)\\
&+\textstyle\int_0^{\mu_1}\xi_i k(s)\,\mathrm{d}s - \xi_i]\big(1-\int_0^{\mu_1}k(s)\,\mathrm{d}s\big)^{-1}\varsigma + \xi_i.
\end{aligned}$$

Theorem 9: *Assume that conditions* (G_2) *and* (G_3) *hold, and let* $\Theta = \tau_1 + \tau_2 + \tau_3 + \tau_4$. *If* $\Theta < 1$, *then the system of equations (1)-(2) has a unique solution* (u_1, u_2) *over* $[0,1]$.

As F is a Banach space and $T: F \times F \to F \times F$, stability is trivial.

Let $x \in [0,1]$, we have . according to the Lipschitz conditions (G_i), for $i = 2, 3$, we obtain

$$\begin{aligned}
\|T_1(v_1, v_2) - T_1(w_1, w_2)\|_{\infty} \leq &\left(\tfrac{|\mathcal{H}|+|\mathcal{H}||K_m|+1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \tfrac{|\mathcal{H}|\Gamma(\natural_1+1)\Gamma'}{\Gamma(\flat_1+\natural_1+1)} + \tfrac{|\Gamma(\natural_1+1)}{(\Gamma(\flat_1+\natural_1+1))^2}\right)\\
&\times\begin{bmatrix}(l_{11}+2h_{11})\|v_1 - w_1\|_{\infty} + l_{12}\|v_2 - w_2\|_{\infty}\\ +(h_{12}+l_{13})\big\|D^{\delta}v_1 - D^{\delta}w_1\big\|_{\infty}\end{bmatrix}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|T_1(v_1, v_2) - T_1(w_1, w_2)\|_{\infty} \leq &\left(\tfrac{|\mathcal{H}|+|\mathcal{H}||K_m|+1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \tfrac{|\mathcal{H}|\Gamma(\natural_1+1)\Gamma'}{\Gamma(\flat_1+\natural_1+1)} + \tfrac{|\Gamma(\natural_1+1)}{(\Gamma(\flat_1+\natural_1+1))^2}\right)\\
&\times\begin{bmatrix}(L_1+H_1)\|v_1 - w_1\|_F\\ +L_1\|v_2 - w_2\|_F\end{bmatrix}\\
\big\|T_1\big(v_1(\varsigma), v_2(\varsigma)\big) - T_1\big(w_1(\varsigma), w_2(\varsigma)\big)\big\|_{\infty} &\leq \tau_1\|(v_1(\varsigma), v_2(\varsigma)) - (w_1(\varsigma), w_2(\varsigma)\|_{F\times F}.
\end{aligned} \tag{12}$$

By applying the Caputo derivative, we arrive at

$$\begin{aligned}
&\mathfrak{D}^{\delta}T_1(v_1(\varsigma), v_2(\varsigma))\\
&\quad= \mathfrak{I}^{\flat_1+\natural_1+\natural_1-\delta}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_1, I^{p}v_1(\varsigma), D^{\delta}v_1(\varsigma))\\
&\quad+\big(\vartheta_1 - \mathfrak{I}^{\flat_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_1, I^{p}v_1, D^{\delta}v_1)\big)
\end{aligned}$$

$$\times \left(\frac{\Gamma(\natural_1+1)}{\Gamma(\natural_1+\natural_1+1)}\frac{\Gamma(\natural_1+\natural_1+1}{\Gamma(\natural_1+\natural_1+1-\delta)}\right)\varsigma^{\natural_1+\natural_1-\delta}$$
$$+\left[\int_0^{\mu_1} k(s)\Im^{\flat_1+\natural_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_1(\varsigma), I^p v_1(\varsigma), D^\delta v_1(\varsigma))\right.$$
$$-\Im^{\flat_1+\natural_1+\natural_1}\mathcal{S}_1\big(1, v_1(1), v_2(1), D^\delta v_1(1)\big), +\mathcal{R}_1(\varsigma, v_1(1), I^p v_1(1), D^\delta v_1(1))$$
$$+\Gamma(\natural_1+1)\big(\vartheta_1-\Im^{\flat_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_1(\varsigma)\big)+\mathcal{R}_1(\varsigma, v_1(\varsigma), I^p v_1(\varsigma), D^\delta v_1(\varsigma))\big)$$
$$\times\left(\int_0^{\mu_1}\frac{k(s)}{\Gamma(\natural_1+\natural_1+1)}\,\mathrm{d}s-\frac{1}{\Gamma(\natural_1+\natural_1+1)}\right)$$
$$+\int_0^{\mu_1}\xi_1 k(s)\,\mathrm{d}s-\xi_1]\big(1-\int_0^{\mu_1}k(s)\,\mathrm{d}s\big)^{-1}\frac{\Gamma(2)}{\Gamma(2-\delta)}\varsigma.$$

applying the same reasoning as in (3.1), and Due to the Lipschitz conditions (G_i), for $i = 2,3$, we have

$$\begin{aligned}&\left\|\mathfrak{D}^\delta T_1(v_1,v_2)-\mathfrak{D}^\delta T_1(w_1,w_2)\right\|_\infty\\&\le\left(\frac{1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)}+\frac{2}{2-\delta}\cdot\frac{|\mathcal{K}|K_m+|\mathcal{K}|}{\Gamma(\flat_1+\natural_1+\natural_1+1)}\right.\\&\left.+\frac{\Gamma(\natural_1+1)}{\Gamma(\natural_1+\natural_1+1)}\cdot\frac{\Gamma(\natural_1+\natural_1+1)}{\Gamma(\natural_1+\natural_1+1-\delta)}+\frac{|\mathcal{K}|\Gamma(\natural_1+1)\Gamma\prime}{\Gamma(\flat_1+\natural_1+1)}\cdot\frac{\Gamma(2)}{\Gamma(2-\delta)}\right)\\&\times[(l_{11}+2h_{11})\|v_1-w_1\|_\infty+l_{12}\|v_2-w_2\|_\infty\\&+(h_{12}+l_{13})\left\|D^\delta v_1-D^\delta w_1\right\|_\infty].\end{aligned}$$

hence

$$\begin{aligned}&\left\|\mathfrak{D}^\delta T_1(v_1,v_2)-\mathfrak{D}^\delta T_1(w_1,w_2)\right\|_\infty\\&\le\left(\frac{1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)}+\frac{2}{2-\delta}\cdot\frac{|\mathcal{K}|K_m+|\mathcal{K}|}{\Gamma(\flat_1+\natural_1+\natural_1+1)}\right.\\&\left.+\frac{\Gamma(\natural_1+1)}{\Gamma(\natural_1+\natural_1+1)}\cdot\frac{\Gamma(\natural_1+\natural_1+1)}{\Gamma(\natural_1+\natural_1+1-\delta)}+\frac{|\mathcal{K}|\Gamma(\natural_1+1)\Gamma\prime}{\Gamma(\flat_1+\natural_1+1)}\cdot\frac{\Gamma(2)}{\Gamma(2-\delta)}\right)\\&\times[(L_1+H_1)\mathrm{P}v_1-w_1\mathrm{P}_F+L_1\mathrm{P}v_2-w_2\mathrm{P}_F].\end{aligned}$$

Consequently, we get

$$\left\|\mathfrak{D}^\delta T_1(v_1,v_2)-\mathfrak{D}^\delta T_1(w_1,w_2)\right\|_\infty\le\tau_2\left\|\big(v_1(\varsigma),v_2(\varsigma)\big)-\big(w_1(\varsigma),w_2(\varsigma)\big)\right\|_{F\times F}. \tag{13}$$

Because of (12) and (13) we have

$$\begin{aligned}\left\|\mathfrak{D}^\delta T_1(v_1,v_2)-\mathfrak{D}^\delta T_1(w_1,w_2)\right\|_F&=\|T_1(v_1,v_2)-T_1(w_1,w_2)\|_\infty\\&+\left\|\mathfrak{D}^\delta T_1(v_1,v_2)-\mathfrak{D}^\delta T_1(w_1,w_2)\right\|_\infty\\&\le(\tau_1+\tau_2)\|(v_1(\varsigma),v_2(\varsigma))-(w_1(\varsigma),w_2(\varsigma))\|_{F\times F}.\end{aligned}$$

The following estimations on T_2 are also available.

$$\begin{aligned}\|T_2(v_1,v_2)-T_2(w_1,w_2)\|_\infty\le&\\&\left(\frac{|\mathcal{K}|+|\mathcal{K}||K_m|+1}{\Gamma(\flat_2+\natural_2+\natural_2+1)}+\frac{|\mathcal{K}|\Gamma(\natural_2+1)\Gamma'}{\Gamma(\flat_2+\natural_2+1)}+\frac{|\Gamma(\natural_2+1)}{(\Gamma(\flat_2+\natural_2+1))^2}\right)\\&\times\begin{bmatrix}(L_2+H_2)\|v_2-w_2\|_F\\+L_2\|v_1-w_1\|_F.\end{bmatrix}\end{aligned}$$

$$\|T_2(v_1(\varsigma), v_2(\varsigma)) - T_2(w_1(\varsigma), w_2(\varsigma))\|_\infty \le \tau_3 \|(v_1(\varsigma), v_2(\varsigma)) - (w_1(\varsigma), w_2(\varsigma)\|_{F\times F}. \tag{14}$$

And

$$\begin{aligned}\left\|\mathfrak{D}^\delta T_2(v_1, v_2) - \mathfrak{D}^\delta T_2(w_1, w_2)\right\|_\infty \\ \le \Big(\frac{1}{\Gamma(\flat_2+\natural_2+\natural_2+1-\delta)} + \frac{2}{2-\delta}\cdot\frac{|\mathcal{K}|K_m+|\mathcal{K}|}{\Gamma(\flat_2+\natural_2+\natural_2+1)} \\ + \frac{\Gamma(\natural_2+1)}{\Gamma(\natural_2+\natural_2+1)}\cdot\frac{\Gamma(\natural_2+\natural_2+1)}{\Gamma(\natural_2+\natural_2+1-\delta)} + \frac{|\mathcal{K}|\Gamma(\natural_2+1)\Gamma\prime}{\Gamma(\flat_2+\natural_2+1)}\cdot\frac{\Gamma(2)}{\Gamma(2-\delta)}\Big) \\ \times [(L_2+H_2)\mathrm{P}v_2 - w_2\mathrm{P}_F + L_2\mathrm{P}v_1 - w_1\mathrm{P}_F].\end{aligned}$$

$$\left\|\mathfrak{D}^\delta T_2(v_1, v_2) - \mathfrak{D}^\delta T_2(w_1, w_2)\right\|_\infty \le (\tau_3+\tau_4)\|(v_1(\varsigma), v_2(\varsigma)) - (w_1(\varsigma), w_2(\varsigma)\|_{F\times F}. \tag{15}$$

Thus

$$\left\|T(v_1, v_2) - T_(w_1, w_2)\right\|_{F\times F} \le (\tau_1+\tau_2+\tau_3+\tau_4)\|(v_1(\varsigma), v_2(\varsigma)) - (w_1(\varsigma), w_2(\varsigma)\|_{F\times F}. \tag{16}$$

3.2 *Stability of solutions to system (1)-(2)*

We first recall the concept of Ulam-Hyers stability, which is essential to the study's analytical framework.

Definition 10: The coupled system (1)-(2) is said to be Ulam-Hyers stable if there exists a real constant $N > 0$ such that the following holds:

For every $\iota_1, \iota_2 > 0$ and for every functions (u_1, u_2) in the space F^2 that satisfies the inequalities:

$$\left|\mathfrak{D}^{\#_1}\mathfrak{D}^{\#_1}\mathfrak{D}^{\#_1}v_1(\varsigma) - \mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_1(\varsigma)\big) - \mathcal{R}_1\big(\varsigma, v_1(\varsigma), I^p v_1(\varsigma), D^\delta v_1(\varsigma)\big)\right| \le \iota_1, \tag{17}$$

$$\left|\mathfrak{D}^{\#_2}\mathfrak{D}^{\#_2}\mathfrak{D}^{\#_2}v_2(\varsigma) - \mathcal{S}_2\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_2(\varsigma)\big) + \mathcal{R}_2(\varsigma, v_2(\varsigma), I^p v_2(\varsigma), D^\delta v_2(\varsigma))\right| \le \iota_2,$$

(under the same boundary conditions as given in (2)), there exists a genuine solution (v_1^*, v_2^*) in F^2 to the original system (1)-(2) for which

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \le N(\iota_1+\iota_2).$$

Let us prove the following theorem concerning the Ulam-Hyers stability of our problem.

Theorem 11: *If the conditions of Theorem (9) are satisfied, then the system (1)-(2) is Ulam -Hyers stable.*

Proof: Let $(v_1, v_2) \in F \times F$ be a solution of equation (28), and denote by $(v_1', v_2') \in F \times F$ the unique solution to equation (1). From (17), it follows that

$$|v_1(x) - \mathfrak{J}^{\flat_1+\natural_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_1(\varsigma), I^p v_1(\varsigma), D^\delta v_1(\varsigma))$$

$$+\big(\vartheta_1 - \mathfrak{I}^{\flat_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_1(\varsigma))\big)\frac{\Gamma(\natural_1+1)}{\Gamma(\natural_1+\natural_1+1)}\varsigma^{\natural_1+\natural_1}$$
$$+[\int_0^{\mu_1} k(s)\mathfrak{I}^{\flat_1+\natural_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_1(\varsigma), I^{p}v_1(\varsigma), D^{\delta}v_1(\varsigma))$$
$$-\mathfrak{I}^{\flat_1+\natural_1+\natural_1}\mathcal{S}_1\big(1, v_1(1), v_2(1), D^{\delta}v_1(1)\big), +\mathcal{R}_1(\varsigma, v_i(1), I^{p}v_i(1), D^{\delta}v_1(1))$$
$$+\Gamma(\natural_1+1)\big(\vartheta_1 - \mathfrak{I}^{\flat_1+\natural_1}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big), +\mathcal{R}_1(\varsigma, v_i(\varsigma), I^{p}v_1(\varsigma), D^{\delta}v_1(\varsigma))\big)$$
$$\times\left(\int_0^{\mu_1}\frac{k(s)}{\Gamma(\natural_1+\natural_1+1)}\,\mathrm{d}s - \frac{1}{\Gamma(\natural_1+\natural_1+1)}\right)$$
$$+\int_0^{\mu_1}\xi_1 k(s)\,\mathrm{d}s - \xi_1]\big(1-\int_0^{\mu_1}k(s)\,\mathrm{d}s\big)^{-1}\varsigma + \xi_1|. \le \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1)}, \tag{18}$$

and

$$|u_2(x) - \mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))$$
$$+\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\big)\frac{\Gamma(\natural_i+1)}{\Gamma(\natural_i+\natural_i+1)}\varsigma^{\natural_i+\natural_i}$$
$$+[\int_0^{\mu_i} k(s)\mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))$$
$$-\mathfrak{I}^{\flat_i+\natural_i+\natural_i}\mathcal{S}_i\big(1, v_1(1), v_2(1), D^{\delta}v_i(1)\big), +\mathcal{R}_i(\varsigma, v_i(1), I^{p}v_i(1), D^{\delta}v_i(1))$$
$$+\Gamma(\natural_i+1)\big(\vartheta_i - \mathfrak{I}^{\flat_i+\natural_i}\mathcal{S}_i\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_i(\varsigma)\big), +\mathcal{R}_i(\varsigma, v_i(\varsigma), I^{p}v_i(\varsigma), D^{\delta}v_i(\varsigma))\big)$$
$$\times\left(\int_0^{\mu_1}\frac{k(s)}{\Gamma(\natural_i+\natural_i+1)}\,\mathrm{d}s - \frac{1}{\Gamma(\natural_i+\natural_i+1)}\right)$$
$$+\int_0^{\mu_1}\xi_i k(s)\,\mathrm{d}s - \xi_i]\big(1-\int_0^{\mu_1}k(s)\,\mathrm{d}s\big)^{-1}\varsigma + \xi_i| \le \frac{\iota_2}{\Gamma(\flat_2+\natural_2+1)}.$$

By (17) and (18), we obtain the following three inequalities

$$\begin{aligned}\|(v_1 - v_1^*\|_\infty &= \|-T_1v_1 + v_1 + T_1v_1 - Y_1v_1\|_\infty\\ &\le \|v_1 - T_1v_1\|_\infty + \|T_1v_1 - T_1v^*\|_\infty\\ &\le \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \tau_1(\|(v_1) - (v_1^*)\|_F),\end{aligned}$$

$$\begin{aligned}\big\|\mathfrak{D}_1^{\delta}(v_1) - \mathfrak{D}_1^{\delta}(v_1^*)\big\|_\infty &\\ \le \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)} &+ \tau_2(\|(v_1) - (v_1^*)\|_F)\end{aligned}$$

So

$$\begin{aligned}\|(v_1 - v_1^*\|_F \le \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)}&\\ +\tau_1 + \tau_2(\|(v_1(\varsigma)) - (v_1^*(\varsigma))\|_F).&\end{aligned} \tag{20}$$

On the other hand, we have

$$\begin{aligned}\|v_2 - v_2^*\|_\infty &= \|-T_2v_2 + v_2 + T_2v_2 - T_2v_2^*\|_\infty\\ &\le \|v_2 - T_2v_2\|_\infty + \|T_2v_2 - T_2v_2^*\|_\infty\\ &\le \frac{\iota_2}{\Gamma(\flat_2+\natural_2+\natural_2+1)} + \tau_3(\|(v_2) - (v_2^*)\|_F),\end{aligned}$$

$$\left\|\mathfrak{D}_2^{\delta}(v_2) - \mathfrak{D}_2^{\delta}(v_2^*)\right\|_{\infty}$$
$$\leq \frac{\iota_2}{\Gamma(\flat_2 + \natural_2 + \natural_2 + 1 - \delta)} + \tau_4(\|(u_2) - (u_2^*)\|_F)$$

Hence,

$$\|(v_2 - (v_2^*)\|_F \leq \frac{\iota_2}{\Gamma(\flat_2+\natural_2+\natural_2+1)} + \frac{\iota_2}{\Gamma(\flat_2+\natural_2+\natural_2+1-\delta)} \quad (21)$$
$$+(\tau_3 + \tau_4)(\|(v_2) - (v_2^*)\|_F).$$

Combining equations (20) and (21) yields the final result

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1)} + \frac{\iota_1}{\Gamma(\flat_1+\natural_1+\natural_1+1-\delta)}$$
$$+ \frac{\iota_2}{\Gamma(\flat_2+\natural_2+\natural_2+1)} + \frac{\iota_2}{\Gamma(\flat_2+\natural_2+\natural_2+1-\delta)}$$
$$+(\tau_1 + \tau_2 + \tau_3 + \tau_4) \times (\|(v_1, v_2) - (v_1^*, v_2^*)\|_{F\times F}).$$

Thus,

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq (\iota_1 + \iota_2) \times \mathcal{A} + \Theta(\|(v_1, v_2) - (v_1^*, v_2^*)\|_{F\times F}),$$

where,

$$\mathcal{A} = \sum_{i=1}^{2} \frac{1}{\Gamma(\flat_i+\natural_i+\natural_i+1)} + \frac{1}{\Gamma(\flat_i+\natural_i+\natural_i+1-\delta)}.$$

Consequently,

$$(1 - \Theta) \times \|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq (\iota_1 + \iota_1) \times \mathcal{A}.$$

Consequently, we find that

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq \frac{\mathcal{A}}{(1-\Theta)} \times (\iota_1 + \iota_2).$$

Finally,

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq N \times (\iota_1 + \iota_2).$$

Consequently, the system defined by equations (1) and (2) is stable in the sense of Ulam-Hyers.

3.3 *Example*

Let consider the problem

$$\begin{cases} \mathfrak{D}^{\frac{3}{4}}\mathfrak{D}^{\frac{1}{2}}\mathfrak{D}^{\frac{1}{4}}v_1(\varsigma) = \mathcal{S}_1\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_1(\varsigma)\big) + \mathcal{R}_1\big(\varsigma, v_1(\varsigma), I^{p}v_1(\varsigma), D^{\delta}v_1(\varsigma)\big), \\ \mathfrak{D}^{\frac{2}{3}}\mathfrak{D}^{\frac{1}{2}}\mathfrak{D}^{\frac{1}{3}}v_2(\varsigma) = \mathcal{S}_2\big(\varsigma, v_1(\varsigma), v_2(\varsigma), D^{\delta}v_2(\varsigma)\big) + \mathcal{R}_2\big(\varsigma, v_2(\varsigma), I^{p}v_2(\varsigma), D^{\delta}v_2(\varsigma)\big), \end{cases}$$

under the conditions:

$$\begin{cases} v_1(0) = 2, v_2(0) = 1, \\ \mathfrak{D}^{\frac{1}{4}}v_1(0) = 0, \mathfrak{D}^{\frac{1}{3}}v_2(0) = 0, \\ v_1(1) = \int_0^{0.3} \frac{1}{5}v_1(\varsigma)\mathrm{d}\varsigma, v_2(1) = \int_0^{0.4} \frac{1}{6}v_2(\varsigma)\mathrm{d}\varsigma, \\ \mathfrak{D}^{\frac{1}{4}}v_1(1) = 0.1, \mathfrak{D}^{\frac{1}{3}}v_2(1) = 0.2, \end{cases}$$

where,

$$\flat_1 = \frac{3}{4}, \natural_1 = \frac{1}{2}, \natural_1 = \frac{1}{4}, \flat_2 = \frac{2}{3}, \natural_2 = \frac{1}{2}, \natural_2 = \frac{1}{3}, \delta = 0.8,$$

and

$$\mathcal{S}_1(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_1(\varsigma)) = \frac{1}{400}\left(\frac{\cos(v_1(\varsigma))}{e^{\varsigma+3}} + \frac{v_2(\varsigma)}{e^{\varsigma+5}(1+|v_2(\varsigma)|)} + \frac{D^\delta v_1(\varsigma)}{e^{\varsigma+4}}\right),$$

$$\mathcal{R}_1(\varsigma, v_1(\varsigma), I^p v_1(\varsigma), D^\delta v_1(\varsigma)) = \frac{1}{500}\left(\frac{\arctan(I^p v_1(\varsigma))}{e^{\varsigma+4}} + \frac{v_1(\varsigma)\cdot D^\delta v_1(\varsigma)}{e^{\varsigma+2}(1+|v_1(\varsigma)\cdot D^\delta v_1(\varsigma)|)}\right),$$

$$\mathcal{S}_2(\varsigma, v_1(\varsigma), v_2(\varsigma), D^\delta v_2(\varsigma)) = \frac{1}{600}\left(\frac{v_1(\varsigma)}{e^{\varsigma+5}(2+|v_1(\varsigma)|)} + \frac{\sin(v_2(\varsigma))}{e^{\varsigma+5}} + \frac{D^\delta v_2(\varsigma)}{e^{\varsigma+5}(1+|D^\delta v_2(\varsigma)|)}\right),$$

$$\mathcal{R}_2(\varsigma, v_2(\varsigma), I^p v_2(\varsigma), D^\delta v_2(\varsigma)) = \frac{1}{700}\left(\frac{\tanh(I^p v_2(\varsigma))}{e^{\varsigma+3}} + \frac{v_2(\varsigma)\cdot D^\delta v_2(\varsigma)}{e^{\varsigma+3}(3+|v_2(\varsigma)\cdot D^\delta v_2(\varsigma)|)}\right).$$

The functions $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{R}_1$, and $\mathcal{R}_2$ satisfy the Lipschitz conditions with constants:

$$H_1 = 0.000677, H_2 = 0.0000711.$$

After detailed calculations, we obtain the following bounds:

$$\tau_1 = 0.002214,\ \tau_2 = 0.003039,\ \tau_3 = 0.000284,\ \tau_4 = 0.000353,$$
$$\Theta = \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0.005890 < 1.$$

Since $\Theta = 0.005890 < 1$, all conditions of our existence and uniqueness theorem are satisfied. Therefore, the coupled system of fractional differential equations admits a unique solution.

3.4 *Example*

Consider the following coupled fractional differential system:

$$\begin{cases} \mathfrak{D}^{0.4}\mathfrak{D}^{0.7}\mathfrak{D}^{1.3}v_1(\varsigma) = \frac{1}{180} \times \left(\frac{e^{-\varsigma}v_1(\varsigma)}{1+|v_1(\varsigma)|} + \frac{\arctan(v_2(\varsigma))}{3+\varsigma^2} + \frac{\mathfrak{D}^{0.8}v_1(\varsigma)}{4+|\mathfrak{D}^{0.8}v_1(\varsigma)|^2}\right) \\ \mathfrak{D}^{0.3}\mathfrak{D}^{0.8}\mathfrak{D}^{1.5}v_2(\varsigma) = \frac{1}{200} \times \left(\frac{\ln(1+|v_1(\varsigma)+v_2(\varsigma)|)}{5+\varsigma^3} - \frac{\sin(\mathfrak{D}^{0.8}v_2(\varsigma))}{6+|\mathfrak{D}^{0.8}v_2(\varsigma)|}\right) \\ v_1(0) = v_1(1) = 2, v_1,(0) = v_1,(1) = 0 \\ v_2(0) = v_2(1) = 4, v_2,(0) = v_2,(1) = 0 \end{cases}$$

where

$$\flat_1 = 0.4, \quad \natural_1 = 0.7, \quad \natural_1 = 1.3,$$
$$\flat_2 = 0.3, \quad \natural_2 = 0.8, \quad \natural_2 = 1.5, \delta = 0.8$$

The nonlinear functions are defined as:

$$\mathcal{S}_1(\varsigma, v_1, v_2, \mathfrak{D}^\delta v_1) = \frac{1}{180} \times \left(\frac{e^{-\varsigma}v_1}{1+|v_1|} + \frac{\arctan(v_2)}{3+\varsigma^2}\right)$$

$$\mathcal{R}_1(\varsigma, v_1, I^p v_1, \mathfrak{D}^\delta v_1) = \frac{1}{180} \times \frac{\mathfrak{D}^\delta v_1}{4+|\mathfrak{D}^\delta v_1|^2}$$

$$\mathcal{S}_2(\varsigma, v_1, v_2, \mathfrak{D}^\delta v_2) = \frac{1}{200} \times \frac{\ln(1+|v_1+v_2|)}{5+\varsigma^3}$$

$$\mathcal{R}_2(\varsigma, v_2, I^p v_2, \mathfrak{D}^\delta v_2) = \frac{1}{200} \times \frac{\sin(\mathfrak{D}^\delta v_2)}{6+|\mathfrak{D}^\delta v_2|}$$

All functions are satisfy Lipschitz conditions with constants:

$$\tau_1 = 0.0028, \tau_2 = 0.0019,$$
$$\tau_3 = 0.0035, \tau_4 = 0.0024$$

So

$$\Theta = \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0.0106 < 1$$

And :

$$\mathcal{A} = 2.1258$$
$$N = \frac{\mathcal{A}}{1-\Theta} = \frac{2.1258}{1-0.0106} = 2.1486$$

Finally, for any $\iota_1, \iota_2 > 0$ and any approximate solution (v_1, v_2) satisfying the perturbation inequalities, there exists an exact solution (v_1^*, v_2^*) such that:

$$\|(v_1 - v_1^*, v_2 - v_2^*)\|_{F\times F} \leq 2.1486 \times (\iota_1 + \iota_2).$$

We conclude that the system is Ulam-Hyers stable, characterized by the stability constant $N = 2.1486$.

4. Traveling wave solutions

In this section, we apply the well known Tanh method [23], in the case of Khalil derivative, see [12] to obtain eventual travelling wave solutions for the following system:

$$\begin{cases} T_\varsigma^{2\flat} u^1(\varsigma, x) + T_x^{4\natural} u^1(\varsigma, x) + G_1(u^1, u^2, T_x^\natural(u^1, u^2)(\varsigma, x), T_x^{2\natural}(u^1, u^2)(\varsigma, x) = 0, \\ T_\varsigma^{2\flat} u^2(\varsigma, x) + T_x^{4\natural} u^2(\varsigma, x) + G_2(u^1, u^2, T_x^\natural(u^1, u^2)(\varsigma, x), T_x^{2\natural}(u^1, u^2)(\varsigma, x) = 0. \end{cases} \tag{22}$$

where G_1, G_2, are two continuous functions and $0 < \flat, \natural \leq 1$, $T_\varsigma^\flat T_x^\natural$ are the conformable fractional derivatives in the sense of Khalil [20], and u is the solution.

We recall now some definitions related the Khalil derivative [20].

Definition 12: The conformable fractional derivative in the sense of Khalil is defined as follows:

$$T_x^\natural u(\varsigma, x) = \frac{\partial^\natural u(\varsigma,x)}{\partial x^\natural} = \lim_{\varepsilon\to 0}\left(\frac{u(\varsigma+\varepsilon\varsigma^{1-\flat},x)-u(x,\varsigma)}{\varepsilon}\right), \varsigma > 0, 0 < \flat \leq 1.$$

We define $T_\varsigma^\# u(\varsigma, x)$ the same way as in the definition above.

Remark: If we take $\flat = \natural = 1$, the derivative of Khalil coincides with the normal derivative and the system (22) takes the following form see [18].

$$\begin{cases} u_{\varsigma\varsigma}^1 + u_{xxxx}^1 + G_1(u^1, u^2, (u^1, u^2)_x, (u^1, u^2)_{xx}) = 0, \\ u_{\varsigma\varsigma}^2 + u_{xxxx}^2 + G_2(u^1, u^2, (u^1, u^2)_x, (u^1, u^2)_{xx}) = 0. \end{cases}$$

Now, We present the overview of tanh method for fractional Khalil derivatives, see [12].

Step 1: We begin by considering the following nonlinear evolutionary system

$$\begin{cases} \mathcal{H}_1\left(u^1, u^2, T_\varsigma^\flat u^1, T_x^\natural u^1, T_\varsigma^\flat u^2, T_x^\natural u^2, T_\varsigma^{2\flat} u^1, T_\varsigma^\flat\left(T_x^\natural u^1\right), T_x^{2\natural} u^1, T_\varsigma^{2\flat} u^2, T_\varsigma^\flat\left(T_x^\natural u^2\right), T_x^{2\natural} u^2, \ldots\right) = 0, \\ \mathcal{H}_2\left(u^1, u^2, T_\varsigma^\flat u^1, T_x^\natural u^1, T_\varsigma^\flat u^2, T_x^\natural u^2, T_\varsigma^{2\flat} u^1, T_\varsigma^\flat\left(T_x^\natural u^1\right), T_x^{2\natural} u^1, T_\varsigma^{2\flat} u^2, T_\varsigma^\flat\left(T_x^\natural u^2\right), T_x^{2\natural} u^2, \ldots\right) = 0. \end{cases} \tag{23}$$

Step 2: We use the following change of variable to transform the fractional derivatives to the ordinary sell

$$Y = \frac{\lambda}{\flat}\varsigma^{\flat} + \frac{\gamma}{\natural}x^{\natural}. \tag{24}$$

Step 3: Now, we introduce a second change of variable

$$Z = tanh(Y). \tag{25}$$

According to (25) we obtain the following changes in the derivatives:

$$\frac{d}{dY} = (1 - Z^2)\frac{d}{dZ},$$
$$\frac{d^2}{dY^2} = -2Z(1 - Z^2)\frac{d}{dZ} + (1 - Z^2)^2\frac{d^2}{dZ^2},$$
$$\frac{d^3}{dY^3} = 2(1 - Z^2)(3Z^2 - 1)\frac{d}{dZ} - 6Z(1 - Z^2)^2,\frac{d^2}{dZ^2} + (1 - Z^2)^3\frac{d^3}{dZ^3}.$$

Step 4: Finally, we look for solutions in the following polynomial form:

$$\begin{cases} u^1(x,\varsigma) = U^1(Y) = L(Z) = \sum_{i=0}^{d_1} c_i Z^i, \\ u^2(x,\varsigma) = U^2(Y) = P(Z) = \sum_{i=0}^{d_2} e_i Z^i. \end{cases} \tag{26}$$

One can obtain the degrees of these polynomials by balancing between the maximum order nonlinear term and the derivative of the maximum order. After that, the solutions can be obtained by solving the resulting system of algebraic equations.

4.1 *Application*

We consider the problem:

$$\begin{aligned} &T_{\varsigma}^{2\flat}u^1 + T_x^{4\natural}u^1 - nT_x^{2\natural}(u^1u^2) - T_x^{2\natural}(u^1)^2 - T_x^{2\natural}(u^1) = 0 \\ &T_{\varsigma}^{2\flat}u^2 + T_x^{4\natural}u^2 - mT_x^{2\natural}(u^1u^2) - T_x^{2\natural}(u^2)^2 - T_x^{2\natural}(u^2) = 0. \end{aligned} \tag{27}$$

We consider $m = n = 1$ to simplify the writing of the solutions.

By the change of first variable which is found in (24), the system (27) becomes

$$\begin{aligned} &\lambda^2 U_{YY}^1 + \gamma^4 U_{YYYY}^1 - \gamma^2(U^1U^2)_{YY} - \gamma^2(U^1)_{YY}^2 - \gamma^2(U_{YY}^1) = 0 \\ &\lambda^2 U_{YY}^2 + \gamma^4 U_{YYYY}^2 - \gamma^2(U^1U^2)_{YY} - \gamma^2(U^1)_{YY}^2 - \gamma^2(U_{YY}^2) = 0, \end{aligned} \tag{28}$$

thanks to integral the system (29), becomes:

$$\begin{aligned} &\lambda^2 U^1 + \gamma^4 U_{YY}^1 - \gamma^2(U^1U^2) - \gamma^2(U^1)^2 - \gamma^2(U^1) = 0 \\ &\lambda^2 U^2 + \gamma^4 U_{YY}^2 - \gamma^2(U^1U^2) - \gamma^2(U^1)^2 - \gamma^2(U^2) = 0, \end{aligned} \tag{29}$$

by using (25) and (26), the system (29) becomes:

$$\lambda^2 L + \gamma^4\left[-2Z(1 - Z^2)\frac{dL}{dZ} + (1 - Z^2)^2\frac{d^2L}{dZ^2}\right] - \gamma^2(LP) - \gamma^2(L)^2 - \gamma^2(L^2), \tag{30}$$

and

$$\lambda^2 P + \gamma^4\left[-2Z(1 - Z^2)\frac{dP}{dZ} + (1 - Z^2)^2\frac{d^2P}{dZ^2}\right] - \gamma^2(PL) - \gamma^2(P)^2 - \gamma^2(P^2). \tag{31}$$

The degree of (30) is given in the form

$$d_1 + 2 = max(2d_1, d_1 + d_2),$$

and The degree of the (31) is

$$d_2 + 2 = max(2d_2, d_1 + d_2),$$

Hence, $d_1 = d_2 = 2$, so the polynomial L and P is degree 2.

On the other hand, we have

$$\begin{aligned}\lambda^2(c_0 + c_1Z + c_2Z^2) + \gamma^4(-2Z(1-Z^2)(c_1 + 2c_2Z) + 2c_2(1-Z^2)^2)\\ -\gamma^2(e_0 + e_1Z + e_2Z^2)(c_0 + c_1Z + c_2Z^2) - \gamma^2(c_0 + c_1Z + c_2Z^2)^2\\ -\gamma^2(c_0 + c_1Z + c_2Z^2) = 0\end{aligned} \tag{32}$$

and

$$\begin{aligned}\lambda^2(e_0 + e_1Z + e_2Z^2) + \gamma^4(-2Z(1-Z^2)(e_1 + 2e_2Z) + 2e_2(1-Z^2)^2)\\ -\gamma^2(e_0 + e_1Z + e_2Z^2)(e_0 + e_1Z + e_2Z^2) - \gamma^2(c_0 + c_1Z + c_2Z^2)^2\\ -\gamma^2(e_0 + e_1Z + e_2Z^2) = 0.\end{aligned} \tag{33}$$

Therefore,

$$\begin{cases} Z^0: 6c_2\gamma^4 - c_2^2\gamma^2 - e_2c_2\gamma^2 = 0,\\ Z^1: 2c_1\gamma^4 - 2c_1c_2\gamma^2 - c_1e_2\gamma^2 - c_2c_1\gamma^2 = 0,\\ Z^2: -\gamma^2c_1^2 - e_1\gamma^2c_1 + c_2\lambda^2 - c_2\gamma^2 - 8c_2\gamma^4 - 2c_0c_2\gamma^2 - c_0e_2\gamma - c_2e_0\gamma^2 = 0,\\ Z^3: c_1\lambda^2 - c_1\gamma^2 - 2c_1\gamma^2 - 2c_1\gamma^4 - 2c_0c_1\gamma^2 - c_0e_1\gamma^2 - c_1e_0\gamma^2 = 0,\\ Z^4: 2c_2\gamma^4 - c_0\gamma^2 + c_0\lambda^2 - c_0^2\gamma^2 - c_0e_0\gamma^2 = 0 \end{cases} \tag{34}$$

and

$$\begin{cases} Z^0: -\gamma^2c^2 - e_2\gamma^2c_2 + 6e_2\gamma^4 = 0,\\ Z^1: 2e_1\gamma^4 - 2c_1c_2\gamma^2 - c_1e_2\gamma^2 - c_2e_1\gamma^2 = 0,\\ Z^2: -\gamma^2c_1^2 - e_1\gamma^2c_1 + e_2\lambda^2 - e_2\gamma^2 - 8e_2\gamma^4 - 2c_0\gamma^2 - c_0e_2\gamma^2 = 0,\\ Z^3: e_1\lambda^2 - e_1\gamma^2 - 2e_1\gamma^4 - 2c_0c_1\gamma^2 - c_0e_1\gamma^2 - c_1e_0\gamma^2 = 0,\\ Z^4: -\gamma^2c_0^2 - e_0\gamma^2c_0 + e_0\lambda^2 - e_0\gamma^2 + 2e_2\gamma^4 = 0. \end{cases} \tag{35}$$

Thanks to Matlab to solve (34), (35), we found the following solutions:

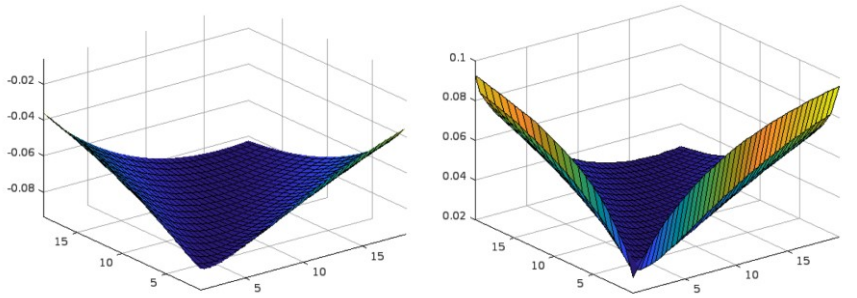
$$\begin{cases} \gamma = \gamma,\\ \lambda = -\gamma,\\ a_0 = -3\gamma^2,\\ a_1 = 0,\\ a_2 = 3\gamma^2.\\ b_0 = \gamma^2,\\ b_1 = 0,\\ b_2 = 3\gamma^2. \end{cases} \tag{36}$$

Finally we obtain the travelling wave solution of (27):

$$u^1(x,\varsigma) = -3\gamma^2 + 3\gamma^2 tanh^2(\tfrac{-\gamma}{\flat}\varsigma^{\flat} + \tfrac{\gamma}{\natural}x^{\natural}), \tag{37}$$

$$u^2(x,\varsigma) = \gamma^2 + 3\gamma^2 tanh^2(\tfrac{-\gamma}{\flat}\varsigma^{\flat} + \tfrac{\gamma}{\natural}x^{\natural}). \tag{38}$$

The plots of the travelling wave solutions we obtained are shown in Figure 1 bellow.



Plot of (37) Plot of (37)

Figure 1

Plot of (37), (38), with $0 \leq x \leq 20, 0 \leq \varsigma \leq 20, \gamma = \frac{1}{6}, \flat = \frac{4}{10}, \natural = \frac{5}{10}$.

5. Conclusion

This study investigates a nonlinear coupled system for beam deflection using Caputo derivatives with novel parameters, and general nonlinearities. We demonstrate that under specific constraints, the system simplifies to a fourth-order ordinary coupled system analogous to classical beam dynamics. Utilizing the Banach contraction principle, we establish the uniqueness of the solution and, under the same assumptions, prove its Ulam-Hyers stability. To validate our findings, we provided an illustrative example for the first main result . An new particular solution of type travelling wave is obtained for the above evolution system involving Khalil derivatives. The graph of the travelling wave is then plotted.

In next research, it would be interesting to investigate the following questions:

1. Analyzing the problem with the incorporation of a Φ-Laplacian operator.
2. Examining the problem posed on an unbounded domain.
3. This problem can be analyzed within the framework of the convex fractional derivative (see [10]).

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