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On class $A_k^{\#}$ operators in semi-Hilbertian spaces

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Abstract

This paper introduces a novel class of operators in semi-Hilbertian spaces, termed class $A_k^{\#}$ operators, which generalizes the well-studied A-hyponormal operators and extends

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recent developments in operator theory. We define an operator Γ to belong to the class $A_k^{\#}$ if there exists a positive integer k such that

$$\Gamma^{\#k}\,\Gamma^k \geq_A (\Gamma\Gamma^{\#})^k$$

Central to this work is the exploration of fundamental properties of this class , its relationship to other operator families. We provide concrete examples to demonstrate the non-triviality and applicability of this class, alongside an in-depth analysis of its stability under tensor product operations. Specifically, we establish sufficient conditions under which the tensor product of two class $A_k^{\#}$ operators preserves the class property. These contributions not unify and extend prior results on operator classes but also offer new insights into their structural and functional.

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1. Introduction

Over the past several decades, the study of operator classes on Hilbert spaces has evolved considerably. Among these, normal operators and hyponormal operators have been thoroughly investigated, forming a central focus of modern operator theory. These classes have been examined in depth by various researchers (see [6],[12], [14])

In very recent years, these classes of operators have been generalized in semi-Hilbertian spaces in this papers and other papers(see [2],[3],[4], [5], [10]).

In this framework, we aim to introduce a novel class of operators in semi-Hilbertian spaces, termed Class $A_k^{\#}$ operators. We begin by defining key terminology. Let $\mathcal{H}$ denote a complex Hilbert space equipped with the inner product $\langle .;. \rangle$ and norm $\|.\|$. The Banach algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})^+$ is the all bounded linear positive operators , i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{\Gamma \in \mathcal{B}(\mathcal{H}); \langle \Gamma v; v \rangle \geq 0, \forall v \in \mathcal{H}\}$$

For any positive operator $A \in \mathcal{B}(\mathcal{H})^+$, a semi-inner product on $\mathcal{H}$ is defined as a semi-definite sesquilinear form, i,e:

$$\begin{aligned} \langle .;. \rangle_A : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (v; v) &\mapsto \langle v; v \rangle_A = \langle Av; v \rangle \end{aligned}$$

the semi-norm induced by $\langle .;. \rangle_A$, which is denoted by $\mathrm{P} \cdot \mathrm{P}_A$ .this makes $(\mathcal{H}; \langle .;. \rangle_A)$ into a semi-Hilbertian space.

Furthermore, $\|.\|_A$ defines a norm if and only if the operator A is injective. Similarly, the semi-normed space $(\mathcal{H}, \|.\|_A)$ is complete precisely when the range of A is closed. Moreover, the subset $\mathcal{B}^A(\mathcal{H})$ is all A-bounded linear operators ,i;e

$$\mathcal{B}^A(\mathcal{H}) = \{\Gamma \in \mathcal{B}(\mathcal{H}); \exists c > 0; \|\Gamma v\|_A \leq c\|v\|_A, \forall v \in \mathcal{H}\}$$

For every $\Gamma \in \mathcal{B}(\mathcal{H})$ its range is denoted by $\mathcal{R}(\Gamma)$, its null space by $\mathcal{N}(\Gamma)$ and its adjoint by Γ^*. If $\mathcal{M} \subset \mathcal{H}$ is a closed subspace, $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$. the closure of $\mathcal{R}(\Gamma)$ will be denoted by $\overline{\mathcal{R}(\Gamma)}$.

Definition 1.1: (see[2]) Let $\Gamma \in \mathcal{B}(\mathcal{H})$, an operator $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of Γ if for every $v, v \in \mathcal{H}$

$$\langle \Gamma v; v\rangle_A = \langle v; \mathcal{T} v\rangle_A$$

i.e., $A\mathcal{T} = \Gamma^* A$.

It is observed that an operator Γ may lack an A-adjoint. However, if $\mathcal{T}$ is an A-adjoint of Γ, there can exist multiple A-adjoints for Γ. Specifically, for any operator $\mathcal{R} \in \mathcal{B}(\mathcal{H})$ satisfying $A\mathcal{R} = 0$, the operators $\mathcal{T} + \mathcal{R}$ also qualifies as an A-adjoint of Γ. The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theory we have that

$$\mathcal{B}_A(\mathcal{H}) = \{\Gamma \in \mathcal{B}(\mathcal{H}); \mathcal{R}(\Gamma^* A) \subset \mathcal{R}(A)\}$$

For any operator $\Gamma \in \mathcal{B}_A(\mathcal{H})$, A-adjoint operator of Γ exists. This operator, denoted as $\Gamma^{\#}$, corresponds to the reduced solution of the equation $AX = \Gamma^* A$, specifically $\Gamma^{\#} = A^{\dagger}\Gamma^* A$ ($A^{\dagger}$ is the Moore-Penrose inverse of A). Thus, $\Gamma^{\#}$ is uniquely defined by this relation.It yields the following results:

$$\mathcal{R}(\Gamma^{\#}) \subset \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(\Gamma^{\#}) = \mathcal{N}(\Gamma^* A).$$

Proposition 1.2: (see [1]) Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$. The following properties hold:

1) $\Gamma^{\#} \in \mathcal{B}_A(\mathcal{H})$.
(2) If $\mathcal{T} \in \mathcal{B}_A(\mathcal{H})$, then $\Gamma\mathcal{T} \in \mathcal{B}_A(\mathcal{H})$ and $(\Gamma\mathcal{T})^{\#} = \mathcal{T}^{\#}\Gamma^{\#}$.
(3) $(\Gamma^{\#})^{\#} = P_{\overline{R(A)}}\Gamma P_{\overline{R(A)}}$ and $[(\Gamma^{\#})^{\#}]^{\#} = \Gamma^{\#}$.
(4) If $\Gamma[\mathcal{N}(A)] \subset \mathcal{N}(A)$, then $P_{\overline{R(A)}}\Gamma = \Gamma P_{\overline{R(A)}}$ and $P_{\overline{R(A)}}\Gamma^{\#} = \Gamma^{\#} P_{\overline{R(A)}}$.

Definition 1.3: (see [9]) An operator Γ is termed A-positive if it satisfies the following:

$$\langle \Gamma v; v\rangle_A \geq 0, \qquad \forall v \in \mathcal{H}$$

i.e, $A\Gamma \in \mathcal{B}(\mathcal{H})^+$.

We note $\Gamma \geq_A 0$.

Definition 1.4: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$.An operator Γ is termed A-selfadjoint when for every $v_1, v_2 \in \mathcal{H}$

$$\langle \Gamma v_1; v_2\rangle_A = \langle v_1; \Gamma v_2\rangle_A$$

equivalent

$$A\Gamma = \Gamma^* A$$

We have this result an operators $\Gamma^{\#}\Gamma$ and $\Gamma\Gamma^{\#}$ are A-selfadjoint.

Definition 1.5: (see [2]) An operator $\Gamma \in \mathcal{B}_A(\mathcal{H})$ is called an A-unitary if

$$\Gamma^{\#}\Gamma = (\Gamma^{\#})^{\#}\Gamma^{\#} = P_{\overline{R(A)}}$$

equivalent

$$\|\Gamma^{\#}v\|_A = \|\Gamma v\|_A = \|v\|_A \ ,\forall v \in \mathcal{H}$$

Definition 1.6: (see [15], [11]) Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$. an operator Γ is called an A-normal operator if

$$\Gamma\Gamma^{\#} = \Gamma^{\#}\Gamma$$

equivalent

$$\mathcal{R}(\Gamma\Gamma^{\#}) \subset \overline{\mathcal{R}(A)} \text{ and } \|\Gamma^{\#}v\|_A = \|\Gamma v\|_A \ ,\forall v \in \mathcal{H}$$

Definition 1.7: (see [7]) An operator $\Gamma \in \mathcal{B}_A(\mathcal{H})$ is called an A-hyponormal if $(\Gamma^{\#}\Gamma - \Gamma\Gamma^{\#}) \in \mathcal{B}_A(\mathcal{H})$
i.e,

$$\Gamma^{\#}\Gamma - \Gamma\Gamma^{\#} \geq_A 0$$

equivalent

$$\|\Gamma v\|_A \geq \|\Gamma^{\#}v\|_A \ ,\forall v \in \mathcal{H}$$

The contents of the paper are as follows.In Section 2, we introduce the notion of class $A_k^{\#}$ operators within the framework of semi-Hilbertian $(\mathcal{H}, \langle .\,;.\rangle_A)$ and establish fundamental properties of this operator class. Section 3 explores tensor product constructions involving Class $A_k^{\#}$ operators, revisiting their structural and functional characteristics.

2. Properties of class $A_k^{\#}$ operators

In this section, we introduce Class $A_k^{\#}$ operators within the framework of semi-Hilbertian spaces and establish key properties of these operators.

Definition 2.1: An operator $\Gamma \in \mathcal{B}_A(\mathcal{H})$ is termed a Class $A_k^{\#}$ if there is a positive integers number k such that

$$(\Gamma^{\#k}\Gamma^k) \geq_A (\Gamma\Gamma^{\#})^k$$

equivalent

$$A[\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^{\#})^k] \geq 0$$

Example 2.2: Let $\Gamma = \begin{pmatrix} 1 & 1-\rho \\ 0 & \rho \end{pmatrix}$; $\forall \rho \in \mathbb{R}$

and $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)^+$, then $A^{\dagger} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

This implies that $\Gamma^{\#} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ Therefore

$$\Gamma^{\#k}\Gamma^{k} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1-\rho^{k} \\ 0 & \rho^{k} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$(\Gamma^{\#}\Gamma)^{k} = \begin{pmatrix} \frac{2-\rho}{2} & \frac{-\rho+2}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} \end{pmatrix}$$

Hence

$$A(\Gamma^{\#k}\Gamma^{k} - (\Gamma\Gamma^{\#})^{k}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\left[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{2-\rho}{2} & \frac{2-\rho}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} \end{pmatrix}\right] \geq 0$$

Then Γ is Class $A_k^{\#}$ operator for all $k \in \mathbb{N}^*$.

Remark 2.3: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$. Here are some of the following results:

1. When $A = I$,Class $I_k^{\#}$ coincide with Class A_k^{*} operator [14].
2. If Γ is Class $A_1^{\#}$, then Γ is A-hyponormal [7].
3. If $\Gamma(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and Γ is Class $A_1^{\#}$ operator, then $\Gamma^{\#}$ is A-hyponormal.
4. If Γ is an A-normal operator, then Γ belongs to the Class $A_k^{\#}$ operator.
5. If A is injective, then every Class $A_1^{\#}$ is A-normal operator.
6. If the range of $\Gamma\Gamma^{\#}$ content the range of A, then Class $A_1^{\#}$ is A-normal operator .

Theorem 2.4: *Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$.then,*

Γ is Class $A_k^{\#}$ operator for some positive integer k if and only if Γ the following conditions
$$\begin{cases} \|\Gamma^{k}v\|_A \geq \left\|(\Gamma\Gamma^{\#})^{\frac{k}{2}}v\right\|_A, & if\ k\ even\ number \\ \|\Gamma^{k}v\|_A \geq \left\|\Gamma^{\#}(\Gamma\Gamma^{\#})^{\frac{k-1}{2}}v\right\|_A & if\ k\ odd\ number \end{cases}$$

Proof: Assume that Γ is a Class $A_k^{\#}$

If k even number

$$\langle(\Gamma\Gamma^{\#})^{k}v;v\rangle_A = \left\langle\overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{k\text{time}}v;v\right\rangle_A$$

$$
\begin{aligned}
&= \left\langle \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{k\text{time}} v; Av \right\rangle \\
&= \langle \Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{*}Av\rangle \\
&= \langle \Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; A\Gamma^{\#}v\rangle \\
&= \langle A\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{\#}v\rangle \\
&= \langle \Gamma^{*}A\Gamma\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{\#}v\rangle \\
&= \langle A\Gamma\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma\Gamma^{\#}v\rangle \\
&\vdots \\
&= \left\langle \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{\frac{k}{2}\text{time}} v; \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{\frac{k}{2}\text{time}} v \right\rangle_A \\
&= \left\langle (\Gamma\Gamma^{\#})^{\frac{k}{2}}v; (\Gamma\Gamma^{\#})^{\frac{k}{2}}v \right\rangle_A \\
&= \left\| (\Gamma\Gamma^{\#})^{\frac{k}{2}} \right\|_A^2
\end{aligned}
$$

other hand that

$$
\begin{aligned}
\langle (\Gamma^{\#})^k\Gamma^k v; v\rangle_A &= \langle A(\Gamma^{\#})^k\Gamma^k v; v\rangle \\
&= \langle \Gamma^{*k}A\Gamma^k v; v\rangle \\
&= \langle A\Gamma^k v; \Gamma^k v\rangle \\
&= \|\Gamma^k v\|_A^2
\end{aligned}
$$

hance

$$
\begin{aligned}
\langle (\Gamma^{\#})^k\Gamma^k - (\Gamma\Gamma^{\#})^k v; v\rangle_A \geq 0 \Leftrightarrow\ & \langle (\Gamma^{\#})^k\Gamma^k v; v\rangle_A \geq \langle (\Gamma\Gamma^{\#})^k v; v\rangle_A \\
\Leftrightarrow\ & \|\Gamma^k v\|_A \geq \left\| (\Gamma\Gamma^{\#})^{\frac{k}{2}} v \right\|_A
\end{aligned}
$$

If k odd number

$$
\begin{aligned}
\langle (\Gamma\Gamma^{\#})^k v; v\rangle_A &= \left\langle \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{k\text{time}} v; v \right\rangle_A \\
&= \left\langle \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{k\text{time}} v; Av \right\rangle \\
&= \langle \Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{*}Av\rangle \\
&= \langle \Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; A\Gamma^{\#}v\rangle \\
&= \langle A\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{\#}v\rangle \\
&= \langle \Gamma^{*}A\Gamma\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma^{\#}v\rangle \\
&= \langle A\Gamma\Gamma^{\#}\cdots\Gamma\Gamma^{\#}v; \Gamma\Gamma^{\#}v\rangle \\
&\vdots
\end{aligned}
$$

$$= \left\langle \Gamma^{\#} \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{\frac{k-1}{2}\text{time}} v; \Gamma^{\#} \overbrace{(\Gamma\Gamma^{\#})\cdots(\Gamma\Gamma^{\#})}^{\frac{k-1}{2}\text{time}} v \right\rangle_A$$

$$= \left\langle \Gamma^{\#}(\Gamma\Gamma^{\#})^{\frac{k-1}{2}} v; \Gamma^{\#}(\Gamma\Gamma^{\#})^{\frac{k-1}{2}} v \right\rangle_A$$

$$= \left\| \Gamma^{\#}(\Gamma\Gamma^{\#})^{\frac{k-1}{2}} v \right\|_A^2$$

hance

$$\langle (\Gamma^{\#})^k \Gamma^k - (\Gamma\Gamma^{\#})^k v; v\rangle_A \geq 0 \Leftrightarrow \langle (\Gamma^{\#})^k \Gamma^k v; v\rangle_A \geq \langle (\Gamma\Gamma^{\#})^k v; v\rangle_A$$

$$\Leftrightarrow \|\Gamma^k v\|_A \geq \left\| \Gamma^{\#}(\Gamma\Gamma^{\#})^{\frac{k-1}{2}} v \right\|_A$$

this achieves the proof.

Theorem 2.5: *Let $\Gamma_1; \Gamma_2 \in \mathcal{B}_A(\mathcal{H})$,such that $\Gamma_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ then $\Gamma_1\Gamma_2^{\#} = \Gamma_2^{\#}\Gamma_1$ if and only if $\Gamma_2\Gamma_1^{\#} = \Gamma_1^{\#}\Gamma_2$.*

Proof: Let $\Gamma_1; \Gamma_2 \in \mathcal{B}_A(\mathcal{H})$,such that $\Gamma_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ then we have

$$P_{\overline{R(A)}}\Gamma_1 = \Gamma_1 P_{\overline{R(A)}} \ ; \ P_{\overline{R(A)}}\Gamma_1^{\#} = \Gamma_1^{\#} P_{\overline{R(A)}} \ ; \ AP_{\overline{R(A)}} = A$$

and

$$P_{\overline{R(A)}}\Gamma_2 = \Gamma_2 P_{\overline{R(A)}} \ ; \ P_{\overline{R(A)}}\Gamma_2^{\#} = \Gamma_2^{\#} P_{\overline{R(A)}}$$

therefore

$$\begin{aligned}
\Gamma_1\Gamma_2^{\#} = \Gamma_2^{\#}\Gamma_1 \ &\Leftrightarrow \langle (\Gamma_1\Gamma_2^{\#})^{\#} v; v\rangle_A = \langle (\Gamma_2^{\#}\Gamma_1)^{\#} v; v\rangle_A; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \langle (\Gamma_2^{\#})^{\#}\Gamma_1^{\#} v; v\rangle_A = \langle \Gamma_1^{\#}(\Gamma_2^{\#})^{\#} v; v\rangle_A; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \langle P_{\overline{R(A)}}\Gamma_2 P_{\overline{R(A)}}\Gamma_1^{\#} v; v\rangle_A = \langle \Gamma_1^{\#} P_{\overline{R(A)}}\Gamma_2 P_{\overline{R(A)}} v; v\rangle_A; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \langle AP_{\overline{R(A)}}\Gamma_2\Gamma_1^{\#} v; v\rangle_A = \langle AP_{\overline{R(A)}}\Gamma_1^{\#}\Gamma_2 v; v\rangle; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \langle A\Gamma_2\Gamma_1^{\#} v; v\rangle = \langle A\Gamma_1^{\#}\Gamma_2 v; v\rangle; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \langle \Gamma_2\Gamma_1^{\#} v; v\rangle_A = \langle \Gamma_1^{\#}\Gamma_2 v; v\rangle_A; \ \forall v, v \in \mathcal{H} \\
&\Leftrightarrow \Gamma_2\Gamma_1^{\#} = \Gamma_1^{\#}\Gamma_2
\end{aligned}$$

The desired result is attained.

In the next lemma, there is a lot of use in this class.

Lemma 2.6: *Let $\Gamma_1, \Gamma_2 \in \mathcal{B}(\mathcal{H})$ satisfy $\Gamma_1 \geq_A \Gamma_2$, and let $\mathcal{T} \in \mathcal{B}_A(\mathcal{H})$.The following properties hold:*

1. $\mathcal{T}^{\#}(\Gamma_1 - \Gamma_2)\mathcal{T} \geq_A 0$.
2. $\mathcal{T}(\Gamma_1 - \Gamma_2)\mathcal{T}^{\#} \geq_A 0$.
3. *If $\mathcal{T}$ is A-selfadjoint , then $\mathcal{T}(\Gamma_1 - \Gamma_2)\mathcal{T} \geq_A 0$.*

Proposition 2.7: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$ satisfy $\Gamma[\mathcal{N}(A)] \subset \mathcal{N}(A)$, $\Gamma^k\Gamma^{\#k} = \Gamma^{\#k}\Gamma^k$ and $(\Gamma\Gamma^\#)^k = (\Gamma^\#\Gamma)^k$. then, Γ is a Class $A_k^\#$ if and only if $\Gamma^\#$ is a Class $A_k^\#$ operator.

Proof: Under the assumption that $\Gamma[\mathcal{N}(A)] \subset \mathcal{N}(A)$.
it follows that

$$P_{\overline{R(A)}}\Gamma = \Gamma P_{\overline{R(A)}} \text{ and } P_{\overline{R(A)}}\Gamma^\# = \Gamma^\# P_{\overline{R(A)}}$$

Suppose Γ is a Class $A_k^\#$ operator. We then prove that $\Gamma^\#$ belongs to the Class $A_k^\#$
we have

$$\Gamma^{\#k}\Gamma^k \geq_A (\Gamma\Gamma^\#)^k$$

and then

$$\begin{aligned}
(\Gamma^\#)^{\#k}\Gamma^{\#k} &= (P_{\overline{R(A)}}\Gamma P_{\overline{R(A)}})^k\Gamma^{\#k} \\
&= P_{\overline{R(A)}}\Gamma^k P_{\overline{R(A)}}\Gamma^{\#k} \\
&= P_{\overline{R(A)}}\Gamma^k\Gamma^{\#k} P_{\overline{R(A)}} \\
&= P_{\overline{R(A)}}\Gamma^{\#k}\Gamma^k P_{\overline{R(A)}} \\
&\geq_A P_{\overline{R(A)}}(\Gamma\Gamma^\#)^k P_{\overline{R(A)}} \\
&= P_{\overline{R(A)}}(\Gamma^\#\Gamma)^k P_{\overline{R(A)}} \\
&= (\Gamma^\# P_{\overline{R(A)}}\Gamma P_{\overline{R(A)}})^k \\
&= (\Gamma^\#(\Gamma^\#)^\#)^k
\end{aligned}$$

hence $\Gamma^\#$ is a Class $A_k^\#$ operator.

Assume that $\Gamma^\#$ is a Class $A_k^\#$ operator and prove that Γ is a Class $A_k^\#$
we have

$$(\Gamma^\#)^{\#k}\Gamma^{\#k} \geq_A (\Gamma^\#(\Gamma^\#)^\#)^k$$

and then

$$\begin{aligned}
(\Gamma^\#)^{\#k}\Gamma^{\#k} \geq_A (\Gamma^\#(\Gamma^\#)^\#)^k &\Rightarrow P_{\overline{R(A)}}\Gamma^k P_{\overline{R(A)}}\Gamma^{\#k} \geq_A (\Gamma^\# P_{\overline{R(A)}}\Gamma P_{\overline{R(A)}})^k \\
&\Rightarrow P_{\overline{R(A)}}\Gamma^k\Gamma^{\#k} P_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(\Gamma^\#\Gamma)^k P_{\overline{R(A)}} \\
&\Rightarrow P_{\overline{R(A)}}\Gamma^{\#k}\Gamma^k P_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(\Gamma\Gamma^\#)^k P_{\overline{R(A)}} \\
&\Rightarrow \Gamma^{\#k}\Gamma^k \geq_A (\Gamma\Gamma^\#)^k
\end{aligned}$$

Hence Γ is a Class $A_k^\#$ operator.

Proposition 2.8: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$ is a Class $A_k^\#$ operator.
$(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v \in \mathcal{N}(A)$ if and only if

$$\begin{cases} \|\Gamma^k v\|_A = \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}}v\right\|_A, & if\ k\ even\ number \\ \|\Gamma^k v\|_A = \left\|\Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}}v\right\|_A & if\ k\ odd\ number \end{cases}$$

Proof:

(1) if k is even number.

($\Rightarrow$) Suppose that

$$\|\Gamma^k v\|_A = \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A$$

Then

$$\begin{aligned}\|\Gamma^k v\|_A = \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A &\Leftrightarrow \langle\Gamma^k v; \Gamma^k v\rangle_A = \langle(\Gamma\Gamma^\#)^{\frac{k}{2}} v; (\Gamma\Gamma^\#)^{\frac{k}{2}} v\rangle_A \\ &\Leftrightarrow \langle\Gamma^{\#k}\Gamma^k v; v\rangle_A = \langle(\Gamma\Gamma^\#)^k v; v\rangle_A \\ &\Leftrightarrow \langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v; v\rangle_A = 0\end{aligned}$$

Since $(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)$ is A -positive by the A-Cauchy Schwarz inequality we have

$$|\langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v; v'\rangle_A|^2 \leq \langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v; v\rangle_A \langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v'; v'\rangle_A = 0$$

Hence

$$\langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v; v'\rangle_A = 0 \ , \forall v; v' \in \mathcal{H}$$

implies that

$$(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v \in \mathcal{N}(A)$$

($\Leftarrow$) Conversely if

$$A(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v = 0$$

this implies that

$$\langle(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k)v; v\rangle_A = 0$$

hence

$$\|\Gamma^k v\|_A = \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A$$

(2) if k is even number. we prove this case in the same way.

Proof is complete.

Proposition 2.9: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$, such that $\Gamma\Gamma^{\#k+1} = \Gamma^{\#k+1}\Gamma$. if Γ is a Class $A_k^\#$ operator, then Γ is a Class $A_{k+1}^\#$ operator

Proof: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$ suppose that Γ is class $A_k^\#$ then

$$\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k \geq_A 0$$

and we have

$$\begin{aligned}\Gamma\Gamma^\#(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^\#)^k) \geq_A 0 \quad &\Rightarrow \Gamma\Gamma^\#\Gamma^{\#k}\Gamma^k - \Gamma\Gamma^\#(\Gamma\Gamma^\#)^k \geq_A 0 \\ &\Rightarrow \Gamma\Gamma^{\#k+1}\Gamma^k - (\Gamma\Gamma^\#)^{k+1} \geq_A 0 \\ &\Rightarrow \Gamma^{\#k+1}\Gamma^{k+1} - (\Gamma\Gamma^\#)^{k+1} \geq_A 0\end{aligned}$$

hence Γ is a Class $A_{k+1}^\#$operator.

Proposition 2.10: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$, such that $\Gamma\Gamma^{\#k+n} = \Gamma^{\#k+n}\Gamma$ for all positive integers number n .then,

$$ClassA_k^{\#} \subseteq ClassA_{k+1}^{\#} \subseteq \ldots \subseteq ClassA_{k+n}^{\#}$$

Proof: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$ suppose that Γ is class $A_k^{\#}$ then

$$\Gamma^{\#k}\Gamma^k \geq_A (\Gamma\Gamma^{\#})^k$$

we have

$$\begin{aligned}
(\Gamma\Gamma^{\#})^n(\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^{\#})^k) \geq_A 0 \quad &\Rightarrow (\Gamma\Gamma^{\#})^n\Gamma^{\#k}\Gamma^k - (\Gamma\Gamma^{\#})^n(\Gamma\Gamma^{\#})^k \geq_A 0 \\
&\Rightarrow (\Gamma\Gamma^{\#})^{n-1}\Gamma\Gamma^{\#k+1}\Gamma^k - (\Gamma\Gamma^{\#})^{k+n} \geq_A 0 \\
&\Rightarrow (\Gamma\Gamma^{\#})^{n-1}\Gamma^{\#k+1}\Gamma^{k+1} - (\Gamma\Gamma^{\#})^{k+n} \geq_A 0 \\
&\Rightarrow (\Gamma\Gamma^{\#})^{n-2}\Gamma\Gamma^{\#k+2}\Gamma^{k+1} - (\Gamma\Gamma^{\#})^{k+n} \geq_A 0 \\
&\Rightarrow (\Gamma\Gamma^{\#})^{n-2}\Gamma^{\#k+2}\Gamma^{k+2} - (\Gamma\Gamma^{\#})^{k+n} \geq_A 0 \\
&\vdots \\
&\Rightarrow \Gamma^{\#k+n}\Gamma^{k+n} - (\Gamma\Gamma^{\#})^{k+n} \geq_A 0
\end{aligned}$$

hence Γ is a Class $A_{k+n}^{\#}$operator for all positive integers number n.

Theorem 2.11: *Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$ such that $\Gamma[\mathcal{N}(A)] \subset \mathcal{N}(A)$ and $(\Gamma\Gamma^{\#})^k = \Gamma^k\Gamma^{\#k}$. the following statements :*

1. Γ, $\Gamma^{\#}$ *are Class $A_k^{\#}$ if and only if* $\|\Gamma^k v\|_A = \|\Gamma^{\#k} v\|_A$.
2. *If A is injective then:*
   Γ *and* $\Gamma^{\#}$ *are Class $A_k^{\#}$ operators if and only if* $\Gamma^{\#k}\Gamma^k = \Gamma^k\Gamma^{\#k}$

Proof: Assume Γ and $\Gamma^{\#}$ are Class $A_k^{\#}$ operators that

$$\Gamma^{\#k}\Gamma^k \geq_A (\Gamma\Gamma^{\#})^k \text{ and } (\Gamma^{\#})^{\#k}\Gamma^{\#k} \geq_A [\Gamma^{\#}(\Gamma^{\#})^{\#}]^k$$

since $\Gamma[\mathcal{N}(A)] \subset \mathcal{N}(A)$ we have

$$\Gamma P_{\overline{\mathcal{R}(\mathcal{A})}} = P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma \ ; \ P_{\overline{\mathcal{R}(\mathcal{A})}}A = AP_{\overline{\mathcal{R}(\mathcal{A})}} = A \ ; \ (\Gamma^{\#})^{\#} = P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma P_{\overline{\mathcal{R}(\mathcal{A})}}$$

therefore for all $v \in \mathcal{H}$

$$\begin{aligned}
&\langle [(\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#}(\Gamma^{\#})^{\#})^k]v; v\rangle_A \geq 0 \\
&\Rightarrow \langle [P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma^k P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma^{\#k} - \left(\Gamma^{\#}P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma P_{\overline{\mathcal{R}(\mathcal{A})}}\right)^k\Big] v; v\rangle_A \geq 0 \\
&\Rightarrow \left\langle P_{\overline{\mathcal{R}(\mathcal{A})}}[\Gamma^k\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^k]P_{\overline{\mathcal{R}(\mathcal{A})}}v; v\right\rangle_A \geq 0 \\
&\Rightarrow \left\langle AP_{\overline{\mathcal{R}(\mathcal{A})}}[\Gamma^k\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^k]P_{\overline{\mathcal{R}(\mathcal{A})}}v; v\right\rangle \geq 0 \\
&\Rightarrow \left\langle A[\Gamma^k\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^k]P_{\overline{\mathcal{R}(\mathcal{A})}}v; v\right\rangle \geq 0 \\
&\Rightarrow \langle [\Gamma^k\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^k]v; v\rangle_A \geq 0 \\
&\Rightarrow \Gamma^k\Gamma^{\#k} \geq_A (\Gamma^{\#}\Gamma)^k \\
&\Rightarrow \Gamma^k\Gamma^{\#k} \geq_A \Gamma^{\#k}\Gamma^k
\end{aligned}$$

On other hand

$$\Gamma^{\#k}\Gamma^{k} \geq_A (\Gamma\Gamma^{\#})^{k} \Leftrightarrow \Gamma^{\#k}\Gamma^{k} \geq_A \Gamma^{k}\Gamma^{\#k}$$

and hence

$$\Gamma^{\#k}\Gamma^{k} \geq_A \Gamma^{k}\Gamma^{\#k} \geq_A \Gamma^{\#k}\Gamma^{k}$$

therefore

$$\|\Gamma^{k}v\|_A = \|\Gamma^{\#k}v\|_A \ \ \forall v \in \mathcal{H}$$

Conversely, assume that $\|\Gamma^{k}v\|_A = \|\Gamma^{\#k}v\|_A \forall v \in \mathcal{H}$. From which it is clear that

$$\langle(\Gamma^{\#k}\Gamma^{k} - \Gamma^{k}\Gamma^{\#k})v; v\rangle_A = 0 \ \ \forall v \in \mathcal{H}$$

then

$$\langle(\Gamma^{\#k}\Gamma^{k} - (\Gamma\Gamma^{\#})^{k})v; v\rangle_A = 0 \ \ \forall v \in \mathcal{H}$$

therefore Γ is a Class $A_k^{\#}$

On other hand

$$\begin{aligned} 0 &= \|\Gamma^{\#k}v\|_A - \|\Gamma^{k}v\|_A \\ &= \langle(((\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#})^{k}\Gamma^{k})v; v\rangle_A \\ &= \langle A((\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^{k})v; v\rangle \\ &= \left\langle AP_{\overline{\mathcal{R}(\mathcal{A})}}((\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#}\Gamma)^{k})P_{\overline{\mathcal{R}(\mathcal{A})}}v; v\right\rangle \\ &= \left\langle\left((\Gamma^{\#})^{\#k}\Gamma^{\#k} - P_{\overline{\mathcal{R}(\mathcal{A})}}(\Gamma^{\#}\Gamma)^{k}P_{\overline{\mathcal{R}(\mathcal{A})}}\right)v; v\right\rangle_A \\ &= \left\langle\left((\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#}P_{\overline{\mathcal{R}(\mathcal{A})}}\Gamma P_{\overline{\mathcal{R}(\mathcal{A})}})^{k}\right)v; v\right\rangle_A \\ &= \langle(((\Gamma^{\#})^{\#k}\Gamma^{\#k} - (\Gamma^{\#}(\Gamma^{\#})^{\#})^{k})v; v\rangle_A \end{aligned}$$

thus, $\Gamma^{\#}$ is a Class $A_k^{\#}$.

2. If we assume that Γ and $\Gamma^{\#}$ are a Class $A_k^{\#}$,from statement (1) it follows that

$$\begin{aligned} \|\Gamma^{k}v\|_A = \|\Gamma^{\#k}v\|_A; \forall v \in \mathcal{H} \quad &\Leftrightarrow \langle\Gamma^{\#k}\Gamma^{k}v; v\rangle_A = \langle\Gamma^{k}\Gamma^{\#k}v; v\rangle_A; \ \ \forall v \in \mathcal{H} \\ &\Leftrightarrow \langle(\Gamma^{\#k}\Gamma^{k} - \Gamma^{k}\Gamma^{\#k})v; v\rangle_A = 0; \ \ \forall v \in \mathcal{H} \\ &\Leftrightarrow A(\Gamma^{\#k}\Gamma^{k} - \Gamma^{k}\Gamma^{\#k}) = 0 \\ &\Leftrightarrow \Gamma^{\#k}\Gamma^{k} - \Gamma^{k}\Gamma^{\#k} = 0 \ \ (A \text{ is injective}). \end{aligned}$$

Theorem 2.12: Let $\Gamma \in \mathcal{B}_A(\mathcal{H})$. then, Γ is Class $A_k^{\#}$ if and only if $\alpha^2\Gamma^{\#k}\Gamma^{k} + 2\alpha(\Gamma\Gamma^{\#})^{k} + \Gamma^{\#k}\Gamma^{k} \geq_A 0, \forall\alpha \in \mathbf{R}$

Proof: If k is even number.

for all $v \in \mathcal{H}$ and $\alpha \in \mathbb{R}$ we have

$$\|\Gamma^k v\|_A^4 \geq \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A^4 \Leftrightarrow \quad 0 \geq \left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A^4 - \|\Gamma^k v\|_A^4$$
$$\Leftrightarrow \quad \alpha^2\|\Gamma^k v\|_A^2 + 2\alpha\left\|(\Gamma\Gamma^\#)^{\frac{k}{2}} v\right\|_A^2 + \|\Gamma^k v\|_A^2 \geq 0$$
$$\Leftrightarrow \quad \alpha^2 < \Gamma^k v; \Gamma^k v >_A + 2\alpha < (\Gamma\Gamma^\#)^{\frac{k}{2}} v; (\Gamma\Gamma^\#)^{\frac{k}{2}} v >_A$$
$$+ < \Gamma^k v; \Gamma^k v >_A \geq 0$$
$$\Leftrightarrow \quad < (\alpha^2\Gamma^{\#k}\Gamma^k + 2\alpha(\Gamma\Gamma^\#)^k + \Gamma^{\#k}\Gamma^k)v; v >_A \geq 0$$
$$\Leftrightarrow \quad \alpha^2\Gamma^{\#k}\Gamma^k + 2\alpha(\Gamma\Gamma^\#)^k + \Gamma^{\#k}\Gamma^k \geq_A 0$$

If k is odd number.
for all $v \in \mathcal{H}$ and $\alpha \in \mathbb{R}$ we have

$$\|\Gamma^k v\|_A^4 \geq \left\|\Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}} v\right\|_A^4$$
$$\Leftrightarrow \quad 0 \geq \left\|\Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}} v\right\|_A^4 - \|\Gamma^k v\|_A^4$$
$$\Leftrightarrow \quad \alpha^2\|\Gamma^k v\|_A^2 + 2\alpha\left\|\Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}} v\right\|_A^2 + \|\Gamma^k v\|_A^2 \geq 0$$
$$\Leftrightarrow \quad \alpha^2 < \Gamma^k v; \Gamma^k v >_A + 2\alpha < (\Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}} v; \Gamma^\#(\Gamma\Gamma^\#)^{\frac{k-1}{2}} v >_A$$
$$+ < \Gamma^k v; \Gamma^k v >_A \geq 0$$
$$\Leftrightarrow \quad < (\alpha^2\Gamma^{\#k}\Gamma^k + 2\alpha(\Gamma\Gamma^\#)^k + \Gamma^{\#k}\Gamma^k)v; v >_A \geq 0$$
$$\Leftrightarrow \quad \alpha^2\Gamma^{\#k}\Gamma^k + 2\alpha(\Gamma\Gamma^\#)^k + \Gamma^{\#k}\Gamma^k \geq_A 0$$

Proposition 2.13: Let $\Gamma_1; \Gamma_2 \in \mathcal{B}_A(\mathcal{H})$ are Class $A_k^\#$ operator such that $\Gamma_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and $\Gamma_1^k\Gamma_2 = \Gamma_2^k\Gamma_1 = \Gamma_1^k\Gamma_2^\# = \Gamma_2^k\Gamma_1^\# = 0$, for all $k \in \mathbb{N}^*$ then $\Gamma_1 + \Gamma_2$ is Class $A_k^\#$ operator.

Proof: According to the hypotheses, we have

$$\Gamma_1^{\#k}\Gamma_1^k - (\Gamma_1\Gamma_1^\#)^k \geq 0 \text{ and } \Gamma_2^{\#k}\Gamma_2^k - (\Gamma_2\Gamma_2^\#)^k \geq 0$$

Therefore

$$(\Gamma_1 + \Gamma_2)^{\#k}(\Gamma_1 + \Gamma_2)^k - ((\Gamma_1 + \Gamma_2)(\Gamma_1 + \Gamma_2)^\#)^k$$
$$= \left(\Gamma_1^{\#k} + \Gamma_2^{\#k} + \sum_{i=1}^{k-1}\binom{i}{k-1}\Gamma_1^{\#i}(\Gamma_2^\#)^{k-1-i}\right)$$
$$\left(\Gamma_1^k + \Gamma_2^k + \sum_{i=1}^{k-1}\binom{i}{k-1}\Gamma_1^i(\Gamma_2)^{k-1-i}\right) - (\Gamma_1\Gamma_1^\# + \Gamma_1\Gamma_2^\# + \Gamma_2\Gamma_1^\# + \Gamma_2\Gamma_2^\#)^k$$
$$= (\Gamma_1^{\#k}\Gamma_1^k + \Gamma_2^{\#k}\Gamma_1^k + \Gamma_1^{\#k}\Gamma_2^k + \Gamma_2^{\#k}\Gamma_2^k)$$
$$- \left((\Gamma_1\Gamma_1^\#)^k + (\Gamma_2\Gamma_2^\#)^k + \sum_{i=1}^{k-1}\binom{i}{k-1}(\Gamma_1\Gamma_1^\#)^i(\Gamma_2\Gamma_2^\#)^{k-1-i}\right)$$
$$= \Gamma_1^{\#k}\Gamma_1^k - (\Gamma_1\Gamma_1^\#)^k + \Gamma_2^{\#k}\Gamma_2^k - (\Gamma_2\Gamma_2^\#)^k$$
$$\geq_A 0.$$

Then $\Gamma_1 + \Gamma_2$ is Class $A_k^\#$ operator.

Proposition 2.14: Let $\Gamma_1 \in \mathcal{B}_A(\mathcal{H})$ and $\Gamma_2 \in \mathcal{B}_A(\mathcal{H})$ such that $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, $\Gamma_1^{\#}\Gamma_2 = \Gamma_2\Gamma_1^{\#}$ and Γ_1 is Class $A_k^{\#}$,then the following statements hold

1. If Γ_2 belongs to A-self adjoint,then $\Gamma_2\Gamma_1$ is Class $A_k^{\#}$ operator
2. If Γ_2 belongs to A-normal operator and $\Gamma_{1,2}[\mathcal{N}(A)] \subseteq \mathcal{N}(A)$,then $\Gamma_2\Gamma_1$ is Class $A_k^{\#}$ operator

Proof:

1. Let Γ_1 is a Class $A_k^{\#}$ then

$$\begin{aligned}
\Gamma_1^{\#k}\Gamma_1^k \geq_A (\Gamma_1\Gamma_1^{\#})^k \Rightarrow\ & \Gamma_2^k\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k \geq_A \Gamma_2^k(\Gamma_1\Gamma_1^{\#})^k\Gamma_2^k \\
\Rightarrow\ & \Gamma_2^k\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k \geq_A \Gamma_2^k(\Gamma_1\Gamma_1^{\#})^k\Gamma_2^k \\
\Rightarrow\ & (\Gamma_2\Gamma_1^{\#})^k(\Gamma_1\Gamma_2)^k \geq_A (\Gamma_2\Gamma_1\Gamma_1^{\#}\Gamma_2)^k \\
\Rightarrow\ & (\Gamma_2^{\#}\Gamma_1^{\#})^k(\Gamma_1\Gamma_2)^k \geq_A (\Gamma_2\Gamma_1\Gamma_1^{\#}\Gamma_2^{\#})^k \\
\Rightarrow\ & (\Gamma_2\Gamma_1)^{\#k}(\Gamma_2\Gamma_1)^k \geq_A (\Gamma_2\Gamma_1(\Gamma_2\Gamma_1)^{\#})^k
\end{aligned}$$

   then $\Gamma_2\Gamma_1$ is a Class $A_k^{\#}$.

2. Let Γ_2 is A-normal operator that is $\Gamma_2\Gamma_2^{\#} = \Gamma_2^{\#}\Gamma_2$ and Γ_1 is a Class $A_k^{\#}$ we get the following relations

$$\begin{aligned}
&\langle[(\Gamma_1\Gamma_2)^{\#k}(\Gamma_1\Gamma_2)^k - ((\Gamma_1\Gamma_2)^{\#}(\Gamma_1\Gamma_2))^k]v; v\rangle_A \\
&= \langle[\Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k - (\Gamma_2^{\#}\Gamma_1^{\#}\Gamma_1\Gamma_2)^k]v; v\rangle_A \\
&= \langle[\Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k - \Gamma_2^{\#k}(\Gamma_1^{\#}\Gamma_1)^k\Gamma_2^k]v; v\rangle_A \\
&= \langle A\Gamma_2^{\#k}[\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k - (\Gamma_1^{\#}\Gamma_1)^k\Gamma_2^k]v; v\rangle \\
&= \langle \Gamma_2^{*k}A[\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k - (\Gamma_1^{\#}\Gamma_1)^k\Gamma_2^k]v; v\rangle \\
&= \langle A[\Gamma_1^{\#k}\Gamma_1^k - (\Gamma_1^{\#}\Gamma_1)^k]\Gamma_2^k v; \Gamma_2^k v\rangle \\
&= \langle[\Gamma_1^{\#k}\Gamma_1^k - (\Gamma_1^{\#}\Gamma_1)^k]\Gamma_2^k v; \Gamma_2^k v\rangle_A \geq 0
\end{aligned}$$

   Then $\Gamma_2\Gamma_1$ Class $A_k^{\#}$

Proposition 2.15: Let $\Gamma_1; \Gamma_2 \in \mathcal{B}_A(\mathcal{H})$ such that $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, $\Gamma_1^{\#}\Gamma_2 = \Gamma_2\Gamma_1^{\#}$ and $\Gamma_1\Gamma_1^{\#}\Gamma_2\Gamma_2^{\#} = \Gamma_2\Gamma_2^{\#}\Gamma_1\Gamma_1^{\#}$. If Γ_1 and Γ_2 are a Class $A_k^{\#}$ operators and $\Gamma_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$,then $\Gamma_1\Gamma_2$ is Class $A_k^{\#}$ operator.

Proof: Let Γ_1 and Γ_2 are Class $A_k^{\#}$ operators.

If k is even number then

$$\begin{aligned}
\|(\Gamma_1\Gamma_2)^k v\|_A &= \left\|\Gamma_1^k\Gamma_2^k v\right\|_A \\
&\geq (\Gamma_1\Gamma_1^{\#})^{\frac{k}{2}}\Gamma_2^k v_A \\
&= \left\|\Gamma_2^k(\Gamma_1\Gamma_1^{\#})^{\frac{k}{2}} v\right\|_A \\
&\geq \left\|(\Gamma_2\Gamma_2^{\#})^{\frac{k}{2}}(\Gamma_1\Gamma_1^{\#})^{\frac{k}{2}} v\right\|_A \\
&= \left\|(\Gamma_1\Gamma_1^{\#}\Gamma_2\Gamma_2^{\#})^{\frac{k}{2}} v\right\|_A
\end{aligned}$$

$$= \left\|(\Gamma_1\Gamma_2\Gamma_1^{\#}\Gamma_2^{\#})^{\frac{k}{2}}v\right\|_A$$
$$= \left\|(\Gamma_1\Gamma_2(\Gamma_1\Gamma_2)^{\#})^{\frac{k}{2}}v\right\|_A$$

then $\Gamma_1\Gamma_2$ is a Class $A_k^{\#}$.

If k is odd number then

$$\begin{aligned}
\|(\Gamma_1\Gamma_2)^k v\|_A &= \left\|\Gamma_1^k\Gamma_2^k v\right\|_A \\
&\geq \left\|\Gamma_1^{\#}(\Gamma_1\Gamma_1^{\#})^{\frac{k-1}{2}}\Gamma_2^k v\right\|_A \\
&= \left\|\Gamma_2^k\Gamma_1^{\#}(\Gamma_1\Gamma_1^{\#})^{\frac{k-1}{2}} v\right\|_A \\
&\geq \left\|\Gamma_2^{\#}(\Gamma_2\Gamma_2^{\#})^{\frac{k-1}{2}}\Gamma_1^{\#}(\Gamma_1\Gamma_1^{\#})^{\frac{k-1}{2}} v\right\|_A \\
&= \left\|\Gamma_2^{\#}\Gamma_1^{\#}(\Gamma_1\Gamma_1^{\#}\Gamma_2\Gamma_2^{\#})^{\frac{k-1}{2}} v\right\|_A \\
&= \left\|(\Gamma_1\Gamma_2)^{\#}(\Gamma_1\Gamma_2\Gamma_1^{\#}\Gamma_2^{\#})^{\frac{k-1}{2}} v\right\|_A \\
&= \left\|(\Gamma_1\Gamma_2)^{\#}(\Gamma_1\Gamma_2(\Gamma_1\Gamma_2)^{\#})^{\frac{k-1}{2}} v\right\|_A
\end{aligned}$$

then $\Gamma_1\Gamma_2$ is a Class $A_k^{\#}$.

We say that Γ_1 is A-unitary equivalent to Γ_2 if there exists an A-unitary U such that $\Gamma_2 = U^{\#}\Gamma_1 U$

Proposition 2.16: Let $\Gamma_2, \Gamma_1 \in \mathcal{B}_A(\mathcal{H})$. If Γ_2 is Class $A_k^{\#}$ operator and Γ_1 is A-unitary equivalent to Γ_2, then Γ_1 is Class $A_k^{\#}$ operator.

Proof: Let Γ_1 is a class $A_k^{\#}$ operator then

$$\begin{aligned}
\Gamma_2^{\#k}\Gamma_2^k \geq_A (\Gamma_2\Gamma_2^{\#})^k \Leftrightarrow\ & (U^{\#})^{\#}\Gamma_1^{\#k}U^{\#}U\Gamma_1^k U^{\#} \geq_A (U\Gamma_1 U^{\#}(U^{\#})^{\#}\Gamma_1^{\#}U^{\#})^k \\
\Leftrightarrow\ & P_{\overline{R(A)}}UP_{\overline{R(A)}}\Gamma_1^{\#k}P_{\overline{R(A)}}\Gamma_1^k U^{\#} \\
& \geq_A (U\Gamma_1 U^{\#}P_{\overline{R(A)}}UP_{\overline{R(A)}}\Gamma_1^{\#}U^{\#})^k \\
\Leftrightarrow\ & P_{\overline{R(A)}}UP_{\overline{R(A)}}(\Gamma_1^{\#k}\Gamma_1^k)U^{\#} \geq_A P_{\overline{R(A)}}UP_{\overline{R(A)}}(\Gamma_1\Gamma_1^{\#})^k U^{\#} \\
\Leftrightarrow\ & (U^{\#})^{\#}(\Gamma_1^{\#k}\Gamma_1^k)U^{\#} \geq_A (U^{\#})^{\#}(\Gamma_1\Gamma_1^{\#})^k U^{\#} \\
\Leftrightarrow\ & \Gamma_1^{\#k}\Gamma_1^k \geq_A (\Gamma_1\Gamma_1^{\#})^k
\end{aligned}$$

then Γ_1 is class $A_k^{\#}$ operator.

3. Tensor Product of Class $A_k^{\#}$ Operators in Semi-Hilbertian Spaces

The tensor product of operators is a mathematical operation that combines two operators into a single operator acting on a tensor product space. If you have two operators Γ_1 and Γ_2 acting on spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, the tensor product operator $\Gamma_1 \otimes \Gamma_2$ acts on the tensor product space $\mathcal{H} \otimes \mathcal{K}$ (see [13] [8]). It is defined as

$$(\Gamma_1 \otimes \Gamma_2)(u_1 \otimes u_2) = (\Gamma_1 u_1) \otimes (\Gamma_2 u_2)$$

These are some of the properties we need in the following theorem

1. $(\Gamma_1 \otimes \Gamma_2)(\Gamma_1 \otimes \Gamma_2)^* = (\Gamma_1 \Gamma_1^* \otimes \Gamma_2 \Gamma_2^*)$
2. $\mu_1 \mu_2 \Gamma_1 \otimes \Gamma_2 = \mu_1 \Gamma_1 \otimes \mu_2 \Gamma_2$,for any real numbers $\mu_{1,2}$
3. $((\Gamma_1 \otimes \Gamma_2)(\Gamma_1 \otimes \Gamma_2)^*)^\rho = (\Gamma_1 \Gamma_1^*)^\rho \otimes (\Gamma_2^* \Gamma_2)^\rho \;\; \forall \rho \in \mathbb{R}^+$

Lemma 3.1: *Let $\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2 \in \mathcal{B}(\mathcal{H})$ and Let $A, B \in \mathcal{B}(\mathcal{H})^+$, such that $\Gamma_1 \geq_A \Gamma_2 \geq_A 0$ and $\Gamma'_1 \geq_B \Gamma'_2 \geq_B 0$, then*

$$(\Gamma_1 \otimes \Gamma'_1) \geq_{A \otimes B} (\Gamma_2 \otimes \Gamma'_2) \geq_{A \otimes B} 0$$

Proposition 3.2: Let $\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2 \in \mathcal{B}(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})^+$ such that Γ_i is A-positive and Γ'_i is B-positive for $i = 1, 2$. If $\Gamma_1 \neq 0$ and $\Gamma'_1 \neq 0$,the following conditions are equivalent:

1. $\Gamma_2 \otimes \Gamma'_2 \geq_{A \otimes B} \Gamma_1 \otimes \Gamma'_1$.
2. There exists $\mu > 0$ such that $\mu \Gamma_2 \geq_A \Gamma_1$ and $\mu^{-1} \Gamma'_2 \geq_B \Gamma'_1$.

Theorem 3.3: *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $\Gamma_1 \in \mathcal{B}_A(\mathcal{H})$,$\Gamma_2 \in \mathcal{B}_B(\mathcal{H})$ .then ,*

Γ_1 is a Class $A_k^\#$ and Γ_2 is a Class $B_k^\#$ if and only if $\Gamma_1 \otimes \Gamma_2$ is a Class $(A \otimes B)_k^\#$.

Proof: Assume that Γ_1 is Class $A_k^\#$ operators and Γ_2 is a Class $B_k^\#$ operators, then

$$\begin{aligned}
[(\Gamma_1 \otimes \Gamma_2)^\#]^k (\Gamma_1 \otimes \Gamma_2)^k &= [\Gamma_1^{\#k} \otimes \Gamma_2^{\#k}](\Gamma_1^k \otimes \Gamma_2^k) \\
&= \Gamma_1^{\#k}\Gamma_1^k \otimes \Gamma_2^{\#k}\Gamma_2^k \\
&\geq_{A \otimes B} (\Gamma_1 \Gamma_1^\#)^k \otimes (\Gamma_2 \Gamma_2^\#)^k \\
&= (\Gamma_1 \Gamma_1^\# \otimes \Gamma_2 \Gamma_2^\#)^k \\
&= [(\Gamma_1 \otimes \Gamma_2)(\Gamma_1^\# \otimes \Gamma_2^\#)]^k \\
&= [(\Gamma_1 \otimes \Gamma_2)(\Gamma_1 \otimes \Gamma_2)^\#]^k.
\end{aligned}$$

therefore $\Gamma_1 \otimes \Gamma_2$ is Class $(A \otimes B)_k^\#$ operators.

Conversely, assume that $\Gamma_1 \otimes \Gamma_2$ is Class $(A \otimes B)_k^\#$ operators. we obtain

$$\begin{aligned}
& [(\Gamma_1 \otimes \Gamma_2)^\#]^k (\Gamma_1 \otimes \Gamma_2)^k \geq_{A \otimes B} [(\Gamma_1 \otimes \Gamma_2)(\Gamma_1 \otimes \Gamma_2)^\#]^k \\
\Leftrightarrow \;& (\Gamma_1^{\#k} \otimes \Gamma_2^{\#k})(\Gamma_1^k \otimes \Gamma_2^k) \geq_{A \otimes B} (\Gamma_1 \Gamma_1^\# \otimes \Gamma_2 \Gamma_2^\#)^k \\
\Leftrightarrow \;& \Gamma_1^{\#k}\Gamma_1^k \otimes \Gamma_2^{\#k}\Gamma_2^k \geq_{A \otimes B} (\Gamma \Gamma^\#)^k \otimes (\Gamma_2 \Gamma_2^\#)^k \\
\Leftrightarrow \;& \Gamma_1^{\#k}\Gamma_1^k \otimes \Gamma_2^{\#k}\Gamma_2^k \geq_{A \otimes B} \mu(\Gamma_1 \Gamma_1^\#)^k \otimes \mu^{-1}(\Gamma_2 \Gamma_2^\#)^k
\end{aligned}$$

By proposition 3.2 there exists $\mu > 0$ such that

$$\begin{cases}
\mu \Gamma_1^{\#k}\Gamma_1^k \geq_A \mu(\Gamma_1 \Gamma_1^\#)^k \\
\text{and} \\
\mu^{-1}\Gamma_2^{\#k}\Gamma_2^k \geq_B \mu^{-1}(\Gamma_2 \Gamma_2^\#)^k
\end{cases}$$

a simple computation then

$$\Gamma_1^{\#k}\Gamma_1^k \geq_A (\Gamma_1\Gamma_1^\#)^k \text{ and } \Gamma_2^{\#k}\Gamma_2^k \geq_B (\Gamma_2\Gamma_2^\#)^k$$

Therefore, Γ_1 is Class $A_k^\#$ operators and Γ_2 is Class $B_k^\#$ operators.

Proposition 3.4: Let $\Gamma_1, \Gamma_2 \in \mathcal{B}_A(\mathcal{H})$, Γ_1 and Γ_2 are Class $A_k^\#$ such that $\Gamma_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$ and $\Gamma_1^\#\Gamma_2 = \Gamma_2\Gamma_1^\#$ then $\Gamma_1\Gamma_2 \otimes \Gamma_1$ and $\Gamma_1\Gamma_2 \otimes \Gamma_2 \in \mathcal{B}_{A\otimes A}(\mathcal{H} \overline{\otimes} \mathcal{H})$ are Class $(A \otimes A)_k^\#$.

Proof:

1. Suppose that Γ_1 and Γ_2 are Class $A_k^\#$,we have

$$\Gamma_1^{\#k}\Gamma_1^k \geq_A (\Gamma_1\Gamma_1^\#)^k \text{ and } \Gamma_2^{\#k}\Gamma_2^k \geq_A (\Gamma_2\Gamma_2^\#)^k$$

and we have that

$$\begin{aligned} \Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k &\geq_A \Gamma_2^{\#k}(\Gamma_1\Gamma_1^\#)^k\Gamma_2^k \\ &= \Gamma_2^{\#k}\Gamma_2^k(\Gamma_1\Gamma_1^\#)^k \\ &\geq_A (\Gamma_2\Gamma_2^\#)^k(\Gamma_1\Gamma_1^\#)^k \\ &= (\Gamma_2\Gamma_2^\#\Gamma_1\Gamma_1^\#)^k \\ &= (\Gamma_2\Gamma_1(\Gamma_2\Gamma_1)^\#)^k \end{aligned}$$

hence

$$\begin{cases} \Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k \geq_A (\Gamma_1\Gamma_2(\Gamma_1\Gamma_2)^\#)^k \geq_A 0 \\ \text{and} \\ \Gamma_1^{\#k}\Gamma_1^k \geq_A (\Gamma_1\Gamma_1^\#)^k \geq_A 0 \end{cases}$$

on other hand

$$\begin{aligned} (\Gamma_1\Gamma_2 \otimes \Gamma_1)^{\#k}(\Gamma_1\Gamma_2 \otimes \Gamma_1)^k &= ((\Gamma_1\Gamma_2)^{\#k} \otimes \Gamma_1^{\#k})((\Gamma_1\Gamma_2)^k \otimes \Gamma_1^k) \\ &= (\Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k \otimes \Gamma_1^{\#k}\Gamma_1^k) \end{aligned}$$

Lemma 3.1 implies that

$$(\Gamma_2^{\#k}\Gamma_1^{\#k}\Gamma_1^k\Gamma_2^k \otimes \Gamma_1^{\#k}\Gamma_1^k) \geq_A (\Gamma_1\Gamma_2(\Gamma_1\Gamma_2)^\#)^k \otimes (\Gamma_1\Gamma_1^\#)^k$$

Then

$$(\Gamma_1\Gamma_2 \otimes \Gamma_1)^{\#k}(\Gamma_1\Gamma_2 \otimes \Gamma_1)^k \geq_A ((\Gamma_1\Gamma_2 \otimes \Gamma_1)(\Gamma_1\Gamma_2 \otimes \Gamma_1)^\#)^k$$

Hence $(\Gamma_1\Gamma_2 \otimes \Gamma_1)$ is Class $(A \otimes A)_k^\#$.

2. In the same way, we may deduce the Class $(A \otimes A)_k^\#$ operators of $\Gamma_1\Gamma_2 \otimes \Gamma_2$

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