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Numerical approximation of Lambert W function in complex domain by unique method of quadratic approximation

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Abstract

This paper presents a novel method for approximating the Lambert W function in the complex plane. By separating real and imaginary components, two equations are derived, which are reduced to a single-variable equation using substitutions. Linear logarithmic approximations and truncated series expansions refine the imaginary part iteratively, while the real part is computed directly using a cotangent relation, avoiding lengthy iterative methods. The approach reveals unique properties of the Lambert W function's branches, with imaginary parts converging to (2k+1/2)π or (2k+3/2)π as k approaches infinity. It overcomes limitations of Newton's and Halley's methods, such as differentiability constraints, while providing infinite solutions in the complex plane. Validation through examples, algorithms, and graphical representations highlights its accuracy. This method offers a robust framework for analysing the Lambert W function, with potential applications in quantum mechanics, thermodynamics, and control systems.

Subject Classification (2010): *33F05, 30E10.*

Keywords: *Lambert W function in complex plane, Natural logarithm, Quadratic approximation, Complex number.*

1. Introduction

The Lambert W function has its roots in the equation $x\,e^x = y$ as in work from the 18th century. Johann Heinrich Lambert first studied the equation $x = x^m + q$ in the year 1758 for its solution [1]. Later, in 1783, Leonhard Euler transformed the equation into $x^a - x^b = (a-b)cx^{a+b}$ and solved w using a convergent series expansion [2]. Once Euler had solved the equation, he then by putting $a = b = 1$ and taking limits, derived the equation $x = c\,ln(x)$. Euler

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obtained a convergent series solution to this equation in terms of c. After taking the derivative with respect to x and with some manipulations, he solved the equation

$$y\, e^{y} = x. \tag{1.1}$$

Written as

$$W(x) = y, \tag{1.2}$$

it was given the nomenclature of the Lambert W Function in x. In general, the function written $we^{w} = z$ holds, if and only if, $z = W_k(z)$ for some integer k. In the real domain, for the function (1.2): (i) when $x \geq 0$, its branch is denoted by W_0, and (ii) when $-1/e \leq x < 0$, the branch is denoted by W_{-1}. In the complex domain a branch W_k pertains to imaginary-part values y in the domain $2k\pi$ to $(2k+1)\pi$ and to the corresponding real-part values $x. W_{-k}$ pertains to value of imaginary part y in domain $-(2k+1)\pi$ to $-2k\pi$ and also corresponding value of real part x [33]. The quadratic approximation method was first introduced for the real domain in Numerical Approximation of Lambert W Function for Real Values by Unique Method of Quadratic Approximation [36]. In the present work, I extend this approach to the complex domain, where branch behaviour and convergence properties reveal new and distinct features.

Application of this function finds use in *viscous flow* [3], *time dependent flow in simple branch hydraulic systems* [4], *neuro-imaging* [5], *chemical engineering* [6] [7], *crystal growth* [8], *material science* [9], *porous media* [10], *Bernoulli's number* and *Todd genus* [11], *statistics* [12] [13], *pooling of tests for infectious diseases* [14] [15] [16], *exact solution of Schrodinger equation* [17], *exact solution of QCD coupling constant* [18], *exact solution of the Einsteins's vacuum equations, Resonances of the delta shell potential* [19], *thermodynamic equilibrium, phase separation of polymer mixtures* [20], *Wein's displacement law in D dimensional universe* [21], *AdS/CFT correspondence* [22] [23], *epidemiology* [24], *determination of the time of flight of the projectiles, electromagnetic surface wave propagation* [25], *general relativity and quantum mechanics* [26].

Function $W_k(x) = x$ has infinitely many values of x that solve this equation in the complex plane. Euler also made mention of complex roots of $x = a^{x}$, when a is a real quantity [27]. However, it was L´emeray who explained how all values of $W_k(x)$ could be calculated [28] [29]. For complex x, Hayes calculated all values of $W_k(x)$ [30]. Thereafter, Wright also calculated all the values of $W_k(x)$ and gave an algorithm to compute these [31] [32]. R. M. Corless et al. provided a method of error analysis for approximating values of$W_k(z)$ [33]. Despite extensive research, there remain opportunities to explore new methods for solving and approximating the Lambert W function.

The objectives of this paper are as follows:

a) To solve for *z* by method of quadratic equation approximation the function $ze^{z} = a + ib$ or $W(a + ib) = z$, where a, b, x, y are given real quantities and $z = x + iy$,

b) To prove existence of infinite solutions to the said function and to determine their precise values by the method of quadratic equation approximation.

This work aims to contribute to the understanding and computation of the Lambert W function, offering a unique approach to solving problems in various scientific and mathematical domains.

2. Numerical approximation By Method of Quadratic Equation Approximation

Equation $ze^z = a + ib$ can be written

$$(x + iy)e^{(x+iy)} = a + ib, \tag{2.1}$$

which, on simplification and equating the real and imaginary parts on the left-hand side with the corresponding parts on the right-hand side, yields,

$$e^x\{x\ cos(y) - ysin(y)\} = a \tag{2.2}$$

and

$$e^x\{y\ cos(y) - x\ sin(y)\} = b \tag{2.3}$$

Let $\cos A = a/(a^2 + b^2)^{1/2}$ and $\sin A = b/(a^2 + b^2)^{1/2}$, Then dividing Equation (2.3) by Equation (2.2) and simplifying results in

$$y(cos\ y\ cos\ A + sin\ y\ sin\ A) = -x(sin\ y\ cos\ A - cos\ y\ sin\ A)$$

or

$$x = -y\ \cot(y - A). \tag{2.4}$$

Putting the value of x given by Equation (2.4) into Equation (2.2) and simplifying further yields

$$ye^{-y\ cot\ (y-A)}\cos A = -a\ sin\ (y - A). \tag{2.5}$$

Substituting $\cos A$ with $a/(a^2 + b^2)^{1/2}$ in Equation (2.5), results in

$$ye^{-y\ cot\ (y-A)} = -(a^2 + b^2)^{1/2}\ sin\ (y - A). \tag{2.6}$$

The general principles of the quadratic approximation method follow those developed in Numerical Approximation of Lambert W Function for Real Values by Unique Method of Quadratic Approximation [36]. For completeness, the essential steps are briefly recalled here, while the modifications necessary for handling complex arguments are derived in detail below.

2.1. *Quadratic Approximation for $W(-a + ib) = x + iy$, when value of real quantity a is given, b is zero, x and y are real quantities*

When $b = 0$, $W(-a + ib) = x + iy$ takes the form $W(-a) = x + iy$ and Equation (2.1) transforms to

$$(x + iy)e^{(x+iy)} = -a, \tag{2.7}$$

Equation (2.4) to

$$x = -y \cot y \tag{2.8}$$

and Equation (2.5) to

$$ye^{-y\cot y} = a\sin y. \tag{2.9}$$

Taking natural logarithm of both sides and rearranging, Equation (2.9) takes the form

$$ln(y) - y\cot y - ln(\sin y) = ln\,(a). \tag{2.10}$$

Now assuming $y = y_1 + c_1$, where c_1 is very small quantity less than one and also $c_1 << y_1$, the Equation (2.10) transforms to

$$ln\,(y_1 + c_1) - (y_1 + c_1)\cot(y_1 + c_1) - ln\{\sin(y_1 + c_1)\} = ln\,a.$$

On expansion,

$$ln\,y_1 + ln(1 + c_1/y_1) - (y_1 + c_1)\frac{1 - \tan y_1 \tan c_1}{\tan y_1 + \tan c_1}$$
$$-ln(\sin y_1 \cos c_1 + \cos y_1 \sin c_1) = ln\,a. \tag{2.11}$$

It is submitted, value of $ln(1 + c_1/y_1)$ approximates $2c_1/(2y_1 + c_1)$, when $y_1 \gg c_1$ [34] and according to series expansion, since c_1 is a small quantity,

$$\tan c_1 = c_1 + c_1^3/3 + c_1^5/5 + \cdots,$$
$$\cos c_1 = 1 - c_1^2/2! + c_1^4/4! - \cdots,$$
$$\sin c_1 = c_1 - c_1^3/3! + c_1^5/5! - \cdots,$$

Therefore, on neglecting c_1^3 and its higher power, being small, Equation (2.11) takes the form

$$\frac{2c_1}{2y_1 + c_1} - (y_1 + c_1)\frac{1 - c_1\tan(y_1)}{\tan(y_1) + c_1} + \frac{2c_1^2}{4 - c_1^2} - \frac{2\,c_1}{2\tan(y_1) + c_1} = ln(a)$$
$$+\ln(\sin y_1) - ln(y_1). \tag{2.12}$$

On multiplying both sides by $(2y_1 + c_1)\{\tan(y_1) + c\}(4 - c_1^2)\{2\tan(y_1) + c_1\}$, yields

$$2c_1(\tan y_1 + c_1)(4 - c_1^2)(2\tan y_1 + c_1) - (y_1 + c_1)(1 - c_1\tan y_1)(2y_1 + c_1)(4 - c_1^2)(2\tan y_1 + c_1) + 2c^2(\tan y_1 + c_1)(2\tan y_1 + c_1)(2y_1 + c_1)$$
$$-2c_1(\tan y_1 + c_1)(2y_1 + c_1)(4 - c_1^2)$$
$$= (\tan y_1 + c_1)(2\tan y_1 + c_1)(2y_1 + c_1)(4 - c_1^2)(\ln + ln\sin y_1 - ln\,y_1). \tag{2.13}$$

On neglecting c_1^3 and its higher power and simplifying, a quadratic equation

$$c_1^2 - c_1\,l_1 - m_1 = 0 \tag{2.14}$$

is obtained, where

$$l_1 = 2\frac{\left\{y_1(y_1 + 5\tan y_1) - 2\tan^2 y_1(y_1^2 + 1) + \ln\left(\frac{a\sin y_1}{y_1}\right)(\tan y_1)(3y_1 + \tan y_1)\right\}}{\left\{-7y_1 + \tan y_1(3y_1^2 + 8y_1\tan y_1 + 2) - \ln\left(\frac{a\sin y_1}{y_1}\right)(2y_1 + 3\tan y_1 - y_1\tan^2 y_1)\right\}}, \tag{2.15}$$

$$m_1 = 4\frac{y_1 \tan y_1\left\{y_1+\tan y_1 \ln\left(\frac{a \sin y_1}{y_1}\right)\right\}}{\left\{-7y_1+\tan y_1(3y_1^2+8\, y_1 \tan y_1+2)-\ln\left(\frac{a\sin y_1}{y_1}\right)(2y_1+3\tan y_1-y_1\tan^2 y_1)\right\}}. \quad (2.16)$$

Out of the two roots $c_1 = l_1/2 \pm (1/2)(l_1^2+4m_1)^{1/2}$ of the quadratic equation (2.14), *the root that has magnitude less than* y_1 *and is the smaller in magnitude is chosen to approximate the real positive value of* $y_1 + c_1 = y_2$. For a better approximation, I now assume $y_3 = y_2 + c_2$ and obtain a quadratic equation,

$$c_2^2 - c_2 l_2 - m_2 = 0, \quad (2.17)$$

where

$$l_2 = 2\frac{\left\{y_2(y_2+5\tan y_2)-2\tan^2 y_2(y_2^2+1)+\ln\left(\frac{a\sin y_2}{y_2}\right)(\tan y_2)(3y_2+\tan y_2)\right\}}{\left\{-7y_2+\tan y_2(3y_2^2+8\, y_2\tan y_2+2)-\ln\left(\frac{a\sin y_2}{y_2}\right)(2y_2+3\tan y_2-y_2\tan^2 y_2)\right\}}, \quad (2.18)$$

$$m_2 = 4\frac{y_2 \tan y_2\left\{y_2+\tan y_2 \ln\left(\frac{a \sin y_2}{y_2}\right)\right\}}{\left\{-7y_2+\tan y_2(3y_2^2+8\, y_2 \tan y_2+2)-\ln\left(\frac{a\sin y_2}{y_2}\right)(2y_2+3\tan y_2-y_2\tan^2 y_2)\right\}}. \quad (2.19)$$

To have precise approximation, this process is iterated n times and that yields

$$y = y_{n+1} = y_n + c_n = y_1 + c_1 + c_2 + c_3 + \cdots + c_n \quad (2.20)$$

and

$$c_n < c_{n-1} < c_{n-2} < \cdots < c_1.$$

Kindly refer to section 3, titled ‘Convergence,’ to see why $c_n < c_{n-1} < c_{n-2} < \cdots < c_1$. On iterating n times, the final quadratic equation obtained is,

$$c_n^2 - c_n l_n - m_n = 0, \quad (2.21)$$

where

$$l_n = 2\frac{\left\{y_n(y_n+5\tan y_n)-2\tan^2 y_n(y_n^2+1)+\ln\left(\frac{a\sin y_n}{y_n}\right)(\tan y_n)(3y_n+\tan y_n)\right\}}{\left\{-7y_n+\tan y_n(3y_n^2+8\, y_n\tan y_n+2)-\ln\left(\frac{a\sin y_n}{y_n}\right)(2y_n+3\tan y_n-y_n\tan^2 y_n)\right\}}, \quad (2.22)$$

$$m_n = 4\frac{y_m \tan y_m\left\{y_m+\tan y_m \ln\left(\frac{a \sin y_m}{y_m}\right)\right\}}{\left\{-7y_n+\tan y_n(3y_n^2+8\, y_n \tan y_n+2)-\ln\left(\frac{a\sin y_n}{y_n}\right)(2y_n+3\tan y_n-y_n\tan^2 y_n)\right\}}. \quad (2.23)$$

In this paper, it is observed $n = 5$ yields results precise to ten decimal points. Once the values of y are known using Equation (2.20), the value of x can be determined using Equation (2.8). The value x will not be determined directly in this paper, as it can be calculated after approximating the value of y as explained. The values of y_1 in subsequent cycles can be assumed as is described in section 2.1.1.3, where the quadratic equation approximation is iterated for subsequent solutions.

2.1.1. Assumption of Value of y_1, When $W(-a) = x + iy$.

2.1.1.1. Assumption from Points of Intersection of the Curves and Ghost Solutions:

To assume the value of y_1, when $W(-a) = x + iy$, I refer to Equation (2.9), which has left-hand side $ye^{-y\cot y}$ and right-hand side $a \sin y$. Obviously, the points of intersection of the curve $f(y) = ye^{-y\cot y}$ with $g(y) = a \sin y$ will be the solutions of Equation (2.9) for values of y.

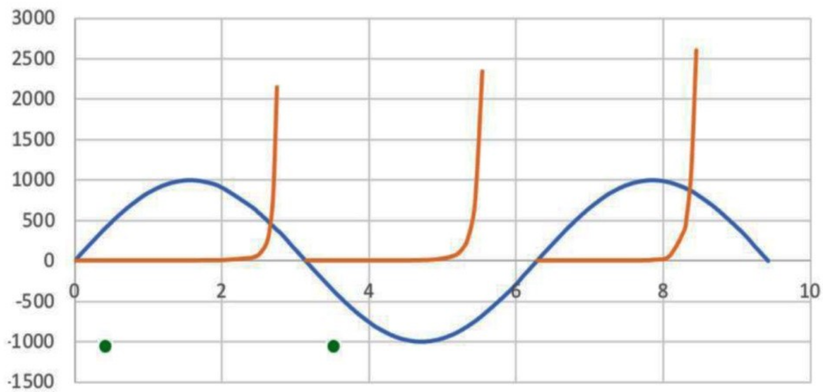

Figure 1

Plot of curves $f(y) = ye^{-y\cot y}$ and $g(y) = a \sin y$, when y is positive

Figure 1 shows the plot of both curves when y is assumed to be a positive value. Figure 2 shows the plot of both curves, when y is assumed a negative value. It is submitted, *assumptions and solutions of the equation here are given for y with positive values. For y with negative value, same solution is applicable, but the result will have a negative sign.* Please note that *values of $f(y)$ are plotted on the x-axis and the corresponding values of $f(y)$ and $g(y)$ on the y-axis.* In Figure 1, curve $f(y) = ye^{-y\cot y}$ rises from value zero exponentially to infinity, when y varies from value zero and tends to approach value π. Once y reaches value π and tends to cross past it, the curve immediately drops from infinity to zero and thereafter, it again rises exponentially from zero, when y increases from value π. Now when y increases from π to 2π, this curve repeats its course, when y varies from value zero to π. The curve thus goes on repeating its form from 2π ot $3\pi, 3\pi$ to 4 pi, and so on, as displayed in Figure 1. Curve $g(y) = a \sin y$ plots a sinusoidal shape that swings from 0 to $+a, +a$ to $0, 0$ to $-a$ and $-a$ to 0 as y varies from 0 to $\pi/2, \pi/2$ to π, π to $3\pi/2$ and $3\pi/2$ to 2π. This shape repeats as y varies from 2π to 4 π so on. Obviously, points of intersections of these two curves are the values of y that satisfy Equations (2.9) and (2.1). At $y = 0, \pi, 2\pi, 3\pi$ so on, two curves also intersect indicating value of x as 0 at all these points. However, at $y = 0$, Equation (2.7) transforms to $xe^x = -a$ and this equation relates to solution of Lambert W Function in real domain, and it is not the subject matter of this paper. At $y = \pi$, Equations (2.2) and (2.3) transform to $e^x(x + i\,\pi) = a$ or $xe^x = a$ and $e^x\pi = 0$. That means a must be equal to zero in all cases but a can be assigned any real value. Therefore, it t leads to absurdity, and thus Equation (2.7) is not defined at $x = \pi$. Similarly, it can be proved that this equation is also not defined at $y = 2\pi, 3\pi, 4\pi$, and so on. That leads to the fact, $y = \pi, 2\pi, 3\,\pi, 4\,\pi$ ... *are ghost intersecting points of the two curves and are not the solutions of the equation.*

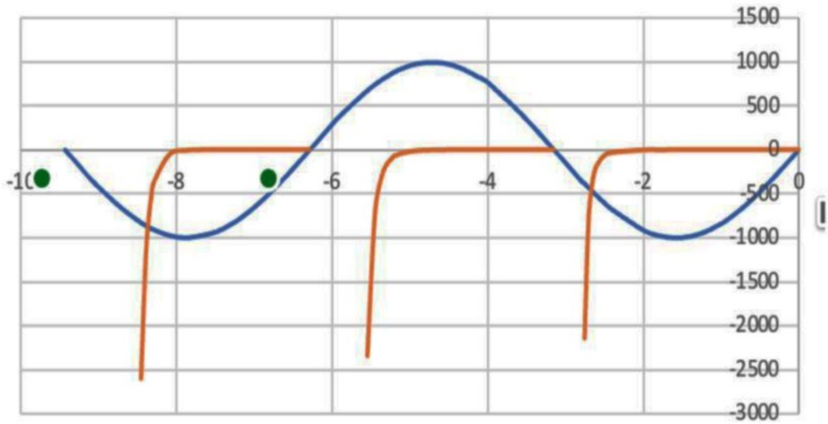

Figure 2

Plot of curves $f(y) = ye^{-y\cot y}$ and $g(y) = a\sin y$, when y is negative

When y is very small, $y\cot y$ tends to 1 and $\sin y$ to y and equation $ye^{-y\cot y} = a\sin y$ takes the form $ye^{-1} = a\,y$. That means, when $a < 1/e$, equation does not have solution for y in the domain of 0 to $\pm\,\pi$. However, it is solvable in the domain $y > 2\,\pi$. Kindly refer to row 1 of Table 1, where $a = 10^{-3}$ and y_1 is assigned value 7 which is more than 2π. Except the points $y = 2\pi, 3\pi, 4\pi, \ldots$, other intersecting points are the solutions of the equation. Value of y (between 0 to π) from intersecting point is assumed as y_1. To have precise values of y, it is necessary to refine these by repeated approximations.

Figure 2 displays the curves when value of y is less than 0. Same phenomenon occurs, except the fact points of intersection of two curves have negative values. Here intersecting points at $y = -\pi, -2\pi, -3\pi, -4\pi, \ldots$ *are ghost solutions of the equation.*

2.1.1.2. Assumption of value of y_1 from shape of curve of $(y) = ye^{-y\cot y}$:

Variation of $f(y) = ye^{-y\cot y}$ with changes in value of y are given in Table 1. Since $a\sin y = ye^{-y\cot y}$, therefore, $ye^{-y\cot y} \leq a$, and maximum value of $ye^{-y\cot y}$ is a. From Table 1, maximum value of y corresponding to maximum value of $ye^{-y\cot y}$, can also be known.

Table 1

Displaying variation of $ye^{-y\cot y}$ with change in y

y	$f(y) = ye^{-y \cot y}$	y	$f(y) = ye^{-y \cot y}$	y	$f(y) = ye^{-y \cot y}$
0	0	π/2	π/2	2.8	7370.1195
0.3	0.1137	1.8	2.7394	2.9	374719.7211
0.6	0.2496	2.1	7.1715	3.05	$(8.047752)10^{14}$
0.9	0.4406	2.4	32.9673	3.10	$(6.9454872)10^{32}$
1.2	0.7526	2.6	195.8491	3.137	$(1.37447458)10^{297}$

Noting the characteristics of curves $a\,sin\,y = ye^{-y\cot y}$, it is observed, value of $ye^{-y\cot y}$ rises as y increases meaning thereby larger value a will call for bigger value of y. It is further observed at $y = \pi/2$, a equals $\pi/2$, therefore, *when* $1/e \le a \le \pi/2$, y varies from 0 to $\pi/2$. For $a > \pi/2$, y will also be bigger than its value at $y = \pi/2$.

I assume $y_1 = \pi/2 + t$, where t is a small quantity; then, using Equation (2.9) and simplifying, approximates $(2/\pi)\,ln(2a/\pi)$. That leads to the fact that an assumption of y_1 can be made as $\pi/2 + t$ when t is small. Substituting value of t, it can be written

$$y_1 = \pi/2 \; + (2/\pi)\ln(2a/\pi). \tag{2.24}$$

It is practically observed that the above said assumption holds good, when $a \le 10$. When $a \ge 30$, I assume $y_1 = \pi - t$, where t a small quantity is, then using Equation (2.9) and simplifying, t approximates $\pi/\{ln(a/\pi)\}$. That means, assumption of y can be made as follow

$$y_1 = \pi - \pi/\{ln(a/\pi)\}. \tag{2.25}$$

It is practically observed that the above assumption holds good when $a \ge 50$. For a between 10 and 50,, the value of y_1 can be assumed 2.0. When $0 < a < 1/e$, Equation (2.9) does not have a solution between 0 and π but it has solutions between 2π and 3π, 4π and 5π and so on.

For finding solution between 2π and 3π, Equation (2.24) changes to

$$y_1 = 5\pi/2 \; + (2/5\pi)\ln(2a/5\pi). \tag{2.26}$$

Accordingly, value of y_1 can be assumed.

2.1.1.3. Subsequent Assumptions in domain 2π to 3π, 4π to 5π, 6π to 7π, so on:

When a precise approximation of the value of y, say α_1, is achieved between 0 and π by the method of iteration, then a rough value of y_1 in the domain 2π to 3π is approximated by the relation

$$y_1 = 2\pi + \alpha_1,$$

in the domain 4π *to* 5π by relation

$$y_1 = 2\pi + \alpha_2,$$

where α_2 is approximated value of y in domain 2π to 3π after iteration, Finally, value of y_1 in the domain $2n\pi$ to $(2n+1)\pi$ is approximated by relation

$$y_1 = 2\pi + \alpha_n.$$

Since initial assumption is rough, therefore, assumption of y_1 between $(2n\,\pi)$ and $(2n+1)\pi$ can also be made by relation

$$y_1 = 2(n-1)\pi + \alpha_1, \tag{2.27}$$

where α_1 is precise approximation in domain 0 to π after iteration. Out of the two methods explained above, the method for assuming the value of y_1 from points of intersection of two curves is preferred, as this method gives a better initial approximation.

2.1.1.4. Infinite Values of x, y Satisfying Function $W(-a) = x + i\,y$:

It is clear from Figures 1 and 2 that Equation (2.9) has solutions for y between 0 and π, between 2π and 3π, and so on. Similarly, it has solutions for y between 0 and $-\pi, -2\pi$ and $-3\pi, -4\pi$ and $-5\,\pi$ so on. The equation does not have solution for y between 0 and π, when $0 < a < 1/e$ but it has solutions for y in the domain $2\,\pi$ and above. Also, the equation does not have solution between between π and $2\pi, 3\pi$ and $4\pi, 5\pi$ and 6π so on. Similarly, it does not have solution for y between $-\pi$ and $-2\pi, -3\pi$, and $-4\pi, -5\,\pi$ and $-6\,\pi$ so on. Also, it does not have solution for y between 0 and $-\pi$, when $-1/e < a < 0$ but it has solutions for y, in the domain -2π and -3π, -4π and -5π so on.

2.1.1.5. Veracity and Truthfulness of Formulae Derived:

To prove veracity of formulae derived, assumption of value y_1 and subsequent approximation of $y_2, y_3, y_4, \dots, y_{n+1}$ for given value of a, I have calculated these values with the help of *Desmos* scientific calculator. These values are mentioned in Table 2. Function $W(-10^{-3}) = x + i\,y$ does not have a solution for y between 0 and π as already explained, therefore, assumption in domain of 2π and 3π is made using Equation (2.26). After iteration, precise approximation 6.9196696588 accurate to ten places of decimal is obtained. In domain, 4π to 5π rough approximation of y_1 is made as 13.18 which roughly equals $2\pi + 6.9196696588$.

This proves Lemma 2.1.

Table 2

Displaying Veracity of Precise Approximation of $y = y_{n+1} = y_n + c_n = y_1 + c_1 + c_2 + \dots + c_n$

Given a	Assumed y_1	Approximated $y_2 = y_1 + c_1$	Approximated $y_3 = y_2 + c_2$	Approximated $y_4 = y_3 + c_3$	Approximated $y_5 = y_4 + c_4$	Approximated $y_6 = y_5 + c_5$	Approximated $y = y_{n+1} = y_n + c_n$	Actual val. of y
10^{-3}	7	6.919547782	6.919669654	6.9196696588	Not needed	Not needed	6.9196696588	6.9196696588
10^{-3}	13.18	13.52213763	13.51466673	13.51360867	13.51362527	13.5136252809	13.5136252809	13.5136252809

1	1	1.375372977	1.337207001	1.3372357014	Not needed	Not needed	1.3372357014	1.33723570
1	7.5	7.588754939	7.58863098	7.588631179	7.588631178473	Not needed	7.588631178473	7.588631178473
10^3	2	2.65661153	2.664198051	2.664198143	2.66419814329	Not needed	2.66419814329	2.66419814329

Lemma 2.1: *Numerical value of y in Lambert W Function $W(-a) = x + iy$ can be approximated using equation $y = y_{n+1} = y_1 + c_1 + c_2 + c_3 \dots + c_n$, where c_n is a root smaller in magnitude out of two roots $l_n/2 \pm 1/2(l_n^2 + 4m_n)^{1/2}$, l_n, m_n are given by equations (2.22), (2.23), a is given real quantity, y_1 is an assumed quantity, $y_{n+1} = y_n + c_n$ and n is the number of times c_1 is iterated to obtain precise value of y. Thereafter, value of x is calculated using equation (2.8).*

2.1.1.6. Solution of $W(-a) = x + i\,y$ between $(2n\,\pi)$ and $(2n+1)\pi$, when integer n is very large (tending to infInity):

Let $y_1 = 2n\pi + \pi/2 + \delta$, where $a \neq 0$, $0 \leq \delta < \pi/2$ and δ can not be equal to $\pi/2$ as that will result in $y = (2n+1)\pi$ which can not be a solution of the equation. Putting this value of $y = 2n\pi + \pi/2 + \delta$ in Equations (2.9) and simplifying yield $e^{a\,sin(\delta)} = 1$ on account of the fact, when n tends to infinity, value of ψ^{ψ} tends to 1, where $\psi = a\,cos(\delta)/(2n\pi + \pi/2 + \delta)$. Consequently, δ tends to 0. However, if δ is replaced by $-\delta$, then $e^{-a\,sin\,\delta} = 1$, which implies $\delta \to 0$. Therefore, function $W(-a) = x + i\,y$ has always solutions at $y = 2n\,\pi + \pi/2 \pm \delta$, when integer $n \to \infty$ and $\delta \to 0$.

2.2. *Quadratic Approximation for $W(a+ib) = x + iy$, when value of real quantity a is given, $b = 0$, x and y are also real quantities*

When $b = 0$, $W(a+ib) = x + iy$ takes the form $W(a) = x + iy$ and Equation (2.1) transforms to

$$(x+iy)e^{(x+iy)} = a, \tag{2.28}$$

Equation (2.4) to

$$x = -y\,cot\,y \tag{2.29}$$

and Equation (2.5) to

$$ye^{-y\,cot\,y} = -a\,sin\,y \tag{2.30}$$

Equations (2.28) and (2.30) are identical to equations (2.7) and (2.9), except that the values of a have opposite signs. That means, method, formulae, equations explained and derived in sections 2.1 are applicable here by replacing a with $-a$.

2.2.1. Assumption of Value of y_1, When $W(a) = x + iy$

2.2.1.1. Assumption from Points of Intersection of the Curves:

In this case, the shape of the curves correspond to Figures 3 and 4, the points of intersection of curves $f(y) = ye^{-y\,cot\,y}$ with $g(y) = -a\,sin\,y$ are the solutions except points at $y = \pm\pi, \pm 2\pi, \pm 3\pi, \dots, \pm n\pi$ which are ghost points as already explained in section 2.1.1.1.

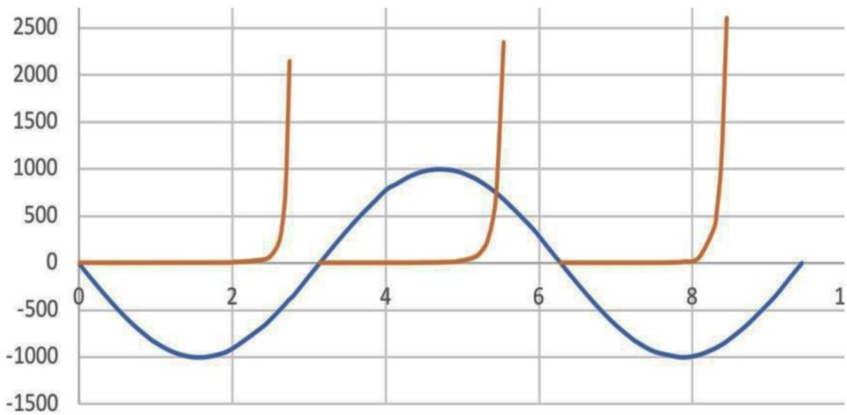

Figure 3

Plot of curves $f(y) = ye^{-y\,cot\,y}$ and $g(y) = -a\,sin\,y$, when y is positive

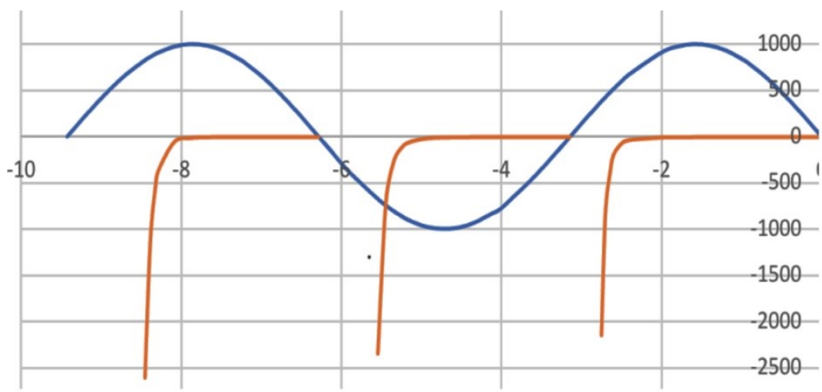

Figure 4

Plot of curves $f(y) = ye^{-y\,cot\,y}$ and $g(y) = -a\,sin\,y$, when y is negative

From the Figures 3 and 4, it is evident that the solutions for y lie between π to 2π, 3π to 4π, 5π to 6π, … , so on and also between $-\pi$ to -2π, -3π to -4π, -5π to -6π, …, so on. The method adopted for approximating the value of y_1 is the same as described in Section 2.1.1.2, except that the values are larger by π. When a is very small (tending to zero), let (π + δ) be the value of y that satisfies the Equation (2.30). In that case, $-a\, sin(\pi+\delta) = (\pi+\delta)$ $e^{-(\pi+\delta)\, cot\,(\pi+\delta)}$ or $(a\, sin\, \delta)/(\pi+\delta) = e^{-(\pi+\delta)\, cot\, \delta}$. This reveals that as a tends to zero, δ must tend to zero, hence a can take values as small as zero, unlike the case discussed in section 2.1.1.1.

2.2.1.2. Subsequent Assumption in domain 3π to 4π, 5π to 6π, 7π to 8π, so on:

Subsequent assumptions are made in the same manner as explained in section 2.1.1.4.

2.2.1.3. Infinite Values of x, y Satisfying Function $W(a) = x + i\, y$:

As explained in section 2.1.1.4, there are also infinite solutions to the above said equation.

2.2.1.4. Veracity and Truthfulness of Formulae Derived:

To prove veracity of formulae derived, assumption of value y_1 and subsequent approximations of $y_2, y_3, y_4, \dots, y_{n+1}$ for given value of a, I have calculated these values with the help of Desmos scientific calculator. These values are mentioned in Table 3. $W(10) = x + i\, y$ has a solution for y between π and 2π calculated as 4.857798356177 and between 3π and 4π calculated as 10.987003940068. Similarly, solution for y for function $W(10^4) = x + i\, y$ was approximated between π and 2π and was found 5.6087386751. No error was found, when approximation was made up to twelve decimal points. Kindly peruse Table 3.

Table 3

Precise Approximation of y for equation $ye^{-y\, cot\, y} = -a\, sin\, y$

Given a	Assumed y_1	Approximated $y_2 = y_1 + c_1$	Approximated $y_3 = y_2 + c_2$	Approximated $y_4 = y_3 + c_3$	Approximated $y_5 = y_4 + c_4$	Approximated $y_6 = y_5 + c_5$	Approximated $y = y_{n+1} = y_n + c_n$	Actual val. of y
-10	4.2	4.949974553	4.857376837	4.857795753	4.857798347	4.857798356177	4.857798356177	4.857798356177
-10	11	10.9870333	10.987003940068	Not needed	Not needed	Not needed	10.987003940068	10.987003940068
-10^4	5.2	5.606938209	5.60873873	5.60873867511	Not needed	Not needed	5.60873867511	5.60873867511

2.2.1.5. n^th Solution of $W(a) = x + i\, y$, When n is Very Large Tending to Infinity:

From explanation already given, there is a solution between $(2n+1)\pi$ to $(2n+2)\pi$, where n is an integer positive or negative. Let that n^{th} solution be

$$y = (2n+1)\pi + \pi/2 + \delta,$$

where $a \neq 0$, $0 \leq \delta < \pi/2$ and δ can not be equal to $\pi/2$ as that will result in $y = (2n+2)\pi$ which can not be the solution of the equation. Putting $y = (2n+1)\pi + \pi/2 + \delta$ in Equation (2.30) and simplifying yield $e^{-a \sin\delta} = 1$ on account of the fact, limit n tending to infinity, value of ψ^{ψ} tends to 1, where $\psi = a\cos(\delta)/(2n\pi + \pi/2 + \delta)$. Consequently, δ tends to 0 as value of a is nonzero. However, if δ is replaced by $-\delta$, then $e^{-a \sin\delta} = 1$, which implies $\delta \to 0$. Therefore, n^{th} solution of $W(a) = x + i\,y$ is always $y = (2n+1)\,\pi + \pi/2 \pm \delta$, when integer $n \to \infty$ and $\delta \to 0$. Derivations made in sections 2.1.1.6 and 2.2.1.5 prove Lemmas 2.2 and 2.3.

Lemma 2.2: *Imaginary part* y *of solution to Lambert W Function* $W(-a) = x + i\,y$ *in complex plane always equals to* $(\pi/2 \pm \delta)$ *added to* $2n\,\pi$, *when* a *is a real positive quantity neither equal to zero nor infinity, positive integer* n *tends to infinity and* δ *tends to zero.*

Lemma 2.3: *Imaginary part* y *of solution to Lambert W Function* $W(a) = x + i\,y$ *in complex plane always equals to* $(\pi/2 \pm \delta)$ *added to* $(2n+1)\pi$, *when* a *is a real quantity neither equal to zero nor infinity, positive integer* n *tends to infinity and* δ *tends to zero.*

2.2.2. Branch Classification

Approximation of y for Lambert W Function is generally found in domain 0 to 2π. However, when approximation of y has been made as (say) Y, approximation for kth branch in domain $2k\pi$ to $(2k+1)\pi$, for that initial approximation is made, $Y + 2\pi(k - K)$, where K is obtained by formula

$$Y/2\pi = K + K_1/K_2,$$

K, K_1, K_2 are integers and $K_1 < K_2$.

2.3. *Quadratic Approximation for* $W(a+ib) = x + i\,y$, *when value of real nonzero quantities* a *and* b *are given and* x *and* y *are also real quantities*

Taking natural logarithm of both sides and rearranging, Equation (2.5) takes the form

$$ln(y) - y\,cot(y - A) - ln\{sin\,(y - A)\} = ln\,(-a/cos\,A). \qquad (2.31)$$

Now assuming $y = y_1 + c_1$, where c_1 is very small quantity, the Equation (2.31) transforms to

$$\ln(y_1 + c_1) - (y_1 + c_1)\cot(y_1 + c_1 - A) - \ln\{\sin(y_1 + c_1 - A)\} = \ln(-a/cos\,A).$$

On expansion,

$$\ln y_1 + \ln\left(1 + \frac{c_1}{y_1}\right) - (y_1 + c_1)\frac{1 - \tan(y_1 - A)\tan c_1}{tan\,(y_1 - A) + tan\,c_1} - \ln\{\,in(y_1 - A)\cos c_1\} + cos\,(y_1 - A) sin\,c_1 = ln\left(\frac{-a}{cos\,A}\right) \quad (2.32)$$

It is submitted, value of $ln(1 + c_1/y_1)$ approximates $2c_1/(2y_1 + c_1)$, when $y_1 \gg c_1$ [34]. On series expansion neglecting c_1^3 and its higher power and following the procedure as detailed in section 2.1, Equation (2.32) transforms to a quadratic equation

$$c_1^2 - c_1\, l_1 - m_1 = 0, \tag{2.33}$$

where $l_1 = p_1/r_1, m_1 = q_1/r_1$ and

$$p_1 = 2[y_1\{y_1 + 5\,tan\,(y_1 - A)\} - 2(y_1^2 + 1)\tan^2(y_1 - A)]$$
$$+2\left[l\,n\left\{\frac{-a\,sin(y_1 - A)}{y_1\,cos\,A}\right\}\,tan\,(y_1 - A)\{3y_1 + tan\,(y_1 - A)\}\right], \tag{2.34}$$

$$q_1 = 4y_1\,tan\,(y_1 - A)\left[y_1 + tan(y_1 - A)\ln\left\{\frac{-a\,sin(y_1 - A)}{y_1\,cos\,A}\right\}\right], \tag{2.35}$$

$$r_1 = -7y_1 + \tan\,(y_1 - A)\,\{3y_1^2 + 8y_1 tan\,(y_1 - A) + 2\}$$
$$-ln\,\left\{\frac{-a\,sin(y_1 - A)}{y_1 cos\,A}\right\}\{2y_1 + 3tan\,(y_1 - A) - y_1 tan^2(y_1 - A)\}. \tag{2.36}$$

Equation (2.33) has two roots i.e. $c_1 = l_1/2 \pm (1/2)(l_1^2 + 4m_1)^{1/2}$ and one root which has magnitude less than y_1 and is also a root with smaller magnitude out of two roots, is chosen for approximation. Value of y_2 then equals $y_1 + c_1$. Thereafter, y_3 is assumed equal to $y_2 + c_2$, where $c_2 = l_2/2 \pm (1/2)(l_2^2 + 4m_2)^{1/2}$, $l_2 = p_2/r_2, m_2 = q_2/r_2$ and

$$p_2 = 2[y_2\{y_2 + 5\,tan\,(y_2 - A)\} - 2(y_2^2 + 1)\tan^2(y_2 - A)]$$
$$+2\left[l\,n\left\{\frac{-a\,sin(y_2 - A)}{y_2\,cos\,A}\right\}\,tan\,(y_2 - A)\{3y_2 + tan\,(y_2 - A)\}\right], \tag{2.37}$$

$$q_2 = 4y_2\,tan\,(y_2 - A)\left[y_2 + tan(y_2 - A)\ln\left\{\frac{-a\,sin(y_2 - A)}{y_2\,cos\,A}\right\}\right], \tag{2.38}$$

$$r_2 = -7y_2 + \tan\,(y_2 - A)\,\{3y_2^2 + 8y_2 tan\,(y_2 - A) + 2\}$$
$$-ln\,\left\{\frac{-a\,sin(y_2 - A)}{y_2 cos\,A}\right\}\{2y_2 + 3tan\,(y_2 - A) - y_2 tan^2(y_2 - A)\}. \tag{2.39}$$

Proceeding in this manner, $y_{n+1} = y_n + c_n$, where $c_n = l_n/2 \pm (1/2)(l_n^2 + 4m_n)^{1/2}$, $l_n = p_n/r_n, m_n = q_n/r_n$ and

$$p_n = 2[y_n\{y_n + 5\,tan\,(y_n - A)\} - 2(y_n^2 + 1)\tan^2(y_n - A)]$$
$$+2\left[ln\left\{\frac{-a\,sin(y_n - A)}{y_n\,cos\,A}\right\}\,tan\,(y_n - A)\{3y_n + tan\,(y_n - A)\}\right], \tag{2.40}$$

$$q_n = 4y_n\,tan\,(y_n - A)\left[y_n + tan(y_n - A)\ln\left\{\frac{-a\,sin(y_n - A)}{y_n\,cos\,A}\right\}\right], \tag{2.41}$$

$$r_n = -7y_n + \tan\,(y_n - A)\,\{3y_n^2 + 8y_n tan\,(y_n - A) + 2\}$$
$$-ln\,\left\{\frac{-a\,sin(y_n - A)}{y_n cos\,Ap}\right\}\{2y_n + 3tan\,(y_n - A) - y_n tan^2(y_n - A)\}. \tag{2.42}$$

To have precise approximation, this process is iterated n times and that yields

$$y = y_{n+1} = y_n + c_n = y_1 + c_1 + c_2 + c_3 + \cdots + c_n.$$

It is proved by practical examples in Table 5 followed by its theory under the section 3 relating to convergence that

$$c_n < c_{n-1} < c_{n-2} < \cdots < c_1.$$

2.3.1. Assumption of Value of y_1

2.3.1.1. Assumption of value of y_1 from shape of curve

Referring to Equation (2.6), I will plot a curve $f_1(y) = ye^{-y\,cot\,(y-A)}$ by taking the value of y on $X\ axis$ and the value of $ye^{-y\,cot\,(y-A)}$ on $Y\ axis$ and another curve $f_2(y) = -(a^2+b^2)^{1/2}\,sin\,(y-A)$ also by taking the value of y on $X\ axis$ and the value of $-(a^2+b^2)^{1/2} sin\,(y-A)$ on Y $axis$. Functions $W(a+i\,b) = x+i\,y, W(-a-i\,b) = x+i\,y, W(a-i\,b) = x+i\,y$ have values of A equal to $tan^{-1}(a/b)$, $\{\pi + tan^{-1}(a/b)\}$, $\{-tan^{-1}(a/b)\}$, $\{\pi - tan^{-1}(a/b)\}$ respectively and their respective graphs are displayed in Figures 5, 6, 7 and 8.

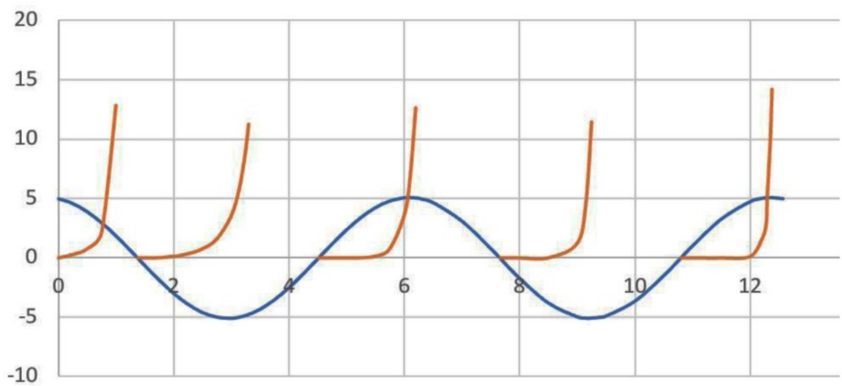

Figure 5

Plot of curves $f_1(y) = ye^{-y\,\cot -A)}$ and $f_2(y) = -(a^2+b^2)^{\frac{1}{2}}\,\sin(y-A)$ for function $W(a+i\,b) = x+iy$

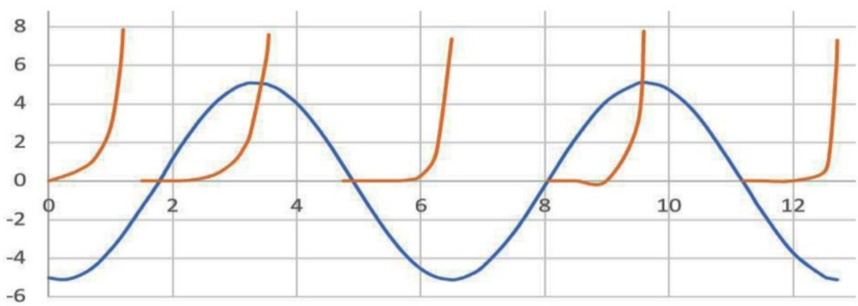

Figure 6

Plot of curves $f_1(y) = ye^{-y\,cot\,(y-A)}$ and $f_2(y) = -(a^2+b^2)^{1/2} sin\,(y-A)$ for function $W(-a-i\,b) = x+iy$

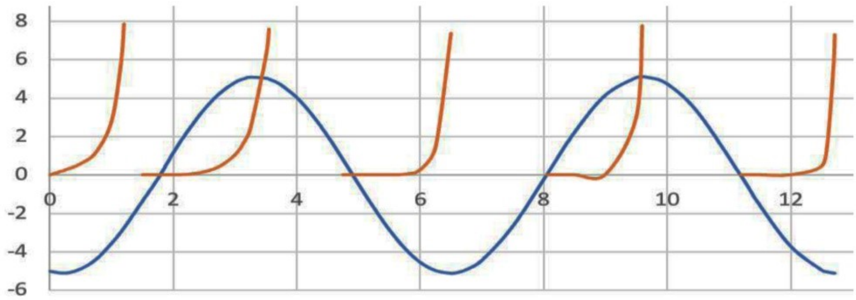

Figure 7

Plot of curves $f_1(y) = ye^{-y\,cot(y-A)}$ and $f_2(y) = -(a^2+b^2)^{1/2} sin\,(y-A)$ for function $W(a - i\,b) = x + i\,y$

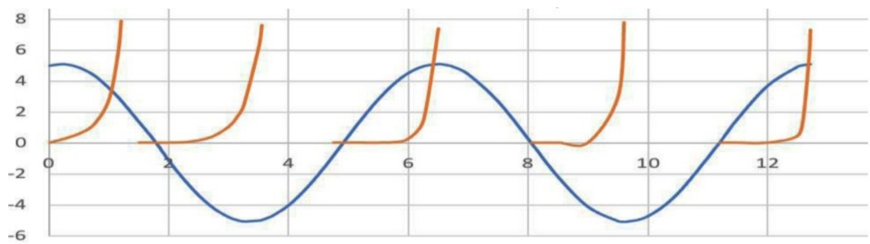

Figure 8

Plot of curves $f_1(y) = ye^{-y\,cot(y-A)}$ and $f_2(y) = -(a^2+b^2)^{1/2} sin\,(y-A)$ for function $W(-a + i\,b) = x + i\,y$

For solving $W(a + i\,b) = x + iy$, it is observed from Figure 5 that at $y = 0$, $f_2(y) = (a^2+b^2)^{1/2}\sin(A)$, where $A = tan^{-1}(b/a)$. At the value of $y = A$, $f_2(y)$ reaches 0 sinusoidally and goes into a negative cycle until $y = \pi + A$; thereafter it goes into a positive cycle until $y = 2\pi + A$. It then repeats these cycles. Curve $f_1(y) = ye^{-y\,\cot(y-A)}$ rises from zero at $y = 0$ to infinity exponentially as y tends to A with left-hand limit. It again rises from zero exponentially as y tends to A from right-hand side, reaching infinity at y tending to $\pi + A$ with left-handed limit. The curve again rises exponentially from zero as y tends to $\pi + A$ from the right-hand side reaching infinity as y tends to $2\pi + A$ with a left-hand limit. For intersection of these two curves, variation of y from 0 to A, $\pi + A$ to $2\pi + A$, $3\pi + A$ to $4\pi + A$ so on are needed. In this paper, initial assumption of value of y is made in half cycle of sine wave from $\pi + A$ to $2\pi + A$. It is further observed at $y = (3\pi/2 + A)$, the curve reaches maximum value of $(a^2+b^2)^{1/2}$. Let $y_1 = 3\pi/2 + t$, where t is a *small quantity*, then

$$-(a^2+b^2)^{1/2}\,sin(3\pi/2 + t) = (3\pi/2 + A + t)e^{-(3\pi/2+A+t)cot(3\pi/2+t)}.$$

On simplification, it leads to

$$(a^2+b^2)^{1/2} = (3\pi/2 + A + t)e^{t(3\pi/2+A)},$$

since $cos(t) = 1, tan\, t = t, t^2 = 0,$ when t has a very small value. On further simplification,

$$t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}.$$

Therefore,

$$y_1 = 3\pi/2 + A + t = 3\pi/2 + A + \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}. \quad (2.43)$$

For $-(a^2+b^2)^{1/2}\, sin(3\pi/2 + t)$ to be in positive half cycle so as to equal $(3\pi/2 + A + t)e^{-(3\pi/2+A+t)cot(3\pi/2+t)}$, it is essential

$$-\pi/2 < \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\} < \pi/2.$$

Although t is a small quantity, its limit is considered between $+\pi/2$ and $-\pi/2$. When value of $t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$ is appreciable and in the vicinity of $-\frac{\pi}{2}, y_1$ is assumed to be slightly more than $(\pi + A)$ (say)$(\pi + A + .1)$. When value of $\left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$ is appreciably large and is in the vicinity of $+\pi/2, y_1$ is assumed a value slightly less than $(2\pi + A)$ say $(2\pi + A - .1)$.

For solving $W(a - i\,b) = x + iy$, $W(-a + i\,b) = x + i\,y$ and $W(-a - i\,b) = x + iy$, Equation (2.43) will still be applicable but A will equal $-tan^{-1}(b/a), \{\pi - tan^{-1}(b/a)\}$, $\{\pi + tan^{-1}(b/a)\}$ respectively. Method of assumption of value for y_1 will be same as in the said case of $W(a + i\,b) = x + i\,y$.

2.3.1.2. Assumption from Points of Intersection of the Curves:

Another way to assume values of y_1 is from the points of intersection of two curves $f_1(y) = ye^{-y\,cot\,(y-A)}$ and $f_2(y) = -(a^2+b^2)^{1/2}\, sin\,(y - A)$. These curves are plotted in Figures 5 to 8. *X* coordinates of their points of intersections are noted. Accordingly, value of *X* coordinate will be initial assumption of value of y_1. Care will have to be exercised not to consider ghost intersection points for the purpose of assumption of value of y_1.

2.3.1.3. Subsequent Assumptions:

Subsequent assumptions for the solution in the next cycle can be made as already explained in section 2.1.1.3.

2.3.1.4. Infinite Values of x, y Satisfying Function $W(a + i\,b) = x + i\,y$:

The function $W(a + i\,b) = x + i\,y$ also has infinite solutions. Kindly refer to section 2.1.1.4.

2.3.2. Veracity and Truthfulness of Formulae Derived

To prove veracity of formulae derived, assumption of value y_1 and subsequent approximations of $y_2, y_3, y_4, \ldots, y_{n+1}$ for given value of $a + ib$, I have calculated the values of y using *Desmos* scientific calculator. I have taken an example considering $a = 1$ and $b = 5$. These values are mentioned in Table 4.

Table 4

Displaying Veracity of Precise Approximation of $y = y_{n+1} = y_n + c_n = y_1 + c_1 + c_2 + \cdots + c_n$ for $W(a + ib) = x + iy$,

Given $a + i\,b$	y_1	$y_2 = y_1 + c_1$	$y_3 = y_2 + c_2$	$y_4 = y_3 + c_3$	$y_5 = y_4 + c_4$	$y_6 = y_5 + c_5$	$y = y_7 = y_6 + c_6$	Actual y
$1 + i\,5$	5.5	6.0199565	6.0565281	6.057079547	6.05725613	6.05729872	6.05729931	6.05729933
$1 + i\,5$	13	12.23684605	12.29744385	12.29730118	12.29730161	12.29730160988	No need	12.29730160988
$-1 + i\,5$	5.9	6.490816373	6.444153254	6.4441624963	No need	No need	No need	6.4441624963
$1 - i\,5$	2.7	3.638990708	3.450084887	3.449981974	3.44998197469	No need	No need	3.44998197469
$-1 - i\,5$	8.7	9.26212772	9.163198178	9.16327806468	No need	No need	No need	9.16327806468

Taking the case of the first row of Table 4 and applying Equation (2.43), the initial assumption calculates to 6.05672068, which is close to the precise approximation of 6.057299331. However, even the initial assumption 5.5 used in Table yielded the precise approximation of 6.057299331. This pertains to the branch corresponding to 2π(1) to 2π(2) i.e. W_1. For W_2 branch, the initial assumption is the precise approximation W_1 plus 2π. However, in Row 2 of the table, it is taken as the whole number 13 instead of 12.34048464 to demonstrate flexibility in the initial assumption, which still leads to the precise approximation of 12.29730160988.

In Row 3, applying Equation (2.43), the initial assumption calculates to 6.443583894, which is close to the precise approximation of 6.441624963. However, the initial assumption 5.9 used in Table, instead of 6.443583894, was employed to prove the flexibility in the initial assumption, ultimately leading to the precise approximation of 6.441624963.

In Row 4, applying Equation (2.43), the initial assumption calculates to 3.465787235, which is close to the precise approximation of 3.44998197469. However, the initial assumption used in the table was taken as 2.7 instead of 3.465787235 to again demonstrate flexibility in the initial assumption, which still leads to the precise approximation of 3.44998197469.

In Row 5, applying Equation (2.43), the initial assumption calculates to 9.163103376, which is close to the precise approximation of 9.16327806468. However, the initial assumption used in the table was taken as 8.7 instead of 9.163103376, again demonstrating flexibility in the initial assumption while still leading to the precise approximation of 9.16327806468. Precise approximations — 6.057299331, 6.441624963, 3.44998197469, 9.16327806468 — for y of Lambert W functions, $(1+i\,5) = x + i\,y, W(-1+i\,5) = x + i\,y, W(1-i\,5) = x + i\,y, W(-1-i\,5) = x + i\,y$, satisfy the Equation (2.6), demonstrating the validity of the method used. This also proves Lemma 2.4.

Lemma 2.4: *Numerical value of* y *in Lambert W Function* $W(a + i\,b) = x + iy$ *is approximated* $y = y_{n+1} = y_1 + c_1 + c_2 + c_3 \ldots + c_n$, *where* c_n *is a root with smaller magnitude out of two roots* $l_n/2 \pm 1/2(l_n^2 + 4m_n)^{1/2}$, $l_n = p_n/r_n, m_n = q_n/r_n$ *and* p_n, q_n, r_n *are given by equations (2.40), (2.41), (2.42),* a *is given real quantity,* y_1 *is an initially assumed quantity,* $A = tan^{-1}(b/a)$, $y_{n+1} = y_n + c_n$, *and* n *is the number of times* c_1 *is iterated to obtain precise value of* y. *Value of* x *is calculated using equation (2.4).*

2.3.3. Branch Classification

Branch classification has already been explained in section 2.2.2 and same is applicable to these cases.

3. Convergence

I will prove convergence of the function $W(-a + i\,b) = x + iy$, when the real quantity a is given, value of b is zero, and when x and y are also real quantities; that transforms the function to $W(-a) = x + iy$ which can be written $(x+iy)e^{(x+iy)} = -a$. As explained in section 2.1, I obtained Equations (2.10), reproduced below

$$ln(y) - y\cot y - ln(\sin y) = ln\,(a).$$

Considering $y = y_1 + c_1$ and simplifying the resultant equation, and assuming $c_1 \ll y_1$, I obtain quadratic equation $c_1^2 - c_1\,l_1 - m_1 = 0$, where l_1 and m_1 are given by Equations (2.22) and (2.23) respectively. The root c_1 having *magnitude* less than y_1 out of two roots $l_1/2 \pm 1/2(l_1^2 + 4m_1)^{1/2}$ of the said quadratic equation will be used to approximate real positive value of $y_1 + c_1$. For precise approximation, it is further iterated and value of $y_3 = (y_2 + c_2)$, $y_4 = (y_3 + y_3), \ldots, y_{n+1} = (y_n + a_n)$ are found. The quadratic approximation is based on formula $ln\{(1 + c_1/y_1)\} \simeq 2c_1/(2y_1 + c_1)$ and series expansion of $\sin c_1, \cos c_1, \tan c_1$, where $c_1 \ll y_1$. Thereafter, value of y_2 obtained as $y_1 + a_1$, will approximate more precisely equation $ln(y) - y\cot y - ln(\sin y) = ln\,(a)$ as compared to $y = y_1$. Similarly, y_3 will approximate more precisely the said equation than $y = y_2$ and finally $y_{n+1} = z_n + a_n$ will approximate the best when integer n is a large number. That means

$$|y_1 + ln\left(\frac{a\sin y_1}{y_1}\right)| > |y_2 + ln\left(\frac{a\sin y_2}{y_2}\right)| > |y_3 + ln\left(\frac{a\sin y_3}{y_3}\right)| > \cdots |y_n + ln\left(\frac{a\sin y_n}{y_n}\right)|. \quad (3.1)$$

Also, a root of equation $c_1^2 - c_1\,l_1 - m_1 = 0$, approximates $-m_1/l_1$, since the quantity c_1 is small, therefore,

$$|-m_1/l_1| > |-m_2/l_2| > |-m_3/l_3| > \cdots |-m_n/l_n| \text{ (3.2) or}$$

$$|c_n| < |c_{n-1}| < |c_{n-2}| < \cdots < |c_1| \tag{3.3}$$

or $|c_1| + |c_2| + |c_3| + \cdots + |c_n|$ converges and thus value of y converges to y_{n+1}, where

$$y_{n+1} = y_1 + c_1 + c_2 + c_3 + \cdots + c_n,$$

value of y_1 is arbitrarily assigned, value of $c_n = (1/2)\left(l_n \pm \sqrt{l_n^2 + 4m_n}\right)$ and value of l_n and m_n are given by Equations (2.22) and (2.23). Similarly, convergence in other cases can also be proved and is not explained here to avoid repetition. *It is also important to state that assumption of value of y_1 must be made so that value of c_1 so found is small otherwise the sum $y_1 + c_1 + c_2 + c_3 + \cdots + c_n$ may diverge.* How assumption of y_1 should be made, has already been explained in relevant subsections of section 2.

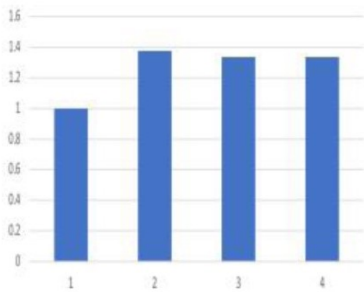

Figure 9

Displaying appr. values of y_1, y_2, y_3, y_4 for function $W(-1) = x + iy$

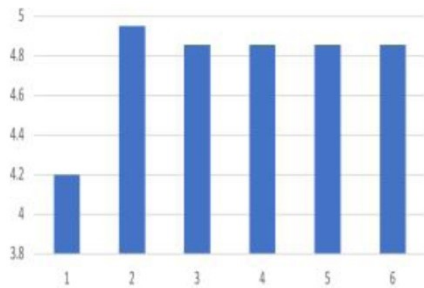

Figure 10

Displaying appr. values of $y_1, y_2, y_3, y_4\ y_5, y_6$ for function $W(10) = x + iy$

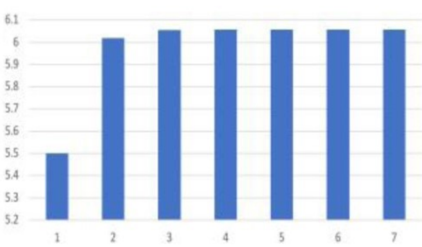

Figure 11

Displaying appr. values of $y_1, y_2, y_3, y_4\ y_5, y_6, y_7$ for function $W(1 + i\,5) = x + iy$

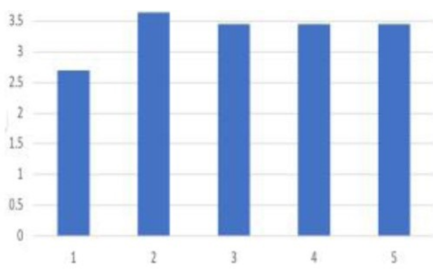

Figure 12

Displaying appr values of y_1, y_2, y_3, y_4, y_5 function $W(1 - i\,5) = x + i$y

For the function $W(-1) = x + iy$, approximated values of are entered in row against $a = -1$ of Table 2. On the basis of these approximated values, bar chart is plotted in Figure 9 to show graphically convergence of $y = y_{n+1} = y_1 + c_1 + c_2 + c_3 + \cdots c_n$ to 1.3372357014. Similarly, functions $W(10) = x + iy$, $W(1 + i\,5) = x + iy$, and $W(1 - i\,5) = x + iy$ are plotted in Figures 10, 11, 12, 13 taking data from Tables 3 and 4 to display graphically convergence of $y = y_{n+1} = y_1 + c_1 + c_2 + c_3 + \cdots c_n$. It is evident from the figures that as number of iterations increases, y approximates more precisely with its actual value. In examples taken up in this paper, maximum six iterations yielded precise result.

4. Computation of x

Referring to Equation (2.4) $x = -y\cot(y - A)$, which is an exact equation and value of x can be calculated straightway by inserting the value of y in it. There is thus no need to approximate the value of x when y has been finely and precisely approximated.

Care needs to be exercised, because a slight error in the approximation of y may be amplified in x on computation due to the behaviour of the cotangent of an angle when the angle is in the vicinity of zero. As far as the equation is concerned, it is exact and true. However, there may be cases where $\cot(y - A)$ reaches zero or infinity values. These are special cases explained in the preceding sections.

4.1. *When $cot(y - A)$ Reaches Zero Value*

When $\cot(y - A)$, then $x = -y\cot(y - A) = 0$ and $(y - A) = \pi/2$ If $x = 0$, then either $y = 0$ or $\cot(y - A) = 0$. If y is assumed zero, then $xe^x = a + i\,b$ and, since x is a real quantity, it cannot have a solution in the complex domain. That means b must be zero and, in that case, it becomes a solution of the Lambert W function in the real domain, which is not the subject of this paper.

Therefore, $\cot(y - A)$ should be zero. Or $y - A = (2m + 1)\pi/2$. Therefore,

$$y = \pm(2m + 1)\pi/2 + A \tag{4.1}$$

where

$$A = tan^{-1}(b/a), \text{ when } W(a + ib) = iy,$$
$$A = \pi - tan^{-1}(b/a), \text{ when } W(-a + ib) = iy,$$
$$A = \pi + tan^{-1}(b/a), \text{ when } W(-a - ib) = iy,$$
$$A = -tan^{-1}(b/a), \text{ when } W(a - ib) = iy,$$

and $m = 0, 1, 2, 3, \ldots$ Since $x = 0$, the function takes the form $iye^{iy} = a + ib$. This can be written as $i\{(2m + 1)\pi/2 + A\}e^{i\{(2m+1)\pi/2+A\}}$ or

$$i\{\pm(2m + 1)\pi/2 + A\}\,cos\{\pm(2m + 1)\pi/2 + A\} - \{\pm(2m + 1)\pi/2 + A\}\,sin\{\pm(2m + 1)\pi/2 + A\} = a + ib$$

Comparing real and imaginary parts and simplifying,

$$\{\pm(2m + 1)\pi/2 + A\}\,cos\{\pm(2m + 1)\pi/2 + A\} = b \tag{4.2}$$

$$\{\pm(2m+1)\pi/2 + A\}\, sin\{\pm(2m+1)\pi/2 + A\} = -a \tag{4.3}$$

$$\{\pm(2m+1)\pi/2 + A\}^2 = a^2 + b^2 \tag{4.4}$$

This proves Lemma 4.1.

Lemma 4.1: *If for a given Lambert W Function* $W(a + i\,b) = x + i\,y$, *the equation* $\{\pm(2m+1)\pi/2 + A\}^2 = a^2 + b^2$, *where* $m = 0, 1, 2, 3, \ldots$ *and* $A = tan^{-1}(b/a)$, *is satisfied, then its exact solution is* $x = 0$, $y = (2m + 1)\pi/2\ + A$.

Corollary 1: *If* $W(i\,b) =\ i\,y$ *and* $\{\pm(m+1)\pi\}^2 = b^2$, *where* $m = 0, 1, 2, 3, \ldots$, *then its exact solution is* $x = 0$ *and* $y = (m + 1)\pi$.

4.1. *When* $cot(y - A)$ *Reaches Infinite Value*

When $\cot(y - A)$ reaches infinite value, then $(y - A) = 0$ and Equation (2.4) by transposition can be written $y = -x\tan(y - A)$ yielding $y = k\pi + A$ and these are ghost solutions in complex plain as explained in section 2.2.1.1.

4.2. *When* $y = (4m + 1)\pi/2$, *and* $m = 0,1,2,3\ldots$*or* $y = (4m - 1)\pi/2$ *and* $m = 1,2,3\ldots$

4.2.1. For solving $W(-a + ib) = x + iy$, when $y = (4m + 1)\pi/2$ and $m = 0, 1, 2, 3\ldots$

Equations (2.2) and (2.3) take the form, $-\,e^x(4m + 1)\pi/2 = -a$, $e^x x = b$. From these equations, $e^x = 2a/(4m + 1)\pi$ and $x = \pi b(4m + 1)/2a$ and that yields

$$(4m + 1)\pi/2 = ae^{-\pi b(4m+1)/2a}. \tag{4.5}$$

4.2.1. For solving $W(a + ib) = x + iy$, when $y = (4m - 1)\pi/2$ and $m = 1, 2, 3\ldots$

Following the same procedure as given above, I get

$$(4m - 1)\pi/2 = ae^{\pi b(4m-1)/2a}. \tag{4.5}$$

The following two corollaries are obtained from the above derivations.

Corollary 2: *If for Lambert W Function* $W(-a + ib) =\ x + iy$, *where a and b are positive real quantities and equation* $(4m + 1)\pi/2 = ae^{-\pi b(4m+1)/2a}$, *is satisfied for specific* m, *where* $m = 0, 1, 2, 3\ldots$ *then* $y = (4m + 1)\pi/2$ *for corresponding* m *and* $x = -(\pi/2)(4m + 1)cot[(4\,m + 1)\pi/2 - \{\pi - tan^{-1}(b/a)\}]$ *(referring to Equation 2.4) or* $x = (\pi b/2a)(4m + 1)$.

Corollary 3: *If for Lambert W Function* $W(a + ib) =\ x + iy$ *where a and b are positive real quantities and equation* $(4m - 1)\pi/2 = ae^{\pi b(4m-1)/2a}$, *is satisfied for specific* m, *where* $m = 1, 2, 3\ldots$ *then* $y = (4m - 1)\pi/2$ *for corresponding m and* $x = -(\pi b/2a)(4m - 1)$.

5. Algorithm For Solving $W(a+ib) = x + i\,y$, When Value of Real Nonzero Quantities a and b are Given and x and y are also Real Quantities

1. Let $W(a+ib) = x + i\,y$, and real values of a and b be given. Check the polarity of a and b. If these have sign $-a$ and sign $+b$ go to 4. If these have sign a and sign $-b$ go to 6. If these have sign $-a$ and sign $-b$ go to 8.
2. Compute $A = tan^{-1}b/a$, and $t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$. If $t \geq \pi/2$ or $t \leq -\pi/2$, then go to 9.
3. Go to 11.
4. Compute $A = \pi - tan^{-1}b/a$, and $t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$, If $t \geq \pi/2$ or $t \leq -\pi/2$, then go to 9.
5. Go to 11.
6. Compute $A = \pi + tan^{-1}b/a$, and $t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$. If $t \geq \pi/2$ or $t \leq -\pi/2$, then go to 9.
7. Go to 11.
8. Compute $A = -tan^{-1}b/a$, and $t = \left(\frac{1}{3\pi/2+A}\right)\ln\left\{\frac{(a^2+b^2)^{1/2}}{3\pi/2+A}\right\}$. If $t \geq \pi/2$ or $t \leq -\pi/2$.
9. If value of t is appreciable or is in the vicinity of $-\pi/2$, then let $y_1 = (\pi + A + \alpha)$ where $\alpha = .1$ (say). Compute $y_1 = (\pi + A + .1)$ and go to 13..
10. If value of t is appreciably large and is in the vicinity of $+\pi/2$, then let $y_1 = (2\pi + A - \alpha)$, where $\alpha = .1$ (say). Compute $y_1 = (2\pi + A - .1)$ and go to 13.
11. Compute $y_1 = 3\pi/2 + A + t$ and go to 13.
12. Define

$$p_n = 2[y_n\{y_n + 5\,tan\,(y_n - A)\} - 2(y_n^2+1)\tan^2(y_n - A)]$$
$$+2\left[ln\left\{\frac{-a\,sin(y_n - A)}{y_n\,cos\,A}\right\}\,tan\,(y_n - A)\{3y_n + tan\,(y_n - A)\}\right],$$
$$q_n = 4y_n\,tan\,(y_n - A)\left[y_n + tan(y_n - A)\ln\left\{\frac{-a\,sin(y_n - A)}{y_n\,cos\,A}\right\}\right],$$
$$r_n = -7y_n + \tan\,(y_n - A)\,\{3y_n^2 + 8y_n tan\,(y_n - A) + 2\}$$
$$-ln\,\left\{\frac{-a\,sin(y_n - A)}{y_n cos\,A}\right\}\{2y_n + 3tan\,(y_n - A) - y_n tan^2(y_n - A)\},$$
$$l_n = p_n/r_n,$$
$$m_n = q_n/r_n$$

c_n is smaller of $l_n/2 \pm 1/2(l_n^2 + 4m_n)^{1/2}$,

$$y_{(n+1)} = y_n + c_n$$

13. Let $y = y_1 + c_1$.
14. Compute p_1, q_1, r_1, l_1, m_1 and c_1.
15. Compute $y_2 = y_1 + c_1$.
16. Compute p_2, q_2, r_2, l_2, m_2 and c_2.
17. Compute $y_3 = y_2 + c_2$.
18. Compute p_3, q_3, r_3, l_3, m_3 and c_3.
19. Compute $y_4 = y_3 + c_3$.
20. Compute p_4, q_4, r_4, l_4, m_4 and c_4.
21. Compute $y_5 = y_4 + c_4$.
22. Compute p_5, q_5, r_5, l_5, m_5 and c_5.
23. Compute $y_6 = y_5 + c_5$.
24. Compute p_6, q_6, r_6, l_6, m_6 and c_6.
25. Compute $y_7 = (y_6 + c_6)/2\pi = K + K_1/K_2$, where K, K_1, K_2 are integers and$K_1 < K_2$. If $K = k$, then go to 28.
26. Check polarity of k, if it is of the form $+k$, then compute $y = y_7 + 2\pi(k - K)$ and go to 13.
27. Since k is of the form $-k$, compute $y = y_7 - 2\pi(k + 1 + K)$ and go to 13.
28. Compute $x = x_7 = -y_7 \cot(y_7 - A)$.
29. Print $y = y_7$ and $x = x_7$.
30. End.

6. Comparison of Methods for Numerical Approximation of Lambert W Function

6.1 *Newton's Method of Approximation*

For approximation of a root of a function $f(x)$, Newton's Method uses the equation

$$x_{n+1} = x_n - f(x_n)/f'(x_n). \tag{4.1}$$

where $f(x_n)$ and $f'(x_n)$ are values of function and its derivative at $x = x_n$. It suffers certain drawbacks. The method fails in cases of non differentiability and also when derivatives are zero or undefined. It converges slowly and breaks down at zero derivatives. It is vulnerable at critical points and breaks down at zero derivatives [35].

6.2 *Halley's Method*

The iterative formula for Halley's method is

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2\{f'(x_n)\}^2 - f(x_n)f''(x_n)}, \tag{4.2}$$

where $f(x_n)$ is the value of the function at x_n, $f'(x_n)$ is the first derivative of the function at x_n and $f''(x_n)$ is the second derivative of the function at x_n. Although this method is a third-order convergence (cubic) method, meaning it converges faster than Newton's method in theory. However, it has its own set of drawbacks. It has computational complexity that requires first and second derivatives, making it difficult for higher-dimensional problems. It further suffers singularity issues and diverges when the second derivative is zero or near zero. It is inefficient for simple functions and adds computational overhead without additional benefits. Its implementation is difficult and creates challenges for systems with fluctuating derivatives. It suffers numerical instability and is susceptible to rounding errors in finite-precision computations.

6.3 *Quadratic Equation Approximation*

Its iterative formula is based on quadratic equation derived from a transformed Lambert W function approximation. It has certain advantages over the said methods. It yields precise results within 3-4 iterations. It does not have differentiability requirement thus eliminating the need for derivatives. It handles critical points efficiently and is unaffected by zero or undefined derivatives. It has broader applicability and solves a wider range of problems. It has the salient feature of fast convergence and achieves high accuracy in fewer iterations. It is easy to apply as finding roots of a quadratic equations falls in basic school education.

7. Results And Conclusions

While the original real-domain formulation of the method [36] established its efficiency, the present analysis demonstrates that its extension to the complex domain uncovers new structural properties of the Lambert W branches, including convergence of the imaginary part to specific multiples of π. The Lambert W Functioning in complex domain $W(a+i\,b)=x+i\,y$ written $(x+i\,y)e^{x+i\,y}=a+i\,b$ can be split into equations $ye^{-y\cot(y-A)}=-(a^2+b^2)^{1/2}\,sin\,(y-A)$ and $x=-y\,\cot(y-A)$, where $A=\tan^{-1}(b/a)$, x, y, a and b are real quantities. Admittedly, former is a transcendental equation involving variable only y. Taking natural logarithm and rearranging transforms the equation to $ln(y)-y\,cot(y-A)-ln\{\sin(y-A)\}=ln\,(-a/\cos A)$. Considering $y=y_1+c_1$, where $c_1\ll y_1$ and y_1 is initial assumption of value to y $ln(1+c_1/y_1)$ approximates $2c_1/(2y_1+c_1)$ and that yields $ln\,y=ln\,y_1+2c_1/(2y_1+c_1)$ [34]. In the same way, quantity $ln\{sin\,(y-A)\}$ can be approximated. Also substituting y with y_1+c_1, in the term $y\,cot(y-A)$ and replacing $sin\,c_1, cos\,c_1, tan\,c_1$ with their corresponding series expansion as c_1 is a small quantity, an algebraic equation in variable c_1 is obtained. In this way, equation $ln(y)-y\,cot(y-A)-ln\{\sin(y-A)\}=ln\,(-a/\cos A)$ is transformed to a quadratic equation in variable c_1 after ignoring terms containing power of c_1 higher than two.

For precise result, approximation is iterated n times using equation $y_{n+1}=y_n+c_n$, where n stands for number of times the approximation is iterated. Once value of c_1 which is a root smaller in magnitude of two roots of the quadratic equation is determined, value of y_2 is then determined by said equation $y_2=y_1+c_1$. Following the procedure, value of y_{n+1} is determined and that equals precise approximation of y. Equation $y_{n+1} = y_n + c_n$ can also be written $y_{n+1} = y_1 + c_1 + c_2 + c_3 + \cdots + c_n$. Based upon approximation of $ln(1 + c_1/y_1)$ with $2c_1/(2y_1 + c_1)$ and convergence of series expansion of $\sin c_1$, $cos\ c_1$, $tan\ c_1$, where value of c_1 is small on account of my assumption of value of y_1, it is proved $c_1 > c_2 > c_3 > \cdots > c_n$. Once value of y stands approximated with desired precision, solution in next cycle is approximated by considering the initial assumption equal to sum of 2π and approximated value in preceding cycle. In this way, infinite solutions are approximated by repeating the process. To corroborate the truthfulness of the method and the formulae derived, exhaustive examples are mentioned in tabular form. It is also observed that imaginary part y of solution of Lambert W Function $W(-a) = x + i\,y$ in complex plane always equals to $(\pi/2 \pm \delta)$ added to $2n\,\pi$, when a is a real positive quantity neither equal to zero nor infinity, positive integer n tends to infinity and δ tends to zero. Similarly, imaginary part y of solution to Lambert W Function $W(a) = x + i\,y$ in complex plane always equals to $(\pi/2 \pm \delta)$ added to $(2n + 1)\pi$, when a is a real quantity neither equal to zero nor infinity, positive integer n tends to infinity and δ tends to zero.

Statements and Declarations

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Author's Contribution

The entire research paper is the sole contribution of the corresponding author.

Data Availability Statement

There is no data-availability statement.

Ethical Approval

Not applicable.

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