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New optical solitons for the Volterra lattice system using the generalized exponential rational function approach

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Abstract

In this study, the Volterra lattice system is investigated as a semi-discrete version of the modified KdV equation. Finding exact solutions for this system is challenging due to its applications in understanding nonlinear phenomena in physics and engineering. In this work, we study the generalised exponential rational function approach (GERFA) to extract exact traveling wave solutions of the Volterra lattice system (VLS). The proposed method provides promising results that allow us to investigate different types of solutions in Volterra lattice systems.

Subject Classification (2020): *35C08, 35Q51.*

Keywords: *Volterra lattice system, Differential-difference equation, Exact traveling Wave solutions, Generalised exponential rational function technique.*

1. Introduction

The study of differential-difference equations (DDEs) is essential for the analysis and modeling of nonlinear physical phenomena, as numerous real-world systems are represented by these equations [1]. A key example of these equations is the Volterra lattice system studied by Suris and Baldwin [2, 3], which possesses many properties and applications [4, 5]. The study of fully discrete dynamical systems continues to be an important area of research in many scientific disciplines. Systems that are often modeled using difference equations. For example, in population growth dynamics, models such as the Leslie model use linear difference equations to effectively describe the growth and asymptotic

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behavior of age-structured populations [6]. In addition, Sattar Nasar Ketab and Shrooq Fadhl Raji have also addressed a crucial aspect of research in discrete dynamical systems by determining the necessary conditions for solution properties such as oscillation and non-oscillation in complex models such as second-order semilinear neutral difference equations [7]. So, whether we are dealing with nonlinear DDEs (NDDEs) or difference equations, the basic goal is the same: to use rigorous mathematical approaches to derive exact solutions and analyze the underlying dynamics. In this work, we focus on using the GERFA to construct new optical solitons for the Volterra lattice system.

The research of exact solutions of DDEs began in the 1950s, following the studies of Fermi et al. in numerous scientific and engineering fields [8]. The widespread use of these equations has led researchers to focus on key integrable lattice models, including models like the Ablowitz Ladik lattice (ABLL) [9], the Volterra lattice (VL) [10], the modified (discrete) KdV [11], the Toda lattice (TL) [12], and the discrete sine-Gordon [13].Therefore, the finding of simplified exact traveling wave solutions is essential for both the validation of the models and the comprehension of the nonlinear behavior of these physical systems.

The VL equation has gained significant attention due to its widespread applications in various fields, including physics. Researchers have shown a great interest in finding simple, convenient, and accurate methods for solving VL equations. Despite the challenges encountered in obtaining acceptable forms and solutions, researchers persevered and successfully solved many VL equations over an extended period of time. The VLS applied in the present study is given below [2, 3]:

$$\begin{cases} \frac{d\omega_n}{dt} - \omega_n(\nu_n - \nu_{n-1}) = 0, \\ \frac{d\nu_n}{dt} - \nu_n(\omega_{n+1} - \omega_n) = 0, \end{cases} \tag{1}$$

Given the wide range of applications of the VLS in various scientific disciplines, numerous researchers have focused on solving these equations. For instance, in 1976, Wadati discussed transformation theories and types of VLS [4].

Suris utilized the Miura transformation in 1998 to study integrable systems of lattice equations and their role in integrable discretizations [5]. C. Uma Maheswari et al. (2025) applied the Variable Separable Method to derive explicit solutions for several nonlinear equations, including DDEs like the space discrete Korteweg-de Vries and TL equations [14], and in 2006, Xie et al. derived some exact traveling wave solutions for the TL and the VL equations [15].

There are numerous advanced and specialized methods available for finding precise solutions, including: modified exponential rational function method [16], laplace adomian decomposition technique [17], Fractional Transformation method [18], Sub-equation method [19], modified Kudryashov approach [20], Auxiliary equation technique [21], Jacobi elliptic function expansion approach [22], Functional Variable method [23], generalized Tanh technique [24], etc.

Given that the GERFA hasn't yet been utilized for the VLS, we aim in this study to leverage this method to obtain exact solutions for this system. This method has been systematically studied by Ghanbari and other researchers, and it has been widely employed to solve various partial differential equations, providing more general solutions [25-32]. The use of $Maple$ software in this method allows for the avoidance of complex and tedious manual calculations, reducing the possibility of errors.

The work is organized as follows:

Section (2) presents a summary of the GERFA for NDDEs. Section (3) focuses on solving the VLS using the proposed method. Lastly, in Section (4) we offer the conclusion.

2. The GERFA for the NDDEs

We study the following general form for a system of NDDEs:

$$\begin{cases} \mathcal{Q}(\omega_{n+k_1}(t),\dots,\omega_{n+k_m}(t),\omega'_{n+k_1}(t),\dots,\omega'_{n+k_m}(t),\dots,\omega^{(N)}_{n+k_1}(t)\dots,\omega^{(N)}_{n+k_m}(t))=0,\\ \mathcal{O}(\nu_{n+k_1}(t),\dots,\nu_{n+k_m}(t),\nu'_{n+k_1}(t),\dots,\nu'_{n+k_m}(t),\dots,\nu^{(N)}_{n+k_1}(t)\dots,\nu^{(N)}_{n+k_m}(t))=0, \end{cases} \tag{2}$$

where $\mathcal{Q}$ and $\mathcal{O}$ represent polynomial equations in ω_n,ν_n and its derivatives and $k_1,\dots,k_m\in\mathbb{Z}$. The three steps of our method are outlined below:

Step 1: We first consider the following transformations.

$$\omega_n(t)=\Omega(\eta_n),\nu_n(t)=\Upsilon(\eta_n),\eta_n=nd+kt+\eta_0, \tag{3}$$

where $d,k\in\mathbb{R}$ and η_0 is an arbitrary constant.

Using the traveling wave transformation, the components of $\omega_{n\pm p}$ and $\nu_{n\pm p}$ are defined by the following relations:

$$\begin{cases} \omega_{n\pm p}=\Omega(\eta_{n\pm p}),\\ \nu_{n\pm p}=\Upsilon(\eta_{n\pm p}),\text{where } \eta_{n\pm p}=\sum_{j=1}^{A}(n\pm p_j)d_j+\sum_{j=1}^{G}k_jt_j+\eta_0, \end{cases} \tag{4}$$

where $A,G\in\mathbb{N}$, and the coefficients $k_1,k_2,\dots,k_G$ and $d_1,d_2,\dots,d_A$ are all constants.

The substitution of these transformations into equation (2) converts the NDDEs into the following ODE:

$$\begin{cases} \mathcal{Q}(\Omega_{n+k_1}(\eta_n),\dots,\Omega_{n+k_m}(\eta_n),\Omega'_{n+k_1}(t),\dots,\Omega'_{n+k_m}(\eta_n),\dots,\Omega^{(N)}_{n+k_1}(t)\dots,\Omega^{(N)}_{n+k_m}(\eta_n))=0,\\ \mathcal{O}(\Upsilon_{n+k_1}(\eta_n),\dots,\Upsilon_{n+k_m}(\eta_n),\Upsilon'_{n+k_1}(\eta_n),\dots,\Upsilon'_{n+k_m}(\eta_n),\dots,\Upsilon^{(N)}_{n+k_1}(\eta_n)\dots,\Upsilon^{(N)}_{n+k_m}(\eta_n))=0, \end{cases} \tag{5}$$

Step 2: Suppose the travelling wave solutions of Eq. (5) are regarded as a rational function expansion of a generalized exponential function:

$$\begin{aligned} \Omega(\eta_n) &= \mathcal{A}_0+\sum_{k=1}^{m_0}\mathcal{A}_k\mathcal{L}^k(\eta_n)+\sum_{k=1}^{m_0}\frac{\mathcal{B}_k}{\mathcal{L}^k(\eta_n)},\\ \Upsilon(\eta_n) &= \mathcal{C}_0+\sum_{k=1}^{m_0}\mathcal{C}_k\mathcal{L}^k(\eta_n)+\sum_{k=1}^{m_0}\frac{\mathcal{D}_k}{\mathcal{L}^k(\eta_n)}, \end{aligned} \tag{6}$$

with

$$\mathcal{L}(\eta_n)=\frac{\theta_1\exp(\mu_1\eta_n)+\theta_2\exp(\mu_2\eta_n)}{\theta_3\exp(\mu_3\eta_n)+\theta_4\exp(\mu_4\eta_n)}, \tag{7}$$

Step 3: The substitution of Eq. (6) into Eq. (5) results in an algebraic polynomial in $\exp(\mu_k \eta_n)$. Setting the coefficients of each power of $\exp(\mu_k \eta_n)$ (where $k = 1, \dots, 4$) to zero yields a large system of algebraic equations. Subsequently, We then utilize $Maple$ software to solve this system and determine the required values for the unknown coefficients.

3. Application

We utilize the method described in Section 2 to obtain new solutions for the VLS, model (1). Now, due to the balancing principle, we can presuppose:

$$\begin{aligned} \Omega(\eta_n) &= \mathcal{A}_0 + \mathcal{A}_1 \mathcal{L}(\eta_n) + \frac{\mathcal{B}_1}{\mathcal{L}(\eta_n)}, \\ \Upsilon(\eta_n) &= \mathcal{C}_0 + \mathcal{C}_1 \mathcal{L}(\eta_n) + \frac{\mathcal{D}_1}{\mathcal{L}(\eta_n)}, \end{aligned} \tag{8}$$

To solve the VLS (1), we first apply the transformations (3) and the shift relations (4) to convert the VLS into an ODE. Subsequently, we substitute the proposed solution (8) into the resulting ODE.

The function $\mathcal{L}(\eta_n)$ defined in Eq. (7) represents a general form, and we can consider different forms of $\mathcal{L}$ by adjusting the parameters θ_k and μ_k as follows:

Group 1:

The first solution set is obtained by assigning the coefficients θ_k and μ_k the values $[-1,1,1,1]$ and $[1,-1,1,-1]$, respectively. For this specific set, Eq. (7) reduces to the $-\tanh$ function.The corresponding system of algebraic equations is given below:

$$\begin{aligned}
&\tanh(d)\mathcal{B}_1(1-\mathcal{D}_1) = 0,\\
&-\mathcal{B}_1(\mathcal{C}_1 - \mathcal{D}_1 + 1)\tanh^2(d) + \tanh(d)\mathcal{A}_0\mathcal{D}_1 - \mathcal{B}_1 = 0,\\
&\tanh(d)(\mathcal{A}_0(\mathcal{C}_1 - \mathcal{D}_1)\tanh(d) + (-\mathcal{A}_1 + \mathcal{B}_1)\mathcal{D}_1 + \mathcal{B}_1\mathcal{C}_1 - \mathcal{A}_1) = 0,\\
&(\mathcal{A}_1(1 + \mathcal{D}_1 - \mathcal{C}_1) + \mathcal{B}_1(\mathcal{C}_1 - \mathcal{D}_1 + 1))\tanh^2(d)\\
&\qquad\qquad +\mathcal{A}_1 - \mathcal{A}_0\tanh(d)(\mathcal{D}_1 + \mathcal{C}_1) + \mathcal{B}_1 = 0,\\
&(\mathcal{A}_0(-\mathcal{C}_1 + \mathcal{D}_1)\tanh(d) + \mathcal{C}_1(\mathcal{B}_1 - \mathcal{A}_1) - \mathcal{A}_1\mathcal{D}_1 + \mathcal{B}_1)\tanh(d) = 0,\\
&\mathcal{A}_1(\mathcal{C}_1 - \mathcal{D}_1 - 1)\tanh^2(d) + \tanh(d)\mathcal{A}_0\mathcal{C}_1 - \mathcal{A}_1 = 0,\\
&\tanh(d)\mathcal{A}_1(1 - \mathcal{C}_1) = 0.\\
&\tanh(d)\mathcal{D}_1(1 + \mathcal{B}_1) = 0,\\
&\mathcal{D}_1 - \mathcal{D}_1(\mathcal{A}_1 - \mathcal{B}_1 - 1)\tanh^2(d) - \tanh(d)\mathcal{B}_1\mathcal{C}_0 = 0,\\
&(\mathcal{C}_0(\mathcal{A}_1 - \mathcal{B}_1)\tanh(d) - (\mathcal{D}_1 - \mathcal{C}_1)\mathcal{B}_1 - \mathcal{A}_1\mathcal{D}_1 - \mathcal{C}_1)\tanh(d) = 0,\\
&(\mathcal{C}_1(-\mathcal{A}_1 + \mathcal{B}_1 - 1) + (\mathcal{A}_1 - 1 - \mathcal{B}_1)\mathcal{D}_1)\tanh^2(d)\\
&\qquad\qquad -\mathcal{D}_1 + \mathcal{C}_0(\mathcal{A}_1 + \mathcal{B}_1)\tanh(d) - \mathcal{C}_1 = 0,\\
&-(\mathcal{C}_0(\mathcal{A}_1 - \mathcal{B}_1)\tanh(d) + (\mathcal{C}_1 - \mathcal{D}_1)\mathcal{A}_1 + \mathcal{B}_1\mathcal{C}_1 + \mathcal{D}_1)\tanh(d) = 0,\\
&(\mathcal{A}_1 - \mathcal{B}_1 + 1)\mathcal{C}_1\tanh^2(d) - \tanh(d)\mathcal{A}_1\mathcal{C}_0 + \mathcal{C}_1 = 0,\\
&\tanh(d)\mathcal{C}_1(1 + \mathcal{A}_1) = 0.
\end{aligned}$$

By solving the algebraic system above, we derive the following states for the unknown coefficients, which lead to various solutions to the equation (1).

Solution collection 1.1:

$$\begin{aligned}\mathcal{A}_0 &= -\frac{\tanh(d)^2+1}{\tanh(d)}, \mathcal{A}_1 = -1, \mathcal{B}_1 = -1,\\ \mathcal{C}_0 &= -\frac{\tanh(d)^2+1}{\tanh(d)}, \mathcal{C}_1 = 1, \mathcal{D}_1 = 1,\end{aligned} \tag{9}$$

Substituting the unknown coefficient values Eq. (9) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\frac{(\tanh(\eta_n)\tanh(d)-1)(\tanh(d)-\tanh(\eta_n))}{\tanh(\eta_n)\tanh(d)}, \tag{10}$$

$$\nu_n(t) = -\frac{(\tanh(\eta_n)+\tanh(d))(1+\tanh(\eta_n)\tanh(d))}{\tanh(\eta_n)\tanh(d)}, \tag{11}$$

where $\eta_n = nd + kt + \eta_0$.

Solution collection 1.2:

$$\begin{aligned}\mathcal{A}_0 &= -\frac{1}{\tanh(d)}, \mathcal{A}_1 = -1, \mathcal{B}_1 = 0,\\ \mathcal{C}_0 &= -\frac{1}{\tanh(d)}, \mathcal{C}_1 = 1, \mathcal{D}_1 = 0,\end{aligned} \tag{12}$$

Substituting the unknown coefficient values Eq. (12) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\frac{1}{\tanh(d)} + \tanh(\eta_n), \tag{13}$$

$$\nu_n(t) = -\frac{1}{\tanh(d)} - \tanh(\eta_n), \tag{14}$$

where $\eta_n = nd + kt + \eta_0$.

Solution collection 1.3:

$$\begin{aligned}\mathcal{A}_0 &= -\frac{1}{\tanh(d)}, \mathcal{A}_1 = 0, \mathcal{B}_1 = -1,\\ \mathcal{C}_0 &= -\frac{1}{\tanh(d)}, \mathcal{C}_1 = 0, \mathcal{D}_1 = 1,\end{aligned} \tag{15}$$

Substituting the unknown coefficient values Eq. (15) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\frac{1}{\tanh(d)} + \tanh(\zeta_n) + \frac{1}{\tanh(\eta_n)}, \tag{16}$$

$$\nu_n(t) = -\frac{1}{\tanh(d)} - \frac{1}{\tanh(\eta_n)}, \tag{17}$$

where $\eta_n = nd + kt + \eta_0$.

Group 2:

The first solution set is obtained by assigning the coefficients θ_k and μ_k the values $[i, -i, 1, 1]$ and $[i, -i, i, -i]$, respectively. For this specific set, Eq. (7) reduces to the $-\tan$ function. The corresponding system of algebraic equations is given below:

$\tan(d)\mathcal{B}_1(-1+\mathcal{D}_1)=0,$
$\mathcal{B}_1(\mathcal{C}_1+\mathcal{D}_1-1)\tan^2(d)-\tan(d)\mathcal{A}_0\mathcal{D}_1+\mathcal{B}_1=0,$
$-\tan(d)(\mathcal{A}_0(\mathcal{C}_1+\mathcal{D}_1)\tan(d)+(-\mathcal{A}_1-\mathcal{B}_1)\mathcal{D}_1+\mathcal{B}_1\mathcal{C}_1-\mathcal{A}_1)=0,$
$(\mathcal{A}_1(\mathcal{C}_1+1+\mathcal{D}_1)+(\mathcal{C}_1-1+\mathcal{D}_1)\mathcal{B}_1)\tan^2(d)$
$+\mathcal{B}_1-\mathcal{A}_0\tan(d)(-\mathcal{C}_1+\mathcal{D}_1)-\mathcal{A}_1=0,$
$(\mathcal{A}_0(-\mathcal{C}_1-\mathcal{D}_1)\tan(d)+\mathcal{C}_1(\mathcal{A}_1+\mathcal{B}_1)-\mathcal{A}_1\mathcal{D}_1-\mathcal{B}_1)\tan(d)=0,$
$\mathcal{A}_1(\mathcal{C}_1+\mathcal{D}_1+1)\tan^2(d)+\tan(d)\mathcal{A}_0\mathcal{C}_1-\mathcal{A}_1=0,$
$\tan(d)\mathcal{A}_1(-1-\mathcal{C}_1)=0.$
$\tan(d)\mathcal{D}_1(-1-\mathcal{B}_1)=0,$
$\mathcal{D}_1(\mathcal{A}_1+\mathcal{B}_1+1)\tan^2(d)+\tan(d)\mathcal{B}_1\mathcal{C}_0-\mathcal{D}_1=0,$
$-\tan(d)(\mathcal{C}_0(\mathcal{A}_1+\mathcal{B}_1)\tan(d)+(\mathcal{C}_1+\mathcal{D}_1)\mathcal{B}_1-\mathcal{A}_1\mathcal{D}_1-\mathcal{C}_1)=0,$
$(\mathcal{C}_1(\mathcal{A}_1-1+\mathcal{B}_1)+\mathcal{D}_1(\mathcal{A}_1+1+\mathcal{B}_1))\tan^2(d)$
$-\mathcal{D}_1-\mathcal{C}_0\tan(d)(\mathcal{A}_1-\mathcal{B}_1)+\mathcal{C}_1=0,$
$-(\mathcal{C}_0(\mathcal{A}_1+\mathcal{B}_1)\tan(d)-\mathcal{D}_1-\mathcal{A}_1(\mathcal{C}_1+\mathcal{D}_1)+\mathcal{B}_1\mathcal{C}_1)\tan(d)=0,$
$(\mathcal{A}_1+\mathcal{B}_1-1)\mathcal{C}_1\tan^2(d)-\tan(d)\mathcal{A}_1\mathcal{C}_0+\mathcal{C}_1=0,$
$\tan(d)\mathcal{C}_1(-1+\mathcal{A}_1)=0.$

By solving the algebraic system above, we derive the following states for the unknown coefficients, which lead to various solutions to the equation (1).

Solution collection 2.1:

$$\begin{aligned}\mathcal{A}_0 &= \frac{\tan(d)^2-1}{\tan(d)}, \mathcal{A}_1=1, \mathcal{B}_1=-1,\\ \mathcal{C}_0 &= \frac{\tan(d)^2-1}{\tan(d)}, \mathcal{C}_1=-1, \mathcal{D}_1=1,\end{aligned} \tag{18}$$

Substituting the unknown coefficient values Eq. (18) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = \frac{(1+\tan(\eta_n)\tan(d))(\tan(d)-\tan(\eta_n))}{(\tan(\eta_n)\tan(d))}, \tag{19}$$

$$\nu_n(t) = \frac{(\tan(\eta_n)+\tan(d))(\tan(\eta_n)\tan(d)-1)}{\tan(\eta_n)\tan(d)}, \tag{20}$$

where $\eta_n = nd + kt + \eta_0$.

Solution collection 2.2:

$$\begin{aligned}\mathcal{A}_0 &= -\frac{1}{\tan(d)}, \mathcal{A}_1=-1, \mathcal{B}_1=0,\\ \mathcal{C}_0 &= -\frac{1}{\tan(d)}, \mathcal{C}_1=1, \mathcal{D}_1=0,\end{aligned} \tag{21}$$

Substituting the unknown coefficient values Eq. (21) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\frac{1}{\tan(d)} - \tan(\eta_n), \tag{22}$$

$$\nu_n(t) = -\frac{1}{\tan(d)} + \tan(\eta_n), \tag{23}$$

where $\eta_n = nd + kt + \eta_0$.

Solution collection 2.3:

$$\begin{aligned} \mathcal{A}_0 &= -\frac{1}{\tan(d)}, \mathcal{A}_1 = 0, \mathcal{B}_1 = -1, \\ \mathcal{C}_0 &= -\frac{1}{\tan(d)}, \mathcal{C}_1 = 0, \mathcal{D}_1 = 1, \end{aligned} \tag{24}$$

Substituting the unknown coefficient values Eq. (24) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega(t) = -\frac{1}{\tan(d)} + \frac{1}{\tan(\eta_n)}, \tag{25}$$

$$\nu_n(t) = -\frac{1}{\tan(d)} - \frac{1}{\tan(\eta_n)}, \tag{26}$$

where $\eta_n = nd + kt + \eta_0$.

Group 3:

The first solution set is obtained by assigning the coefficients θ_k and μ_k the values $[-1,-1,1,-1]$ and $[1,-1,1,-1]$, respectively. For this specific set, Eq. (7) reduces to the $-\coth$ function.The corresponding system of algebraic equations is given below:

$$\begin{aligned}
&\coth(d)\mathcal{B}_1(1-\mathcal{D}_1) = 0,\\
&-\coth^2(d)\mathcal{B}_1 + \coth(d)\mathcal{A}_0\mathcal{D}_1 - (\mathcal{C}_1 + 1 - \mathcal{D}_1)\mathcal{B}_1 = 0,\\
&((-\mathcal{A}_1 + \mathcal{B}_1)\mathcal{D}_1 + \mathcal{B}_1\mathcal{C}_1 - \mathcal{A}_1)\coth(d) + \mathcal{A}_0(\mathcal{C}_1 - \mathcal{D}_1) = 0,\\
&\coth^2(d)(\mathcal{A}_1 + \mathcal{B}_1) - \mathcal{A}_0\coth(d)(\mathcal{C}_1 + \mathcal{D}_1) + (-\mathcal{C}_1 + 1 + \mathcal{D}_1)\mathcal{A}_1\\
&\qquad + \mathcal{B}_1(\mathcal{C}_1 + 1 - \mathcal{D}_1) = 0,\\
&((\mathcal{A}_1 - \mathcal{B}_1)\mathcal{C}_1 + \mathcal{A}_1\mathcal{D}_1 - \mathcal{B}_1)\coth(d) - \mathcal{A}_0(\mathcal{C}_1 - \mathcal{D}_1) = 0,\\
&-\coth^2(d)\mathcal{A}_1 + \coth(d)\mathcal{A}_0\mathcal{C}_1 + (\mathcal{C}_1 - 1 - \mathcal{D}_1)\mathcal{A}_1 = 0,\\
&\coth(d)\mathcal{A}_1(1-\mathcal{C}_1) = 0.\\
&\coth(d)\mathcal{D}_1(1+\mathcal{B}_1) = 0,\\
&\coth^2(d)\mathcal{D}_1 - \coth(d)\mathcal{B}_1\mathcal{C}_0 - \mathcal{D}_1(\mathcal{A}_1 - 1 - \mathcal{B}_1) = 0,\\
&((\mathcal{C}_1 - \mathcal{D}_1)\mathcal{B}_1 - \mathcal{A}_1\mathcal{D}_1 - \mathcal{C}_1)\coth(d) + \mathcal{C}_0(\mathcal{A}_1 - \mathcal{B}_1) = 0,\\
&-(\mathcal{C}_1 + \mathcal{D}_1)\coth^2(d) + \mathcal{C}_0\coth(d)(\mathcal{A}_1 + \mathcal{B}_1) + (-\mathcal{A}_1 - 1 + \mathcal{B}_1)\mathcal{C}_1\\
&\qquad + \mathcal{D}_1(\mathcal{A}_1 - 1 - \mathcal{B}_1) = 0,\\
&((-\mathcal{C}_1 + \mathcal{D}_1)\mathcal{A}_1 - \mathcal{B}_1\mathcal{C}_1 - \mathcal{D}_1)\coth(d) - \mathcal{C}_0(\mathcal{A}_1 - \mathcal{B}_1) = 0,\\
&\quad \coth^2(d)\mathcal{C}_1 - \coth(d)\mathcal{A}_1\mathcal{C}_0 + (\mathcal{A}_1 - \mathcal{B}_1 + 1)\mathcal{C}_1 = 0,\\
&\quad \coth(d)\mathcal{C}_1(1+\mathcal{A}_1) = 0.
\end{aligned}$$

By solving the algebraic system above, we derive the following states for the unknown coefficients, which lead to various solutions to the equation (1).

Solution collection 3.1:

$$\begin{aligned} \mathcal{A}_0 &= -\coth(d), \mathcal{A}_1 = 0, \mathcal{B}_1 = -1, \\ \mathcal{C}_0 &= -\coth(d), \mathcal{C}_1 = 0, \mathcal{D}_1 = 1, \end{aligned} \tag{27}$$

Substituting the unknown coefficient values Eq. (27) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\coth(d) + \frac{1}{\coth(\eta_n)}, \tag{28}$$

$$\nu_n(t) = -\coth(d) - \frac{1}{\coth(\eta_n)}, \tag{29}$$

where $\eta_n = nd + kt + \eta_0$.

Solution collection 3.2:

$$\begin{aligned} \mathcal{A}_0 &= \frac{-\coth(d)^2+1}{\coth(d)}, \mathcal{A}_1 = -1, \mathcal{B}_1 = -1, \\ \mathcal{C}_0 &= \frac{-\coth(d)^2+1}{\coth(d)}, \mathcal{C}_1 = 1, \mathcal{D}_1 = 1, \end{aligned} \tag{30}$$

Substituting the unknown coefficient values Eq. (30) into the assumed solutions Eq. (8), and utilizing the traveling wave transformation (3) along with Eq. (4), the final solutions ω_n and ν_n are obtained in the following form:

$$\omega_n(t) = -\coth(d) - \frac{1}{\coth(d)} + \coth(\eta_n) + \frac{1}{\coth(\eta_n)}, \tag{31}$$

$$\nu_n(t) = -\coth(d) - \frac{1}{\coth(d)} - \coth(\eta_n) - \frac{1}{\coth(\eta_n)}, \tag{32}$$

where $\eta_n = nd + kt + \eta_0$.

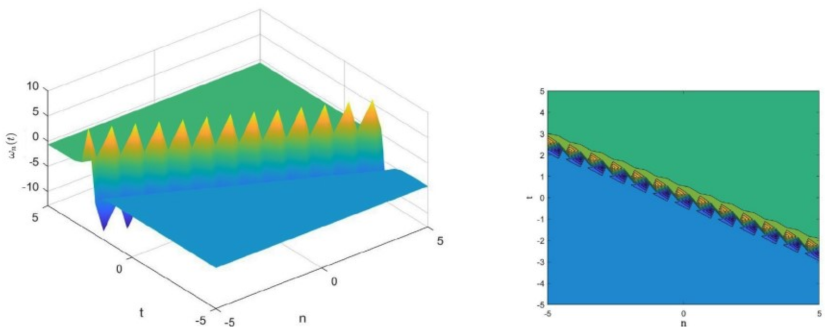

Figure 1
The 3-dimensional representation and contour plot of the solution (10) considering the given parameters $d = 0.5, c = 1, \zeta_0 = 0$.

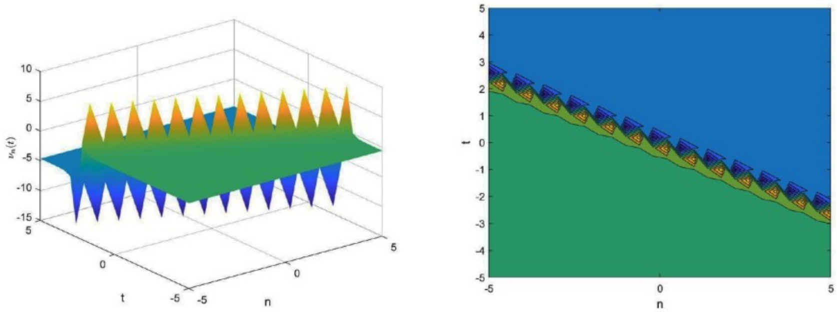

Figure 2
The 3-dimensional representation and contour plot of the solution (11) considering the given parameters $d = 0.5, c = 1, \zeta_0 = 0$.

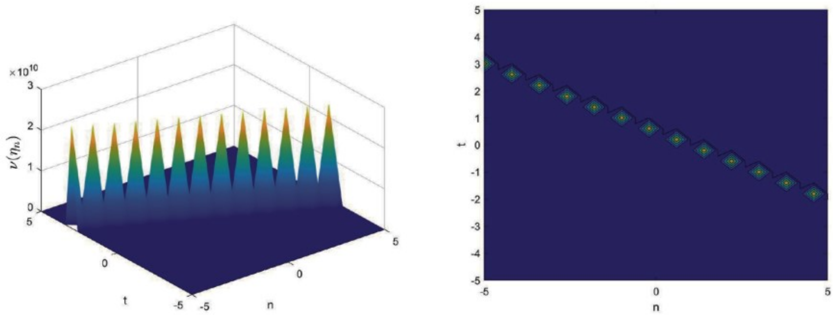

Figure 3
The 3-dimensional representation and contour plot of the solution (17) considering the given parameters $d = 0.5, c = 1, \zeta_0 = 0$.

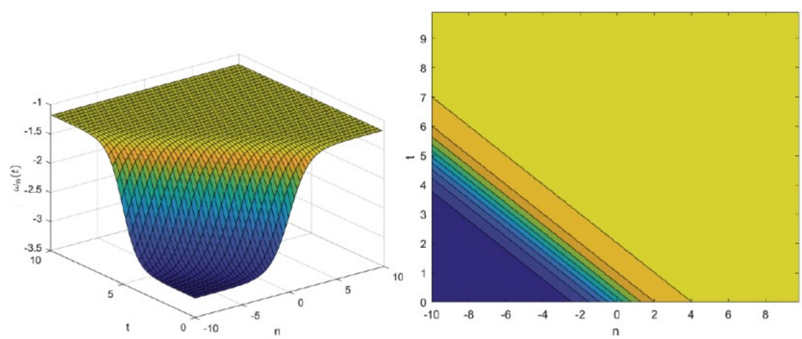

Figure 4
The 3-dimensional representation and contour plot of the solution (31) considering the given parameters $d = 0.5, c = 1, \zeta_0 = 0$.

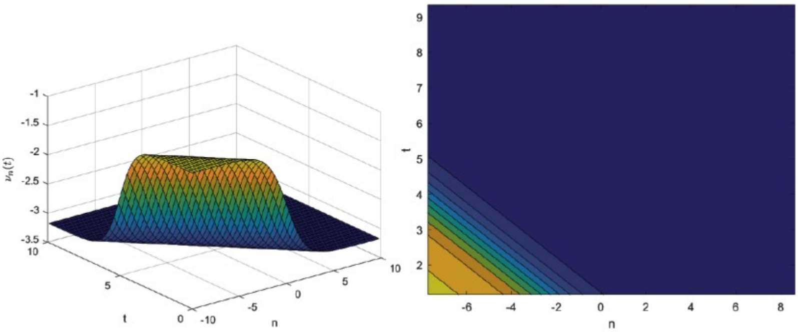

Figure 5
The 3-dimensional representation and contour plot of the solution (32) considering the given parameters $d = 0.5, c = 1, \zeta_0 = 0$.

Conclusion

In this research, we presented a novel methodology aimed at constructing exact solutions for nonlinear lattices. The primary focus is on investigating the VL, The contour and 3-dimensional plots of the traveling wave solutions obtained whit arbitrary given parameters, Indicative of the validity and effectiveness of our approach (Figs. 1-5). The computation procedure employed in this study reveals the crucial role played by computer algebra in accurately solving DDEs associated with nonlinear lattices. The proposed method offers a valuable tool for researchers and practitioners in their quest for accurate solutions and deeper insights into the behavior of such systems.

Declarations

Ethical Approval. The authors declare that there is no animal studies in this work.

Conflict of Interest. The authors declare that they have no conflict of interest.

Author Contributions. S.H. gave the first idea of investigation. The concept and modeling was done by M.E. The methodology.

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Data Availability. There is no data set used.

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