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Modeling and solving applied differential equations in Physics by Gamma interval functions

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Abstract

A new function, namely the Gamma Interval Function is defined first. The function is used in modelling physically inspired problems of first and second order differential equations. The differential equations corresponding to different subdomains are solved and the solutions are combined to each other by imposing the physical constraints of the problem. It is shown that the solutions can be expressed as a single function with the aid of Gamma Interval Functions. Sample problems are selected in the area of applied mechanics such as dynamics and mechanical vibrations. This new approach presented here may inspire further work related to modelling of the mathematical physics problems.

Subject Classification: *34A05, 34A12, 76G25, 70J30.*

Keywords: *Gamma interval functions, Ordinary differential equations, Applied mechanics, Mathematical modeling.*

1. Introduction

Ordinary differential equations play a vital role in modeling physical problems. Applications of ordinary differential equations in dynamics, vibrations, heat transfer, mechanics of materials, fluids are vast and the fundamental problems possessing analytical solutions were discussed in detail in textbooks on differential equations [1,2]. Nonlinear dynamical problems were extensively treated by analytical, semi-analytical and numerical methods [3-5]. The tremendous amount of publications on modeling physical systems by ordinary differential equations cannot be reviewed within the context of a single

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paper. For some mathematical models derived from physical problems, see [6-16] for example.

In this work, the new Gamma Interval Function is proposed for modelling ordinary differential equations. With the aid of the functional representation, some terms may be neglected/included or changed in character in a subdomain of the problem. In real world problems, the physical conditions may not remain the same within the whole domain and Gamma Interval functions can be applied to the models to represent the changes in a single mathematical model. The resulting solutions for such problems occur as piecewise continuous solutions and Gamma Interval Functions can also be employed to unite the solutions in a single expression. Two first order and five second order ordinary differential equations are given as examples of the modelling of such problems. Physical problems are selected from the areas of dynamics and mechanical vibrations. This work is a pioneering work on the topic and may inspire further applications in modelling and solutions of physical systems in a more compact and convenient way.

2. The Gamma Interval Function

The conditions of physical problems may change over the domain of interest and this requires variations in the physical model resulting in the variations of the solutions. It would be good to express the mathematical model as a single expression inheriting the differences in the model as well as expressing these piecewise solutions as a compact single solution valid over the whole domain of interest. In order to achieve these goals, a new Gamma Interval Function is proposed with properties

$$\gamma[a,b)(x) = \begin{cases} 1 & a \le x < b \\ 0 & x < a, x \ge b \end{cases} \tag{2.1}$$

where a is the left limit (included) and b is the right limit (excluded) for the function being unity. The argument is the independent variable x but depending on the differential equation considered, the argument may also be the dependent variable or even its derivatives or a physical parameter of the system. For the limits a and b, indeed there are four possibilities one of which is shown above and the other three being $\gamma[a,b](x)$, $\gamma(a,b](x)$ and $\gamma(a,b)(x)$. As usual, the brackets represent inclusion of the limiting value to the domain and parentheses represent exclusion of the limiting value.

Sample problems of ordinary differential equations will be given in the subsequent section about the usage of the function in mathematical modeling of the physical problems as well as expressing the solutions in a more compact form.

3. First Order Ordinary Differential Equations

Two sample problems will be treated in this section with the first being a simple prototype equation and the second being the free fall of an object with variations in the drag force models.

Example Problem 1

Consider the first order linear differential equation with gamma interval function

$$y' + \gamma[a,b](x)y = 0\,, \quad y(0) = \alpha\,, \quad 0 < a < b \tag{3.1}$$

The differential equation possesses three parts to be solved separately.

i) $0 \le x < a$

For this part, the equation is $y' = 0$, with a solution $y_1 = \alpha$ in view of the initial condition.

ii) $a \le x \le b$

The equation reads $y' + y = 0$, with a solution $y_2 = c_2 e^{-x}$. If one looks for a continuous solution as in the case of physical systems, then one may require $y_1(a) = y_2(a)$ leading to a solution $y_2 = \alpha e^{a-x}$.

iii) $x > b$

The equation is again $y' = 0$ and the continuous solution yields $y_3 = \alpha e^{a-b}$.

Uniting the three piecewise solutions and expressing them in a compact form with the aid of gamma interval functions yield

$$y(x) = \alpha\gamma[0,a)(x) + \alpha e^{a-x}\gamma[a,b](x) + \ \alpha e^{a-b}\gamma[b,\infty)(x) \tag{3.2}$$

which is a solution valid within the whole domain of interest. The plot is given in Figure 1 for $\alpha = 2, a = 2, b = 5$. As can be seen in the junction points, the function is continuous but not differentiable.

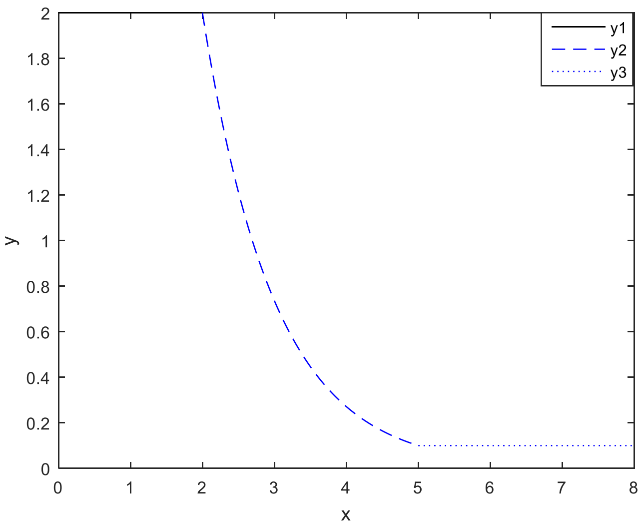

Figure 1
Plot of Solution (3.2) (Problem 1)

Example Problem 2 (Free Falling Object)

The free-falling object under the influence of gravity is examined in this problem. It is well known that for velocities below a critical value, the drag force can be taken a linear function of the velocity whereas above a critical value, it is a quadratic function. The mathematical model for the problem can be expressed in terms of Gamma Interval Functions

$$m\frac{dv^*}{dt^*} + k_1 v^* \gamma[0, v_c)(v^*) + k_2 v^{*2} \gamma[v_c, \infty)(v^*) = mg \text{ , } v^*(0) = 0 \quad (3.3)$$

where m is the mass, g is the gravitational constant, v^* is the dimensional velocity, t^* is the dimensional time and v_c is the critical velocity where the drag force switches from a linear relationship to a quadratic relationship. The equation can be cast into a nondimensional form by defining

$$t = \frac{t^*}{T}, \quad v = \frac{v^*}{v_c}. \quad (3.4)$$

The dimensionless equation is

$$\frac{dv}{dt} + \varepsilon_1 v \gamma[0,1)(v) + \varepsilon_2^2 v^2 \gamma[1, \infty)(v) = 1 \text{ , } v(0) = 0 \quad (3.5)$$

with the quantities defined by

$$\varepsilon_1 = \frac{k_1 v_c}{mg}, \quad \varepsilon_2^2 = \frac{k_2 v_c^2}{mg}, \quad T = \frac{v_c}{g} \quad (3.6)$$

Note that the dimensionless quantity ε_1 represents the ratio of the linear drag force to gravity force and ε_2^2 represents the ratio of quadratic drag force to gravity force at the critical speed. T represents the hypothetical time to reach the critical velocity in the absence of drag forces.

To find the solution for the problem, two different regions should be solved separately.

i) $v \in [0,1)$

The equation to be solved is

$$\frac{dv}{dt} + \varepsilon_1 v = 1 \text{ , } \quad v(0) = 0 \quad (3.7)$$

whose solution is

$$v = \frac{1}{\varepsilon_1}(1 - e^{-\varepsilon_1 t}) \text{ , } \quad 0 \leq t < t_c \quad (3.8)$$

which is valid for time values up to the critical time t_c for which the dimensionless velocity becomes $v = 1$. Inserting $v = 1$ into (3.8) and solving for t_c

$$t_c = -\frac{1}{\varepsilon_1} ln(1 - \varepsilon_1) \quad (3.9)$$

ii) $v \in [1, \infty)$

The equation to be solved is

$$\frac{dv}{dt} + \varepsilon_2^2 v^2 = 1 \quad (3.10)$$

whose solution is

$$v = \frac{1-c_2 e^{-2\varepsilon_2 t}}{\varepsilon_2\left(1+c_2 e^{-2\varepsilon_2 t}\right)} \tag{3.11}$$

Imposing the physical condition $v(t_c) = 1$, the solution is

$$v = \frac{1-\frac{1-\varepsilon_2}{1+\varepsilon_2}e^{-2\varepsilon_2(t-t_c)}}{\varepsilon_2\left(1+\frac{1-\varepsilon_2}{1+\varepsilon_2}e^{-2\varepsilon_2(t-t_c)}\right)} \tag{3.12}$$

Combining both parts of the solutions into a single expression, the velocity in its final form is

$$v = \frac{1}{\varepsilon_1}\left(1 - e^{-\varepsilon_1 t}\right)\gamma[0,t_c)(t) + \frac{1-\frac{1-\varepsilon_2}{1+\varepsilon_2}e^{-2\varepsilon_2(t-t_c)}}{\varepsilon_2\left(1+\frac{1-\varepsilon_2}{1+\varepsilon_2}e^{-2\varepsilon_2(t-t_c)}\right)}\gamma[t_c,\infty)(t) \tag{3.13}$$

where the parameters are defined in (3.6) and (3.9). The plot of the dimensionless velocity versus dimensionless time is given in Figure 2 for the parameter values of $\varepsilon_1 = 0.3$, $\varepsilon_2 = 0.5$ and $t_c = 1.1889$.

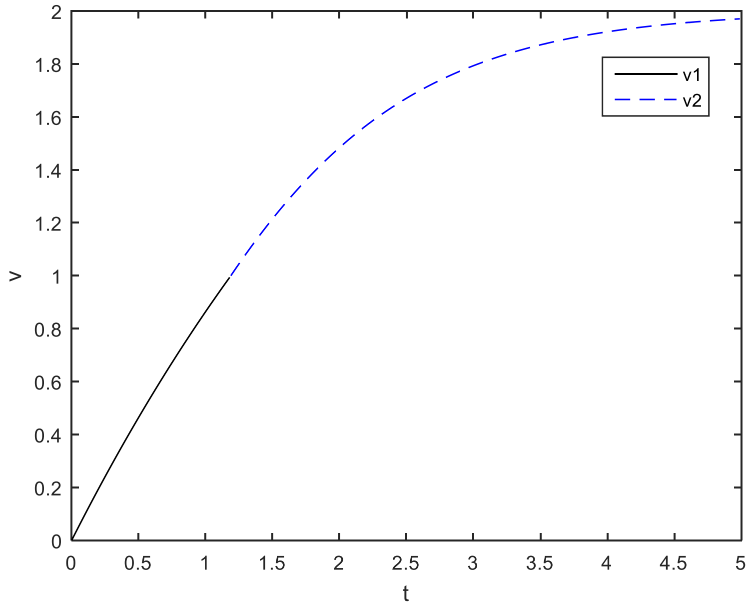

Figure 2
Velocity of a free-falling object (Problem 2)

As the velocity attains higher values, the drag force balances the gravitational force and the velocity tends to a constant value in accordance with the observations. For a given object, the calculation of the dimensionless parameters requires some type of experimentation since the drag coefficients depend on the shape factor, viscosity, density and cross-sectional area vertical to the falling direction.

4. Second Order Ordinary Differential Equations

Five sample problems emanating from physical applications will be treated in this section using the gamma interval functions.

Example Problem 3 (Partially Damped Oscillator)

Consider the free vibration problem

$$\ddot{x} + x + 2\varepsilon\gamma[2\pi, 4\pi](t)\dot{x} = 0\ ,\ x(0) = 0\ ,\ \dot{x}(0) = 0 \qquad (4.1)$$

in which the object is subject to viscous damping only in the time interval $t \in [2\pi, 4\pi]$. The differential equation has three separate parts to be solved.

i) $t \in [0,2\pi)$

For this part, the equation is $\ddot{x} + x = 0$, and applying the initial conditions, the solution is

$$x_1 = sint \qquad (4.2)$$

ii) $t \in [2\pi, 4\pi]$

The differential equation for this part is $\ddot{x} + x + 2\varepsilon\dot{x} = 0$ for which the solution is

$$x_2 = e^{-\varepsilon t}\left(c_1 cos\sqrt{1-\varepsilon^2}t + c_2 sin\sqrt{1-\varepsilon^2}t\right) \qquad (4.3)$$

The physics of the problem requires a continuous differentiable solution for which

$$x_1(2\pi) = x_2(2\pi),\ \dot{x}_1(2\pi) = \dot{x}_2(2\pi). \qquad (4.4)$$

Applying the conditions, the solution is

$$x_2 = \frac{e^{-\varepsilon(t-2\pi)}}{\sqrt{1-\varepsilon^2}} sin\sqrt{1-\varepsilon^2}(t-2\pi) \qquad (4.5)$$

iii) $t \in [4\pi, \infty)$

The equation $\ddot{x} + x = 0$ is solved subject to the conditions $x_2(4\pi) = x_3(4\pi), \dot{x}_2(4\pi) = \dot{x}_3(4\pi)$,

$$x_3 = \frac{e^{-2\varepsilon\pi}}{\sqrt{1-\varepsilon^2}}(sin\sqrt{1-\varepsilon^2}2\pi\ cost +$$
$$\left(-\varepsilon sin\sqrt{1-\varepsilon^2}2\pi + \sqrt{1-\varepsilon^2}cos\sqrt{1-\varepsilon^2}2\pi\right)sint). \qquad (4.6)$$

Combining all solutions in a single expression yields the final solution

$$x(t) = sint\ \gamma[0,2\pi)(t) + \frac{e^{-\varepsilon(t-2\pi)}}{\sqrt{1-\varepsilon^2}} sin\sqrt{1-\varepsilon^2}(t-2\pi)\gamma[2\pi, 4\pi](t)$$
$$+\frac{e^{-2\varepsilon\pi}}{\sqrt{1-\varepsilon^2}}(sin\sqrt{1-\varepsilon^2}2\pi\ cost +$$
$$\left(-\varepsilon sin\sqrt{1-\varepsilon^2}2\pi + \sqrt{1-\varepsilon^2}cos\sqrt{1-\varepsilon^2}2\pi\right)sint)\gamma[4\pi, \infty)(t) \qquad (4.7)$$

The plot of the function is given in Figure 3.

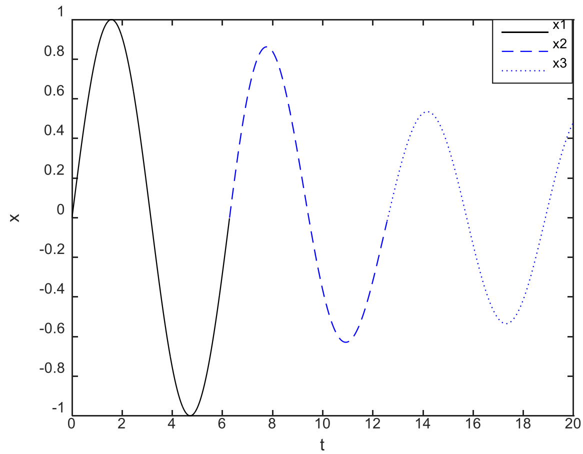

Figure 3
Partially Damped Oscillator (Problem 3)

The initial and final parts are undamped constant amplitude solutions whereas damping is effective in the middle part resulting in a reduction in amplitude.

Example Problem 4 (Retarded Hardening Model)

Consider the free vibration problem

$$\ddot{u} + u + \varepsilon\gamma[2\pi, \infty)(t)u^2 = 0 \text{ , } u(0) = 0 \text{ , } \dot{u}(0) = 1 \tag{4.8}$$

in which the quadratic nonlinearities appear after $t \geq 2\pi$. The domain is separated into two parts

i) $t \in [0,2\pi)$

For this part, the equation is $\ddot{u} + u = 0$, and applying the initial conditions, the solution is

$$u_1 = sint \tag{4.9}$$

ii) $t \in [2\pi, \infty)$

The differential equation for this part is $\ddot{u} + u + \varepsilon u^2 = 0$ subject to the continuity conditions $u_2(2\pi) = u_1(2\pi)$, $\dot{u}_2(2\pi) = \dot{u}_1(2\pi)$. A regular perturbation solution can be constructed for this case yielding

$$u_2 = sint + \varepsilon\left(\frac{2}{3}sin\left(t - \frac{3\pi}{2}\right) - \frac{1}{2} - \frac{1}{6}cos2t\right) \tag{4.10}$$

Combining both solutions in a single expression yields the final solution

$$u(t) = sint\, \gamma[0,2\pi)(t) + \\ \left(sint + \varepsilon\left(\frac{2}{3}sin\left(t - \frac{3\pi}{2}\right) - \frac{1}{2} - \frac{1}{6}cos2t\right)\right)\gamma[2\pi, \infty)(t) \tag{4.11}$$

The plot of the function is given in Figure 4 for $\varepsilon = 0.2$.

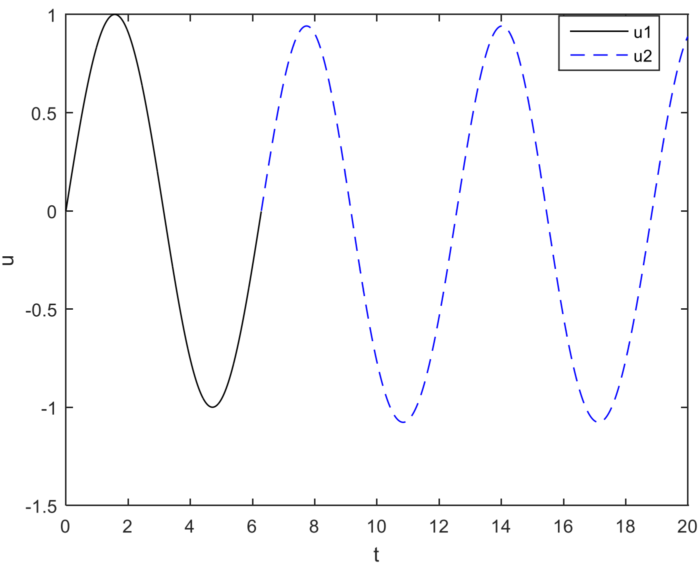

Figure 4
Retarded Hardening Model (Problem 4)

Example Problem 5 (Curvature Change)

Assume that, due to the loading conditions, the curvature of a slender beam can be modelled in nondimensional form in the domain $\left[-1,\frac{3}{2}\right]$ as follows

$$\frac{y''}{\left(1+\gamma[-1,\frac{1}{2})(x)y'^2\right)^{3/2}} = -1 \ , \ y(0) = 1 \ , \ y'(0) = 0 \qquad (4.12)$$

in which the slope effects are dominant in the region $x \in \left[-1,\frac{1}{2}\right)$ and cannot be neglected. Separating the domain into two parts

i) $x \in [-1,\frac{1}{2})$

For this part, the equation is $\frac{y''}{(1+y'^2)^{3/2}} = -1$, and the conditions $y(0) = 1$, $y'(0) = 0$ remain in this interval. Only the positive solution is considered which is

$$y_1 = \sqrt{1-x^2} \ . \qquad (4.13)$$

ii) $x \in [\frac{1}{2},\frac{3}{2}]$

The differential equation for this part is $y'' = -1$ subject to the continuity conditions $y_2\left(\frac{1}{2}\right) = y_1(\frac{1}{2})$, $y_2'\left(\frac{1}{2}\right) = y_1'\left(\frac{1}{2}\right)$. The solution is

$$y_2 = -\frac{1}{2}x^2 + \frac{\sqrt{3}-2}{2\sqrt{3}}x + \frac{16-\sqrt{3}}{8\sqrt{3}} \tag{4.14}$$

The solution over the whole domain is then

$$y(x) = \sqrt{1-x^2}\,\gamma[-1,\tfrac{1}{2})(x) + \left(-\frac{1}{2}x^2 + \frac{\sqrt{3}-2}{2\sqrt{3}}x + \frac{16-\sqrt{3}}{8\sqrt{3}}\right)\gamma[\tfrac{1}{2},\tfrac{3}{2}](x) \tag{4.15}$$

The plot of the function is given in Figure 5.

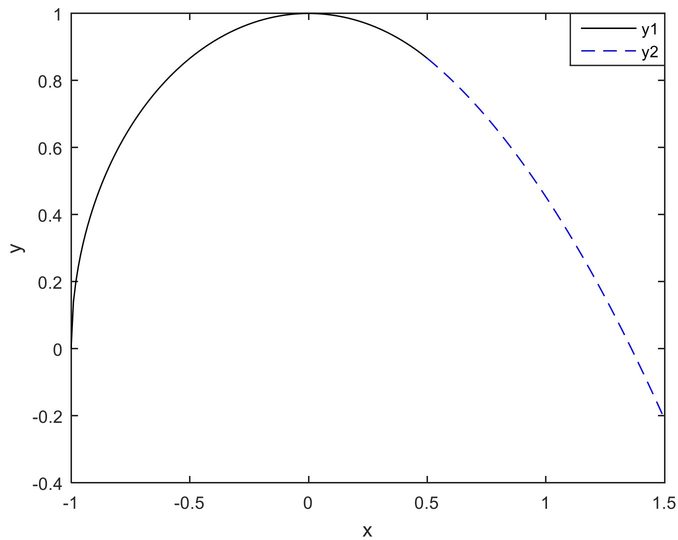

Figure 5
Curvature Change Model (Problem 5)

Example Problem 6 (Slowing down of an Object)

In this example, the argument of the gamma interval function is not the independent variable, rather the argument is the derivative of the dependent function. Consider the problem

$$\ddot{y} + \dot{y}^{1+\gamma[\frac{1}{2},1](\dot{y})} = 0\ ,\ y(0) = 0\ ,\ \dot{y}(0) = 1 \tag{4.16}$$

which is a simplified dimensionless form of a mass slowing down under the influence of drag forces in the absence of gravitational driving force. Assume that, when the velocity is in the range $[\frac{1}{2},1]$, the drag force is proportional to the $\dot{y}^2$ and in the range $[0,\frac{1}{2}]$ it is proportional to $\dot{y}$. The two cases will be solved separately

i) $\dot{y} \in [\frac{1}{2},1]$

For this part, the equation is $\ddot{y} + \dot{y}^2 = 0$, and the mass slows down from the dimensionless velocity 1 to ½. Both conditions apply for this region and the solution is

$$y_1 = ln(1+t)\ . \tag{4.17}$$

To find the critical time in which the velocity reduces to ½, $\frac{1}{1+t_c} = \frac{1}{2}$ $\rightarrow\ t_c = 1$. Hence this solution is valid in the time domain $[0,1]$.

ii) $\dot{y} \in [0, \frac{1}{2})$

The differential equation for this part is $\ddot{y} + \dot{y} = 0$ subject to the continuity conditions $y_2(1) = y_1(1)$, $\dot{y}_2(1) = \dot{y}_1(1)$. The solution is

$$y_2 = ln2 + \frac{1}{2} - \frac{1}{2}e^{1-t} \tag{4.18}$$

which is valid for the time domain $[1, \infty)$. The displacement over the whole time domain is then

$$y(t) = ln(1+t)\,\gamma[0,1](t) + \left(ln2 + \frac{1}{2} - \frac{1}{2}e^{1-t}\right)\gamma[1,\infty)(t) \tag{4.19}$$

The plot of the function is given in Figure 6.

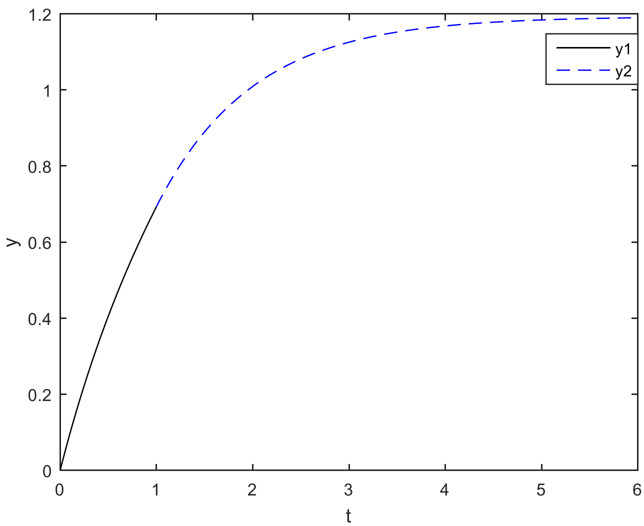

Figure 6
Slowing down of an Object (Problem 6)

The total travelling distance is found when time tends to infinity and the asymptotic limit is $y_t = ln2 + \frac{1}{2}$. Note that although we started with a gamma interval function in terms of the derivative of the dependent variable, at the end, we used an interval function in terms of the independent variable to express the solution.

Example Problem 7 (Spring-Mass System on a Frictional Surface)

The spring-mass system on a frictional surface can be modelled in terms of the gamma interval functions as follows

$$\ddot{x} + \omega^2 x = \mu g \sum_{k=0}^{N} (-1)^k \gamma[k\frac{\pi}{\omega}, (k+1)\frac{\pi}{\omega})(t)\ ,\ x(0) = x_0,\ \dot{x}(0) = 0 \quad (4.20)$$

where $\omega = \sqrt{\frac{K}{M}}$ and the frictional force alternates its sign every half period. A mathematical model in terms of the sgn functions is given in [6]. The solution here is established starting with the initial conditions and solving for $k = 0$. Using the continuity conditions, the next solution is found for $k = 1$. The process repeats itself and terminates at the specific N+1 value for which the amplitude of oscillations became negative which is impossible. Skipping the details, the final solution can be expressed in terms of the Gamma Interval Functions as follows

$$x(t) = \sum_{k=0}^{N} \left\{ \left(x_0 - (2k+1)\frac{\mu g}{\omega^2} \right) \cos\omega t + (-1)^k \frac{\mu g}{\omega^2} \right\} \gamma[k\frac{\pi}{\omega}, (k+1)\frac{\pi}{\omega})(t) \quad (4.21)$$

Since the amplitudes decrease by $\frac{2\mu g}{\omega^2}$ each half cycle, then the calculations should be terminated for the specific N value when $x_0 - (N+1)\frac{2\mu g}{\omega^2} < 0$. A plot of the function is given in Figure 7 for the specific values of $x_0 = 1$, $\omega = 1$, $g = 9.81$, $\mu = 0.015$ which makes $N = 3$.

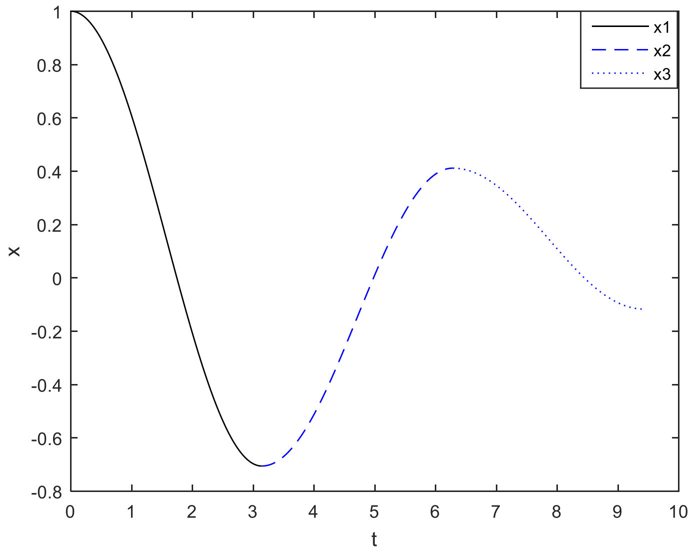

Figure 7
Spring-Mass System on a Frictional Surface (Problem 7)

5. Concluding Remarks

This is a pioneering work on the usage of Gamma Interval Functions in modelling and expressing solutions of differential equations. A number of practical physical problems are treated in order to display the potential of the new approach. The embedded Gamma Interval Function to the differential equation alters the nature of the differential equation leading to piecewise solutions. These solutions can be united to each other by applying the continuity conditions arising from the physical constraints. For a purely mathematical model however, these physical constraints do not exist and should be implemented by the researcher which may introduce a flexibility in the choices. One should be cautious about the existence and uniqueness of the solutions as the imposed conditions may not yield a solution at all or may result in more than one solution. Depending on the location of the interval function, the degree of the equation may become lower for a subdomain which may restrict the number of conditions imposed at this interval. The boundary layer type equations possess the mentioned property which means that the gamma interval functions may also be used in treating such problems. Although the argument of a Gamma Interval Function is usually the independent variable, it may not be the case always. The interval functions may be functions of the dependent variable and its derivatives as well. It may be a function of the physical parameter of the system also.

This introductory work can be extended in a number of ways. Many more involved ordinary differential models in mathematical physics can be expressed by this approach and the solutions of such systems can be represented as a single unique function. Boundary layer type problems can be treated by this approach also. Existence and uniqueness of solutions can be studied for such models. The extension to partial differential equations can be made but this requires a substantial change in the Gamma Interval Function definition, since, instead of one-dimensional intervals, one has to consider regions of two and higher order dimensions.

6. Declarations

Data Availability- No additional data is reported with this work.

Competing Interests- Author declares no competing interests.

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