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Invariant solutions of a European option pricing equation under a continuous-time capital asset pricing model

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Abstract

This work investigates a nonlinear European option pricing equation under a continuous-time Capital Asset Pricing Model. In particular, we determine its point symmetries and thereby find analytical solutions using the Lie symmetry approach. In our investigation, we vary the beta coefficient, a measure of an asset's systematic risk. We examine the model for both a non-zero and a zero market beta in our study. We show that in the first instance, this model is invariant under a three-dimensional Lie algebra, and in the latter situation, it is invariant under a six-dimensional Lie algebra. For both the non-zero and zero market beta models, we get reductions and with various parameter settings, interesting graphical solutions are investigated.

Subject Classification (2020): *35K05, 35K15, 37C79.*

Keywords: *Option pricing, Lie symmetries, Exact solutions.*

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1. Introduction

H. Markowitz established the groundwork for asset pricing models being the first to devise a specific measure of portfolio risk and to calculate a portfolio's expected return and risk [8]. The Capital Asset Pricing Model (CAPM) is based on Markowitz's approach of portfolio selection and developed by Sharpe [14] and Lintner [7], becoming a significant instrument in pricing risky securities, evaluation of portfolio performance, portfolio diversification, valuation of investments, and portfolio strategy selection. Particularly, in determining whether or not to add assets to a well-diversified portfolio, the model can be used to determine the required rate of return of the asset. The Black-Scholes equation can be extended to give the price of an option under a continuous-time CAPM. In this paper we analyse an option pricing partial differential equation (PDE) based on the continuous-time CAPM where the solution to this model gives the price of a European option. In the continuous-time CAPM the evolution of the market price, M_t, and asset price S_t are given as:

$$\frac{dM_t}{M_t} = \mu d_t + \sigma_m dW_t^{(1)}, \tag{1.1}$$

$$\frac{dS_t}{S_t} = \beta \frac{dM_t}{M} t + \sigma dW_t^{(2)}, \tag{1.2}$$

where μ is the expected rate of return, σ_m and σ are positive constant volatilities, β is the slope, and $W_t^{(1)}$ and $W_t^{(2)}$ are uncorrelated Brownian motions. From (1.1), we see that market price is governed by a geometric Brownian motion and in (1.2) the asset's expected return is proportional to the market's return. Under a continuous-time CAPM, the PDE derived by [13] depicts the evolution of the price of a European option, $V := V(S, M, \tau)$, depending on underlying asset price , S, market price, M, and time-to-maturity τ:

$$\begin{aligned} \frac{\partial V}{\partial \tau} = {} & \frac{1}{2}(\sigma_m^2 \beta^2 + \sigma^2) S^2 \frac{\partial^2 V}{\partial S^2} + \beta \sigma_m^2 M S \frac{\partial^2 V}{\partial S\, \partial M} + \\ & \frac{1}{2}\sigma_m^2 M^2 \frac{\partial^2 V}{\partial M^2} + r\left(S \frac{\partial V}{\partial S} + M \frac{\partial V}{\partial M} - V\right), \\ & 0 < \tau \leq T, S, M \in \Re_+, \end{aligned} \tag{1.3}$$

where r is the risk-free interest.

The β of a stock is vital to the development and performance of stock portfolios showing the extent of the asset's risk in comparison to the market's systematic risk. β statistics for a single stock provides an investor an estimate of how much risk the stock will add to a supposedly well-diversified portfolio. A stock with β of 1.0 added to a portfolio doesn't raise risk, but it also doesn't enhance the chance of the portfolio providing an excess return. A β greater than 1.0 means that the security's price is more volatile than the market. Adding a security with said β will increase its risk but expected return may potentially be increased. New technology companies usually have a beta greater than one. With the same logic, a β less than 1.0 means the stock is less volatile than the market, therefore, adding the stock makes the portfolio less risky. Negative β's can occur, indicating that the investment moves in the opposite direction of the market measure.

The use of Lie symmetry techniques to the study of PDEs is very successful, and there is a wealth of literature on the subject [5, 3, 10, 4, 15]. In fact, the technique has been utilised in finding invariant solutions to other financial engineering models such as the Black-Scholes-Merton model for European options [12], the fractional Black-Scholes equation [2], and the Asian option [1] etc. This method has also been extended to many areas in science, for example, an interesting application to a pre-cancerous cell population model can be found in [9].

The purpose of this work is to use method of Lie symmetries to study the continuous-time CAPM options pricing model, equation (1.3), for β being a constant and β being equal to zero. We use this approach to identify invariant solutions to this PDE systematically, using first-order invariants that we construct and use to reduce the PDE. Section 2 incorporates a symmetry analysis of (1.3) with a constant beta. The derivation and application of Lie point symmetries to (1.3), commutator relations, PDE reductions and novel invariant solutions of the model are presented here. Section 3 is dedicated to the analysis of this model with no systematic risk i.e. $\beta = 0$, in the same manner as was done in section 2. Where appropriate, we provide plots of these solutions. Finally, in the last section we conclude our study.

2. Analysis of the European option pricing model with a constant market β

The CAPM equation (1.3), is invariant under a three-dimensional Lie algebra consisting of the symmetry vector fields, as determined by Lie theory:

$$X_1 = M\frac{\partial}{\partial M}, \tag{2.1}$$

$$X_2 = S\frac{\partial}{\partial S}, \tag{2.2}$$

$$X_3 = V\frac{\partial}{\partial V}. \tag{2.3}$$

In addition to the above three symmetries, there is a fourth, the infinite number of solution symmetries (trivial symmetries). For (2.1)-(2.3), calculations of the Lie brackets show that all commutators vanish.

2.1 *Reductions and invariant solutions for constant* β

We include the solutions for the CAPM using the symmetries detailed above. To solve for the European option price function $V(S, M, \tau)$, point symmetries corresponding to it are required. Such symmetries can be used to systematically reduce the order of the model and find exact solutions. We plot selected solutions in the analysis below.

We use the symmetry X_1 to reduce equation (1.3) to

$$\begin{aligned} &S^2\left(\frac{\partial^2}{\partial S^2}K(\tau,S)\right)\beta^2 a^2 + S^2\left(\frac{\partial^2}{\partial S^2}K(\tau,S)\right)\sigma^2 + \\ &2\, rS\frac{\partial}{\partial S}K(\tau,S) - 2\, rK(\tau,S) - 2\,\frac{\partial}{\partial \tau}K(\tau,S), \end{aligned} \tag{2.4}$$

where $V(S,M,\tau)=K(\tau,S)$. Reduced PDE (2.4), admits X_2 which we use to reduce again and get the solution

$$V(S,M,\tau)=C_1\mathrm{e}^{-r\tau}, \tag{2.5}$$

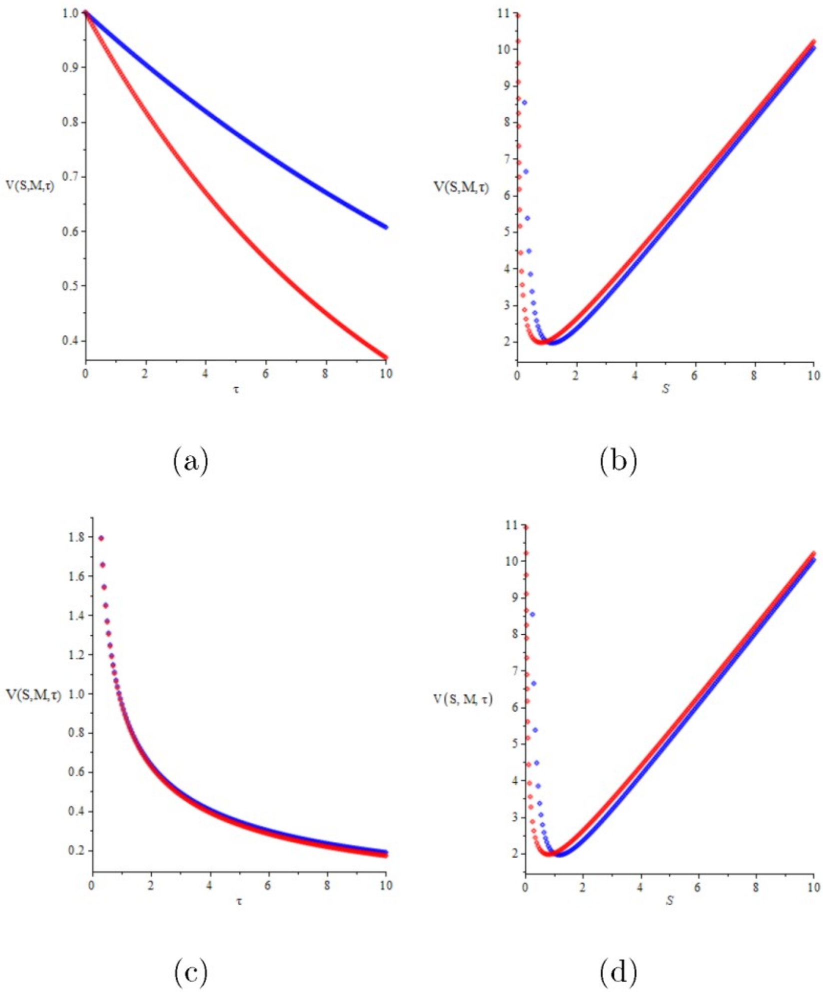


Figure 1

Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1=C_2=1$, and the ranges $0\leq\tau\leq 10$ and $0\leq S\leq 10$. We choose the parameter values of $\sigma_m=0.25$ and $\sigma=0.05$ for the solutions graphed and a value of $r=0.05$ except for the first plot: (a) solution of (2.5) where $r=0.05$ (blue plot) and $r=0.10$ (red plot); (b) solution of (2.6) where $\beta=1.0$ (blue) and $\beta=1.5$ (red); (c) solution of (2.7) where $\beta=1.0$ (blue) and $\beta=2.0$ (red); and, fig:ex1-d solution of (2.8) where $\beta=1.0$ (blue) and $\beta=1.5$ (red).

where C_1 is an arbitrary constant. Secondly, equation (2.4), admits the vector field $\frac{\partial}{\partial \tau}$, which is then used to get the solution

$$V(S,M,\tau) = C_1 S + C_2 S^{-2\frac{r}{\beta^2 {\sigma_m}^2+\sigma^2}}, \tag{2.6}$$

where C_i,s are arbitrary constants throughout the paper. A more complicated symmetry vector field admitted by (2.4) is

$$\left(-{\sigma_m}^2\beta^2\tau - \sigma^2\tau + 2\,r\tau + 2\ln(S)\right)K\frac{\partial}{\partial K} + (-2\,\beta^2\tau{\sigma_m}^2 - 2\,\sigma^2\tau)X_2,$$

and gives the price of the European option as

$$V(S,M,\tau) = \frac{C_1}{\sqrt{\tau}}\mathrm{e}^{-1/8\frac{\tau\left(\beta^2{\sigma_m}^2+\sigma^2+2\,r\right)^2}{\beta^2{\sigma_m}^2+\sigma^2}}. \tag{2.7}$$

The application of a linear combination of symmetries admitted by (2.4), $X_2 + \frac{\partial}{\partial \tau}$, yields

$$V(S,M,\tau) = C_1\, S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r+\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}} + C_2\, S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r-\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}}. \tag{2.8}$$

In Figure 1 we plot solutions (2.5)-(2.8) by varying either the risk-free interest rates or the β's.

Continuing our analysis in the same manner, X_2 reduces our model of study to get the reduced PDE

$${\sigma_m}^2M^2\frac{\partial^2}{\partial M^2}K(\tau,M) + 2\,rM\frac{\partial}{\partial M}K(\tau,M) - 2\,rK(\tau,M) - 2\,\frac{\partial}{\partial \tau}K(\tau,M), \tag{2.9}$$

where $V(S,M,\tau) = K(\tau,M)$. Application of $\left(-{\sigma_m}^2\tau + 2\,r\tau + 2\ln(M)\right)K\frac{\partial}{\partial K} - 2\,X_1\tau{\sigma_m}^2$ admitted by (2.9) yields the option price function

$$V(S,M,\tau) = \frac{C_1}{\sqrt{\tau}}\mathrm{e}^{-1/8\frac{\left({\sigma_m}^2+2\,r\right)^2\tau}{{\sigma_m}^2}}. \tag{2.10}$$

Similarly using $({\sigma_m}^4 + 4\,r{\sigma_m}^2 + 4\,r^2)K\frac{\partial}{\partial K} - 8\,\frac{\partial}{\partial \tau}{\sigma_m}^2$ we get

$$V(S,M,\tau) = C_1 M^{1/2\frac{{\sigma_m}^2-2\,r}{{\sigma_m}^2}} + C_2 M^{1/2\frac{{\sigma_m}^2-2\,r}{{\sigma_m}^2}}\ln(M). \tag{2.11}$$

In Figure 2 we plot these solutions.

Similarly, with X_2 followed by $\frac{\partial}{\partial \tau}$, we get the solution

$$V(S,M,\tau) = C_1 M^{1/2\frac{{\sigma_m}^2-2\,r+\sqrt{\left({\sigma_m}^2+2\,r\right)^2}}{{\sigma_m}^2}} + C_2 M^{-1/2\frac{-{\sigma_m}^2+2\,r+\sqrt{\left({\sigma_m}^2+2\,r\right)^2}}{{\sigma_m}^2}}. \tag{2.12}$$

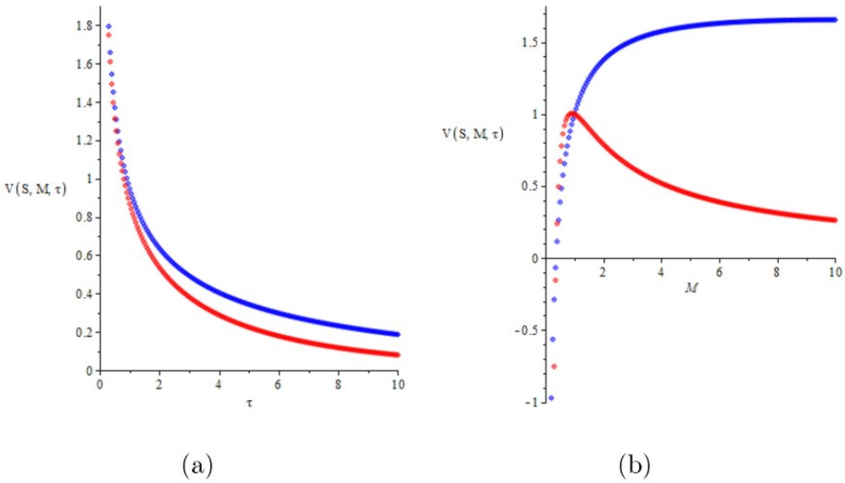


Figure 2
We select the parameter values $C_1 = C_2 = 1$ and $\sigma_m = 0.25$, with the ranges $0 \leq t \leq 10$ and $0 \leq M \leq 100$, while changing r: (a) solution of (2.10) where $r = 0.05$ (blue) and $r = 0.10$ (red); (b) solution of (2.11) where $r = 0.05$ (blue) and $r = 0.10$ (red).

In Figure 3 we plot (2.12) to examine how it evolves.

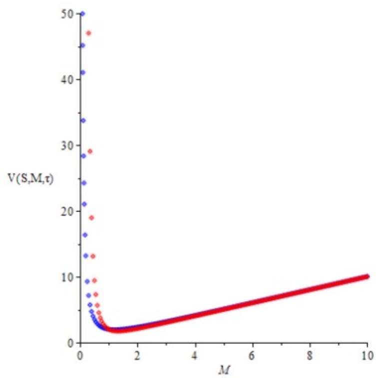


Figure 3
Evolution of the analytical solution (2.12) in the range $0 \leq M \leq 10$; we select the parameter values $C_1 = C_2 = 1$ and $\sigma_m = 0.25$ in the solution while varying r, with $r = 0.05$ (blue) and $r = 0.10$ (red).

A reduction by $c_1X_1 + c_2X_3$ followed by X_2 leads to

$$V(S, M, \tau) = C_1 \mathrm{e}^{-1/2\frac{(c_1 - c_2)(c_2\sigma_m{}^2 + 2\,rc_1)\tau}{c_1{}^2}}. \qquad (2.13)$$

A reduction by $c_1X_2 + c_2X_3$ and X_1 gives the following invariant solution

$$V(S,M,\tau) = C_1 \mathrm{e}^{-1/2\,\frac{(c_1-c_2)(\beta^2 c_2 \sigma_m{}^2+\sigma^2 c_2+2\,r c_1)\tau}{c_1{}^2}}. \tag{2.14}$$

Lastly, a reduction by $c_1X_1 + c_2X_2 + c_3X_3$ gives

$$V(S,M,\tau) = C_1 \mathrm{e}^{-1/2\,\frac{(c_2-c_3)(\beta^2 c_3 \sigma_m{}^2+\sigma^2 c_3+2\,r c_2)\tau}{c_2{}^2}}. \tag{2.15}$$

3. Analysis of the European option pricing model with zero systematic risk

In this section we let $\beta = 0$ in (1.3). This gives a zero-beta portfolio that has been designed to have no systematic risk. Risk-less assets have betas of zero. The projected return on a zero-beta portfolio would be the same as the risk-free rate. There is zero correlation with market movements. Due to the lack of market exposure, zero-beta portfolios are unlikely to attract investor attention in bull markets, as they would under-perform diversified market portfolios. But it may attract attention in bear markets. An example of such an asset is cash whose value remains constant given no inflation. Our model with this condition is now

$$\begin{aligned} &-\frac{\partial V}{\partial t} + 1/2\,\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 1/2\,\sigma_m{}^2 M^2 \frac{\partial^2 V}{\partial M^2} + \\ &r\left(S\frac{\partial V}{\partial S}V + M\frac{\partial V}{\partial M} - V\right) = 0. \end{aligned} \tag{3.1}$$

The model, (3.1), is invariant under a larger Lie algebra i.e. the six-dimensional Lie algebra consisting of the Lie point symmetries:

$$X_1, X_2, X_3, \tag{3.2}$$

$$X_4 = \frac{\partial}{\partial \tau}, \tag{3.3}$$

$$Y_1 = 2\,X_1 \tau \sigma_m{}^2 + X_3\big(\sigma_m{}^2\tau - 2\,r\tau - 2\ln(M)\big), \tag{3.4}$$

$$Y_2 = -2\,X_2 \tau \sigma^2 + X_3\big(-\sigma^2\tau + 2\,r\tau + 2\ln(S)\big). \tag{3.5}$$

In Table 1, we present the Lie brackets of the Lie symmetries of (3.1) derived above.

Table 1

Commutator table for $\boldsymbol{\beta = 0}$.

[,]	X_1	X_2	X_3	X_4
X_4	0	0	0	0
Y_1	$2X_3$	0	0	$-2\sigma_m^2 X_1 - (\sigma_m^2 - 2r)X_3$
Y_2	0	$-2X_3$	0	$2\sigma^2 X_2 - (-\sigma^2 + 2r)X_3$

3.1 *Reductions and invariant solutions for* $\beta = 0$

We derive the following reductions and invariant solutions using the same mathematical procedure as in the previous section. We use the symmetry X_1 to reduce model (3.1) to

$$\sigma^2 S^2 \frac{\partial^2}{\partial S^2} K(\tau,S) + 2\, rS \frac{\partial}{\partial S} K(\tau,S) - 2\, rK(\tau,S) - 2\, \frac{\partial}{\partial \tau} K(\tau,S), \tag{3.6}$$

where $V(S,M,\tau) = K(\tau,S)$ as a result of the reduction. From (3.6), its symmetry X_2 can be used to obtain (2.5). The third symmetry of the reduced equation (3.6) is $(\sigma^4 + 4\, r\sigma^2 + 4\, r^2)K \frac{\partial}{\partial K} - 8 \frac{\partial}{\partial \tau} \sigma^2$ from which the invariant solution found is

$$V(S,M,\tau) = C_1 S^{-1/2 \frac{-\sigma^2 + 2\, r}{\sigma^2}} + C_2 S^{-1/2 \frac{-\sigma^2 + 2\, r}{\sigma^2}} \ln(S). \tag{3.7}$$

Now (3.6) has a fourth symmetry vector field $\left(-\sigma^2 \tau + 2\, r\tau + 2 \ln(S)\right) K \frac{\partial}{\partial K} - 2X_2 \tau \sigma^2$ with the following solution

$$V(S,M,\tau) = \frac{C_1}{\sqrt{\tau}} \mathrm{e}^{-1/8 \frac{(\sigma^2 + 2\, r)^2 \tau}{\sigma^2}}. \tag{3.8}$$

In Figure 4 we plot these two solutions.

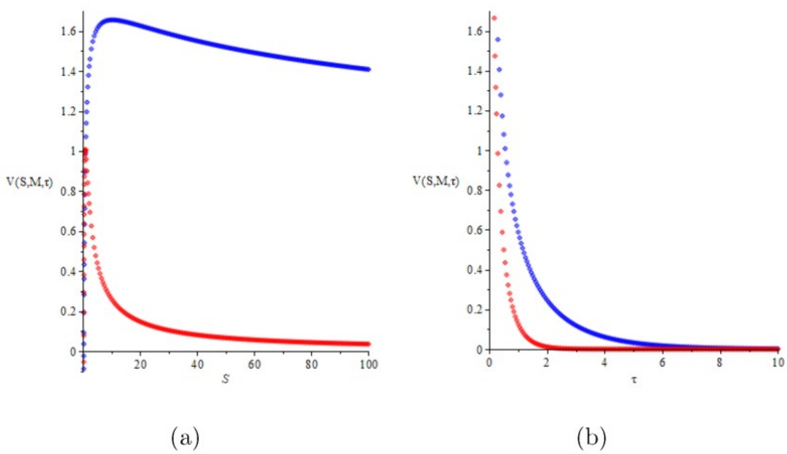


Figure 4

Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = 1$ with the ranges $0 \le \tau \le 10$ and $0 \le S \le 100$: (a) solution of (3.7) with $\sigma = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (3.8) with $\sigma = 0.05$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$.

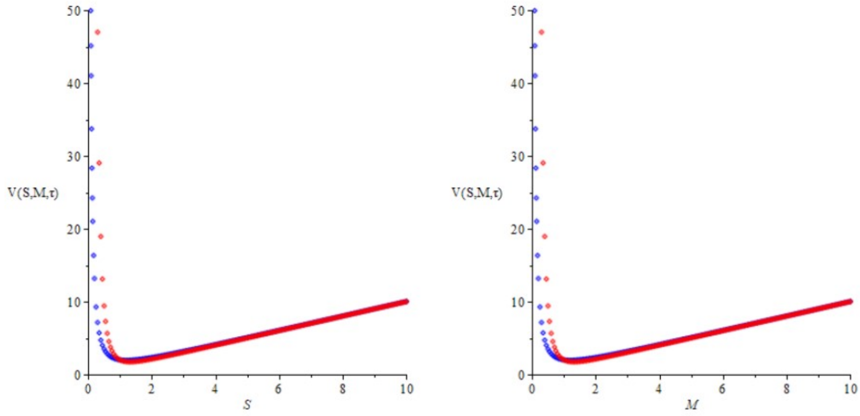
Reduction with X_4 followed by X_1 provides the solution

$$V(S,M,\tau) = C_1 S + C_2 S^{-2 \frac{r}{\sigma^2}}, \tag{3.9}$$

or if followed by the symmetry X_2, we have

$$V(S,M,\tau) = C_1 M + C_2 M^{-2 \frac{r}{\sigma_m{}^2}}. \tag{3.10}$$

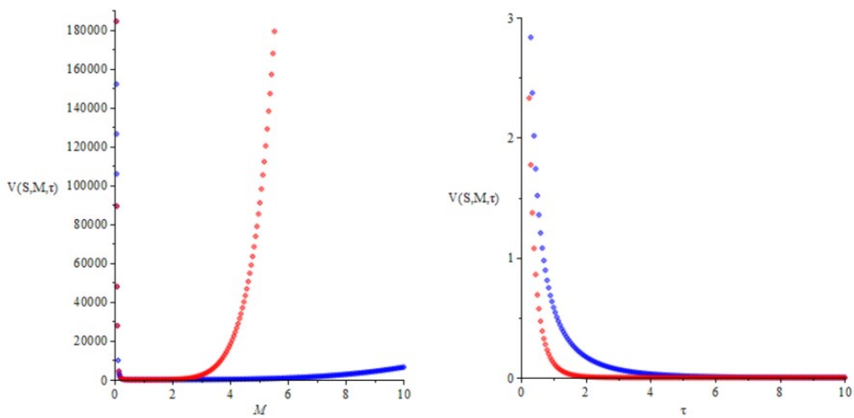
Figure 5 shows the evolution of these solutions for selected parameter values.



(a) (b)

Figure 5

We choose the parameter values $C_1 = C_2 = 1$ for the ranges $0 \leq M \leq 10$ and $0 \leq S \leq 10$: (a) solution of (3.9) with $\sigma = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (3.10) with $\sigma_m = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$;



(a) (b)

Figure 6

Evolution of $V(S, M, \tau)$. We select the parameter values $C_1 = C_2 = 1$ for the ranges $0 \leq M \leq 10$ and $0 \leq \tau \leq 10$ with $\sigma_m = 0.25; \sigma = 0.05$: (a) solution of (3.11) where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (3.12) where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$.

A reduction with Y_2 and the symmetry $\frac{K\frac{\partial}{\partial K}}{\tau} - 2\,\frac{\partial}{\partial \tau}$ gives us

$$V(S,M,\tau) = C_1 M^{1/2\frac{\sqrt{(\sigma^2+\sigma_m{}^2)(\sigma^2\sigma_m{}^2+4r^2)}+(\sigma_m{}^2-2r)\sigma}{\sigma_m{}^2\sigma}} + C_2 M^{1/2\frac{-\sqrt{(\sigma^2+\sigma_m{}^2)(\sigma^2\sigma_m{}^2+4r^2)}+(\sigma_m{}^2-2r)\sigma}{\sigma_m{}^2\sigma}}. \tag{3.11}$$

Lastly, from Lie point symmetry of the reduced PDE, $\left(-K\sigma_m{}^2 t + 2\,Kr\tau + 2\,K\ln(M)\right)\frac{\partial}{\partial K} - 2\tau X_1\sigma_m{}^2$ we have

$$V(S,M,\tau) = \frac{C_1}{\tau}\mathrm{e}^{-1/8\frac{\tau(\sigma^2+\sigma_m{}^2)(\sigma^2\sigma_m{}^2+4r^2)}{\sigma^2\sigma_m{}^2}}. \tag{3.12}$$

In Figure 6 these invariant solutions are visually presented. Reducing with $c_1X_2 + c_2Y_2$ and X_1, we obtain

$$V(S,M,\tau) = \frac{1}{2}\frac{C_1\sqrt{2}}{\sqrt{\sigma^2\tau}}\mathrm{e}^{-1/8\frac{\tau(\sigma^2+2r)^2}{\sigma^2}}. \tag{3.13}$$

This solution is plotted in Figure 7.

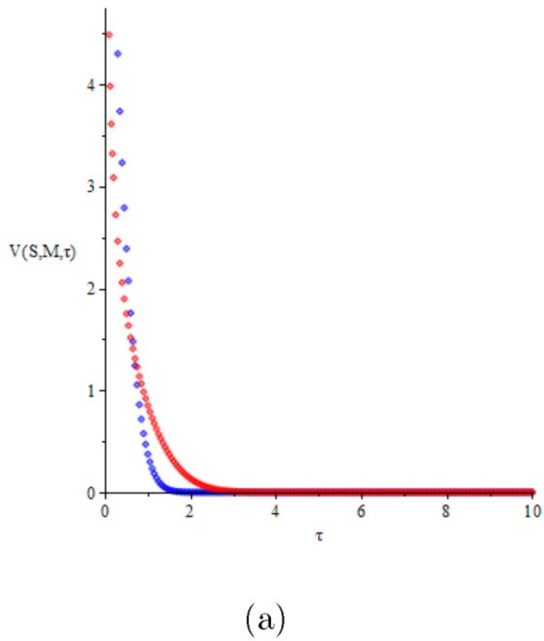


Figure 7
We let $C_1 = 1$ and $r = 0.05$ and the range $0 \le \tau \le 10$: (a) solution of (3.13) for $\sigma = 0.25$ (blue) and for $\sigma = 0.5$ (red);

Finally, reduction with Y_1 and X_2 provides

$$V(S,M,\tau) = \frac{1}{2}\frac{C_1\sqrt{2}}{\sqrt{\sigma_m{}^2\tau}}\mathrm{e}^{-1/8\frac{\tau\left(\sigma_m{}^2+2r\right)^2}{\sigma_m{}^2}}. \tag{3.14}$$

4. Conclusion

The CAPM is a critical component of portfolio management. Despite the presence of more advanced asset pricing and portfolio selection methodologies, the CAPM is widely utilized due to its ease of use. The European option price under a continuous-time CAPM can be derived. In this paper we have successfully performed a symmetry analysis for the European option pricing model under a continuous-time CAPM, considering both constant and zero market betas. We discovered that this PDE admits several invariant solutions for zero and non-zero betas. Commutator relations were also derived in which, for beta equal to a constant, all commutators vanished. We provided several plots for our solutions given different levels of beta, risk-free interest rates, and volatilities.

When β is constant in our model, solution (2.5) shows that the option price decreases exponentially over time for a given interest rate. This could be explained by time decay which states that as the time to the expiration date approaches, the value of an option decreases. We observe that time decay accelerates as we increase τ because as the expiration date gets closer there is less time for an investor to profit from the option. In other solutions we see that the option price generally increases with market price or with asset price, as expected. We conclude that symmetry analysis provides valuable insights into option pricing.

Declarations

Conflict of interest: We have no conflict of interest to declare.
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