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Infinitely many solutions for a Neumann elliptic system with critical exponents and hardy potentials

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Abstract

Using concentration estimates along with the Fountain Theorem and its dual form, we establish the existence of two infinite and disjoint sets of solutions for the following Neumann elliptic system involving critical Sobolev exponents, a Hardy potential, and a concave-convex nonlinearity:

$$\begin{cases} -\Delta w_1 - \beta \frac{w_1}{|x|^2} = \frac{2\alpha_1}{\alpha_1+\beta_1}|w_1|^{\alpha_1-2} w_1 |w_2|^{\beta_1} \\ \qquad\qquad + \frac{2\alpha_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2-2} w_1 |w_2|^{\beta_2} & \text{in } \Omega, \\ \qquad -\Delta w_2 - \beta \frac{w_2}{|x|^2} = \frac{2\beta_1}{\alpha_1+\beta_1}|w_1|^{\alpha_1} |w_2|^{\beta_1-2} w_2 \\ \qquad\qquad + \frac{2\beta_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2} |w_2|^{\beta_2-2} w_2 & \text{in } \Omega, \\ \partial_v w_1 = \partial_v w_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing the origin, with $N \geq 7$, $\alpha_1 + \beta_1 = 2^*$, $\overline{\beta} = \frac{(N-2)^2}{4}$, $\beta \in [0, \overline{\beta} - 4)$, $2^* - \sqrt{1-\frac{\beta}{\overline{\beta}}} < \alpha_2 + \beta_2 < 2$, and $2^* := \frac{2N}{N-2}$ denotes the critical Sobolev exponent.

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1. Introduction

This paper investigates the existence of infinitely many solutions to the following elliptic problem subject to Neumann boundary conditions:

$$\begin{cases} -\Delta w_1 - \beta \frac{w_1}{|x|^2} &= \frac{2\alpha_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1-2} w_1 |w_2|^{\beta_1} \\ & \quad + \frac{2\alpha_2}{\alpha_2+\beta_2} |w_1|^{\alpha_2-2} w_1 |w_2|^{\beta_2} \quad \text{in } \Omega, \\ -\Delta w_2 - \beta \frac{w_2}{|x|^2} &= \frac{2\beta_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1} |w_2|^{\beta_1-2} w_2 \\ & \quad + \frac{2\beta_2}{\alpha_2+\beta_2} |w_1|^{\alpha_2} |w_2|^{\beta_2-2} w_2 \quad \text{in } \Omega, \\ \partial_\nu w_1 = \partial_\nu w_2 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $1 < \alpha_2 + \beta_2 < 2$, , and the parameters $\alpha_1, \beta_1 > 1$ are such that $\alpha_1 + \beta_1 = 2^*$, where $2^* = \frac{2N}{N-2}$ represents the critical Sobolev exponent for $N \geq 3$. The vector ν denotes the outward normal to the boundary $\partial\Omega$. Throughout this paper, we consider Ω to be a smooth and bounded open domain in $\mathbb{R}^N$ that contains the origin.

The corresponding energy functional to (1) is

$$\begin{aligned} I(w_1, w_2) &= \frac{1}{2} \int_\Omega \left(|Dw_1|^2 + |Dw_2|^2 - \beta \frac{|w_1|^2 + |w_2|^2}{|x|^2} \right) dx \\ &\quad - \frac{2}{\alpha_1+\beta_1} \int_\Omega |w_1|^{\alpha_1} |w_2|^{\beta_1} dx - \frac{2}{\alpha_2+\beta_2} \int_\Omega |w_1|^{\alpha_2} |w_2|^{\beta_2} dx. \end{aligned} \tag{2}$$

However, we still have the following inequalitie : for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for all $w \in H^1(\Omega)$,

$$\int_\Omega \frac{w^2}{|x|^2} dx \leqslant \frac{1+\varepsilon}{\beta} \int_\Omega |Dw|^2 dx + C(\varepsilon) \int_\Omega |w|^2 dx.$$

Under the assumptions $\beta = 0$, $\alpha_2 = 2$, $\beta_2 = 0$, and $w_1 = w_2$, problem (1) simplifies to the following scalar semilinear elliptic equation:

$$\begin{cases} -\Delta w = |w|^{2^*-2} w + \mu w, & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

In the celebrated paper [7] Cao and Yan showed that under certain assumptions on N and $\partial\Omega$ the problem (3) has infinitely many solutions. Comte and Knaap in [10] proved that if $N = 3$, there exists a nontrivial solution of (3) for every $\mu \in \mathbb{R}^+ \backslash \sigma(-\Delta)$, where $\sigma(-\Delta)$ is the set of eigenvalues of $-\Delta$ in $\left\{ w \in H^1(\Omega) | \frac{\partial w}{\partial \nu}|_{\partial\Omega} = 0 \right\}$. In recent years the nonlinear Neumann problem involving critical Sobolev exponent has been widely studied in [8, 9, 11, 14]. Subsequently, for the Dirichlet problem of (3), Devillanova and Solimini [12] established the existence of infinitely many solutions to problem (3) when $N \geq 7$. Their key idea was to prove the strong convergence of approximating solutions. The main tool in their analysis is the derivation of suitable estimates for these approximating solutions in a carefully defined "safe region," followed by

the application of a local Pohozaev identity. Building on similar techniques, Han [13] proved that if $N > \frac{2(q+1)}{q-1}$, then problem (3) admits infinitely many solutions with positive energy. Further affirmative results related to the above open problem can be found in [3, 6]. However, to the best of our knowledge, no straightforward answer exists for the corresponding question concerning problem (1). The main result of this paper is stated below.

Theorem 1.1: *If we assume that* $N \geq 7$, $\beta \in [0, \bar{\beta} - 4)$ *and* $2^* - \sqrt{1 - \frac{\beta}{\bar{\beta}}} < \alpha_2 + \beta_2 < 2$ *then*

(i) *There exists a sequence of solutions* $(w_{1,k}, w_{2,k})_k$ *of (1) such that* $I(w_{1,k}, w_{2,k}) < 0$ *and* $I(w_{1,k}, w_{2,k}) \to 0$ *as* $k \to +\infty$.

(ii) *There exists a sequence of solutions* $(\widetilde{w}_{1,k}, \widetilde{w}_{2,k})_k$ *of (1) such that* $I(\widetilde{w}_{1,k}, \widetilde{w}_{2,k}) > 0$ *and* $I(\widetilde{w}_{1,k}, \widetilde{w}_{2,k}) \to +\infty$ *as* $k \to +\infty$.

When $\alpha_2 = 2$, $\beta_2 = 0$, and $w_1 = w_2$, the method introduced in [12] was also employed by Daomin Cao and Shusen Yan in [5] to show that if $N \geq 7$, $a > 0$, and $\mu \in [0, \bar{\mu} - 4)$, then problem (1) admits infinitely many solutions. For additional related results, we refer the reader to [2, 4, 17]. Our main approach follows the same general strategy as that used in [5]. However, a significant difficulty arises when applying the Moser iteration technique, due to the lack of a reverse Hölder inequality in the case $1 < \alpha_2 + \beta_2 < 2$. To overcome this issue, we introduce a new argument that works for the subcritical term $|u|^{\alpha_2+\beta_2}$ with $1 < \alpha_2 + \beta_2 < 2^*$ (see the proof of Proposition 2.5 below). This paper is organized as follows. In Section 2, we establish the strong convergence of approximating solutions of (1) in $H^1(\Omega)$. In Section 3, using the Fountain Theorem and its dual form [1, 19], we prove Theorem 1.1.

To conclude this introduction, we clarify some notations used throughout the paper. We denote the norms in the spaces $H^1(\Omega)$ and $L^p(\Omega)$ (for $1 \leq p < \infty$) by $\|w\|_{H^1(\Omega)} := \left(\int_\Omega |Dw|^2 dx + |w|^2 dx\right)^{\frac{1}{2}}$, $\|w\|_p := \left(\int_\Omega |w|^p dx\right)^{\frac{1}{p}}$, respectively. Throughout the paper, C, C_1, C_2, ... will denote generic positive constants, which may vary from line to line. In addition, for convenience, we define the norm on the product space $H^1(\Omega) \times H^1(\Omega)$ by $\|(w_1, w_2)\| := \|w_1\|^2_{H^1(\Omega)} + \|w_2\|^2_{H^1(\Omega)}$.

2. Strong convergence of approximating solutions in $H^1(\Omega)$.

The study focuses on the following perturbed problem:

$$\begin{cases} -\Delta w_1 - \beta \frac{w_1}{|x|^2+\varepsilon} = \frac{2(\alpha_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon}|w_1|^{\alpha_1-2-\varepsilon}w_1|w_2|^{\beta_1-\varepsilon} \\ \qquad\qquad + \frac{2\alpha_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2-2}w_1|w_2|^{\beta_2} \quad \text{in } \Omega, \\ -\Delta w_2 - \beta \frac{w_2}{|x|^2+\varepsilon} = \frac{2(\beta_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon}|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-2-\varepsilon}w_2 \\ \qquad\qquad + \frac{2\beta_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2}|w_2|^{\beta_2-2}w_2 \quad \text{in } \Omega, \\ \partial_\nu w_1 = \partial_\nu w_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

Here, $\varepsilon > 0$ is a small constant satisfying $0 < \varepsilon < \min\{\alpha_1 - 1, \beta_1 - 1\}$. A pair of functions $(w_1, w_2) \in H^1(\Omega) \times H^1(\Omega)$ is referred to as a weak solution of problem (4) if the following condition holds for all test functions $\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega) \times H^1(\Omega)$

$$\begin{aligned} &\int_\Omega \left(Dw_1 D\varphi_1 + Dw_2 D\varphi_2 - \beta \frac{w_1\varphi_1}{|x|^2+\varepsilon} - \beta \frac{w_2\varphi_2}{|x|^2+\varepsilon} \right) dx \\ &- \frac{2(\alpha_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon} \int_\Omega |w_1|^{\alpha_1-2-\varepsilon}w_1|w_2|^{\beta_1-\varepsilon}\varphi_1 dx \\ &- \frac{2(\beta_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon} \int_\Omega |w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-2-\varepsilon}w_2\varphi_2 dx \\ &- \frac{2\alpha_2}{\alpha_2+\beta_2} \int_\Omega |w_1|^{\alpha_2-2}|w_2|^{\beta_2}w_1\varphi_1 dx \\ &- \frac{2\beta_2}{\alpha_2+\beta_2} \int_\Omega |w_1|^{\alpha_2}|w_2|^{\beta_2-2}w_2\varphi_2 dx = 0. \end{aligned}$$

We define the energy functional related to problem (4) over $H^1(\Omega) \times H^1(\Omega)$ by

$$\begin{aligned} I^\varepsilon(w_1, w_2) &= \frac{1}{2}\int_\Omega \left(|Dw_1|^2 + |Dw_2|^2 - \beta \frac{|w_1|^2+|w_2|^2}{|x|^2+\varepsilon} \right) dx \\ &- \frac{2}{\alpha_1+\beta_1-2\varepsilon} \int_\Omega |w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon} dx - \frac{2}{\alpha_2+\beta_2} \int_\Omega |w_1|^{\alpha_2}|w_2|^{\beta_2} dx. \end{aligned} \tag{5}$$

We begin by introducing some notations and terminology that will be used throughout the remainder of the work. Let (w_1, w_2) be a solution to problem (4), and define $\widetilde{w}_1 := |w_1|$, $\widetilde{w}_2 = |w_2|$, extended by zero outside Ω. It follows that $\widetilde{w}_1, \widetilde{w}_2 \in H^1(\mathbb{R}^N)$. Then, for any $\varphi \in H^1(\mathbb{R}^N)$ such that $\varphi \geq 0$, we have:

$$\begin{aligned} &\int_{\mathbb{R}^N} D\widetilde{w}_1 D\varphi = \int_\Omega D|w_1|.D\varphi dx \\ &= \int_{\partial\Omega} \varphi \frac{\partial|w_1|}{\partial n} ds - \int_\Omega |w_1|^{-1}w_1 div(Dw_1)\varphi dx \\ &\leq \int_{\mathbb{R}^N} \left(\beta \frac{\widetilde{w}_1+\widetilde{w}_2}{|x|^2+\varepsilon} + \frac{2(\alpha_1-\varepsilon)}{2^*-2\varepsilon} (\widetilde{w}_1 + \widetilde{w}_2)^{2^*-1-2\varepsilon} \right. \\ &\left. + \frac{2\alpha_2}{\alpha_2+\beta_2} (\widetilde{w}_1 + \widetilde{w}_2)^{\alpha_2+\beta_2-1} \right) \varphi dx, \end{aligned}$$

which holds in the distributional sense

$$-\Delta\widetilde{w}_1 - \beta\frac{\widetilde{w}_1+\widetilde{w}_2}{|x|^2+\varepsilon} \leq \frac{2(\alpha_1-\varepsilon)}{2^*-2\varepsilon}(\widetilde{w}_1+\widetilde{w}_2)^{2^*-1-2\varepsilon} + \frac{2\alpha_2}{\alpha_2+\beta_2}(\widetilde{w}_1+\widetilde{w}_2)^{\alpha_2+\beta_2-1}.$$

Similarly,

$$-\Delta\widetilde{w}_2 - \beta\frac{\widetilde{w}_1+\widetilde{w}_2}{|x|^2+\varepsilon} \leq \frac{2(\beta_1-\varepsilon)}{2^*-2\varepsilon}(\widetilde{w}_1+\widetilde{w}_2)^{2^*-1-2\varepsilon} + \frac{2\beta_2}{\alpha_2+\beta_2}(\widetilde{w}_1+\widetilde{w}_2)^{\alpha_2+\beta_2-1}.$$

It follows that

$$-\Delta(\widetilde{w}_1+\widetilde{w}_2) - \beta\frac{\widetilde{w}_1+\widetilde{w}_2}{|x|^2+\varepsilon} \leq 3(\widetilde{w}_1+\widetilde{w}_2)^{2^*-1} + A \text{ in } \Omega, \tag{6}$$

where $\partial_v(\widetilde{w}_1+\widetilde{w}_2) = 0 \text{on } \partial\Omega$, and A is a positive constant. Therefore, in the next section, we restrict our attention to estimating solutions of (6) in the space $H^1(\mathbb{R}^N)$.

Definition 2.1: Let us consider a sequence $\{(w_{1,n}, w_{2,n})\}$ as given.

(i) We say that the sequence $\{(w_{1,n}, w_{2,n})\}$ is controlled if the sum $w_{1,n} + w_{2,n}$ is a solution to problem (6).

(ii) We refer to the sequence $\{(w_{1,n}, w_{2,n})\}$ as balanced if each pair $\{(w_{1,n}, w_{2,n})\}$ solves problem (4) with $\varepsilon = \varepsilon_n > 0$ where $\varepsilon_n \to 0$ as $n \to \infty$.

Remark 1: Every balanced sequence $(w_{1,n}, w_{2,n})$ is also a controlled sequence. Moreover, we may assume without loss of generality that any controlled sequence $(w_{1,n}, w_{2,n})$ is nonnegative in Ω, i.e., $w_{1,n}, w_{2,n} \geq 0$ in Ω, since each function can be replaced by its absolute value.

From the appendix in [16], we have the following decomposition of approximating solutions.

Proposition 2.2: Let $0 \in \overline{\Omega}$, and let $\{(w_{1,n}, w_{2,n})\} \subset H^1(\Omega) \times H^1(\Omega)$ be a Palais--Smale sequence for the functional I (as defined in (2)). Then, possibly after passing to a subsequence, the following decomposition holds:

$$\begin{aligned} w_{1,n} &= w_{1,\infty} + \sum_{i=1}^{k_1+k_2} (\sigma_n^i)^{\frac{N-2}{2}} W_{1,i}\left(\sigma_n^i(x-x_n^i)\right) \\ &\quad + \sum_{i=k_1+k_2+1}^{k_1+k_2+k_3} (\sigma_n^i)^{\frac{N-2}{2}} W_{1,i}(\sigma_n^i x) + \omega_{1n}, \\ w_{2,n} &= w_{2,\infty} + \sum_{i=1}^{k_1+k_2} (\sigma_n^i)^{\frac{N-2}{2}} W_{2,i}\left(\sigma_n^i(x-x_n^i)\right) \\ &\quad + \sum_{i=k_1+k_2+1}^{k_1+k_3} (\sigma_n^i)^{\frac{N-2}{2}} W_{2,i}(\sigma_n^i x) + \omega_{2n}, \end{aligned} \tag{7}$$

where $(w_{1,\infty}, w_{2,\infty})$ is a solution to problem (1) and $\|\omega_{in}\|_{H^1(\Omega)} \to 0$ as $n \to \infty$ for $i = 1, 2$.

For the profile indices:

- For $i = 1, \dots, k_1$, if $\operatorname{dist}(x_n^i, \partial\Omega) \to \infty$ and $\sigma_n^i \operatorname{dist}(x_n^i, \partial\Omega) \to \infty$, then $(W_{1,i}, W_{2,i}) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solves the limit problem (Q_β^∞).
- For $i = k_1 + 1, \dots, k_1 + k_2$, if $\operatorname{dist}(x_n^i, \partial\Omega) \to 0$ and $\sigma_n^i \operatorname{dist}(x_n^i, \partial\Omega) \to \infty$, then $(W_{1,i}, W_{2,i}) \in D^{1,2}(\mathbb{R}+^N) \times D^{1,2}(\mathbb{R}+^N)$ solves the limit problem (P_0^∞).
- For $i = k_1 + k_2 + 1, \dots, k_1 + k_2 + k_3$, the pair $(W_{1,i}, W_{2,i}) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solves:

- (Q_β^∞) for some $\beta \in (0, \bar{\beta})$ if $0 \in \Omega$,
- (P_β^∞) for some $\beta \in (0, \bar{\beta})$ if $0 \in \partial\Omega$,

where the problems at infinity are given by:

$$\left(Q_\beta^\infty\right) \begin{cases} -\Delta w_1 - \beta \frac{w_1}{|x|^2} = \frac{2\alpha_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1-2} w_1 |w_2|^{\beta_1}, \\ -\Delta w_2 - \beta \frac{w_2}{|x|^2} = \frac{2\beta_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1} |w_2|^{\beta_1-2} w_2, \end{cases} \text{in } \mathbb{R}^N,$$

with the decay condition $w_1(x), w_2(x) \to 0$ as $|x| \to \infty$.

$$\left(P_\beta^\infty\right) \begin{cases} -\Delta w_1 - \beta \frac{w_1}{|x|^2} = \frac{2\alpha_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1-2} w_1 |w_2|^{\beta_1}, \\ -\Delta w_2 - \beta \frac{w_2}{|x|^2} = \frac{2\beta_1}{\alpha_1+\beta_1} |w_1|^{\alpha_1} |w_2|^{\beta_1-2} w_2, \end{cases} \text{in } \mathbb{R}_+^N,$$

with the Neumann boundary condition $\partial_\nu w_1 = \partial_\nu w_2 = 0$ on $\partial\mathbb{R}_+^N$.

Here, $\mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$, and $\alpha_1, \beta_1 > 1$ with $\alpha_1 + \beta_1 = 2^*$.

The sequence $\{(w_{1,n}, w_{2,n})\}$ is said to be concentrating if the limit in () is attained strongly in the H^1 topology. Among the various bubbles arising in (), we select the one with the lowest concentration speed, denoted by σ_n, which concentrates at the point $x_n = x_n^i$. One can always choose a constant $\bar{C} > 0$ such that the region $\mathcal{A}_n^1 := \left(B_{(\bar{C}+5)\sigma_n^{-\frac{1}{2}}}(x_n) \setminus B_{\bar{C}\sigma_n^{-\frac{1}{2}}}(x_n) \right) \cap \Omega$, contains no concentration point of the sequence (u_n, v_n) for any $n \in \mathbb{N}$. This region is referred to as a safe region, as it lies sufficiently far from all concentration points.

We now introduce two thinner subsets, defined as follows:

$$\mathcal{A}_n^2 := \left(B_{(\bar{C}+4)\sigma_n^{-\frac{1}{2}}}(x_n) \setminus B_{(\bar{C}+1)\sigma_n^{-\frac{1}{2}}}(x_n) \right) \cap \Omega,$$

and

$$\mathcal{A}_n^3 := \left(B_{(\bar{C}+3)\sigma_n^{-\frac{1}{2}}}(x_n) \setminus B_{(\bar{C}+2)\sigma_n^{-\frac{1}{2}}}(x_n) \right) \cap \Omega.$$

Given $p_2 < 2^* < p_1$, together with parameters $\alpha > 0$ and $\sigma > 0$, we consider the relation below:

$$\begin{cases} \|w_1\|_{p_1} \leq \alpha, \\ \|w_2\|_{p_2} \leq \alpha \sigma^{\frac{N}{2^*} - \frac{N}{p_2}}. \end{cases} \tag{8}$$

We define $\|w\|_{p_1,p_2,\sigma}$ as the infimum of all $\alpha > 0$ for which there exist functions w_1 and w_2 such that condition (8) is satisfied and $|w| \leq w_1 + w_2$.

Based on (8), we have the following two results pertaining to controlled concentrating sequences.

Proposition 2.3: Let $\{(w_{1,n}, w_{2,n})\}_{n\in\mathbb{N}}$ be a controlled sequence. For any exponents $p_1, p_2 \in \left(\frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}+\sqrt{\bar{\beta}-\beta}}, \frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}-\sqrt{\bar{\beta}-\beta}}\right)$, with $p_2 < 2^* < p_1$, there exists a constant $C > 0$, depending only on p_1, p_2, N, and t, such that

$$\left\| |w_{1,n}| + |w_{2,n}| \right\|_{p_1,p_2,\sigma_n} \leq C,$$

where σ_n denotes the smallest concentration rate among all bubbles appearing in the sequence $(w_{1,n}, w_{2,n})$.

Lemma 2.4: *Assume that Ω is convex and that $0 \in \overline{\Omega}$. Let $\{(w_{1,n}, w_{2,n})\}_{n\in\mathbb{N}}$ be a controlled concentrating sequence. Then, for any $p_1 \in \left(\frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}+\sqrt{\bar{\beta}-\beta}}, \frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}-\sqrt{\bar{\beta}-\beta}}\right)$ with $0 \leq t < \bar{t}$, there exists a constant $C > 0$, independent of n, such that for sufficiently large n and any radius*

$$r \in \left[\bar{C}\sigma_n^{-\frac{1}{2}}, (\bar{C}+5)\sigma_n^{-\frac{1}{2}}\right],$$

the following estimate holds:

$$\frac{1}{r^{N-1}} \int_{\Omega\cap\partial B_r(x_n)} w_n \, dS \leq C \, \sigma_n^{\frac{N}{2p_1}},$$

where $w_n(x) = |w_{1,n}(x)| + |w_{2,n}(x)|$.

We set $\Phi_n(x) := w_n\left(\sigma_n^{-1/2}x\right), x \in \Omega_n$, where $\Omega_n := \left\{x : \sigma_n^{-1/2}x \in \Omega\right\}$.

Using inequality (6), it is straightforward to verify that Φ_n, extended by zero outside Ω, satisfies the following relation:

$$-\Delta\Phi_n \leq \sigma_n^{-1}\left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right) \text{in } \mathbb{R}^N. \tag{9}$$

From (9) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} D\Phi_n D\varphi \mathrm{d}x \leq \sigma_n^{-1}\int_{\mathbb{R}^N}\left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right)\varphi \mathrm{d}x, \\ &\quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0. \end{aligned} \tag{10}$$

We have the following estimate:

Proposition 2.5: Let Ω be a convex domain with $0 \in \overline{\Omega}$, and let $\{(w_{1,n}, w_{2,n})\}$ be a controlled concentrating sequence. Then, for any

$$p_1 \in \left(\frac{2^*\sqrt{\beta}}{\sqrt{\beta}+\sqrt{\bar{\beta}-\beta}}, \frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}-\sqrt{\bar{\beta}-\beta}}\right), \text{ with } 0 \le \beta < \bar{\beta},$$

there exists a constant $C > 0$, independent of n, such that

$$\int_{A_n^2} |w_n|^{2t^2}\, dx \le C\lambda_n^{-\frac{N}{2}+\frac{Nt^2}{p_1}},$$

where $t := \frac{2^*}{2}$.

Proof: Fix $y \in \mathcal{A}_n^2$ and let $0 < r < R \le 1$. Set $z_n := \sigma_n^{1/2} y$, and let $\chi \in \mathcal{C}_0^\infty(B(z_n, R))$ be a cut-off function satisfying $0 \le \chi \le 1$, $\chi = 1$ on $B(z_n, r)$, and $|D\chi| \le \frac{1}{R-r}$. For any $t > 1$ and $M > 0$, we define the test function

$$\varphi := \chi^2 \Phi_n\, \widetilde{\Phi}_n^{2(t-1)},$$

where $\widetilde{\Phi}_n := \min\{\Phi_n, M\}$. In what follows, the symbol C denotes various positive constants that are independent of n, r, R, β, and M. Note that $D\Phi_n D\widetilde{\Phi}_n = |D\widetilde{\Phi}_n|^2$ and $D\Phi_n D\widetilde{\Phi}_n \Phi_n \widetilde{\Phi}_n^{2(t-1)-1} = |D\widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(t-1)}$.

$$\begin{aligned}
&\int_{\mathbb{R}^N} D\Phi_n D\varphi dx = \int_{\mathbb{R}^N} D\Phi_n.\left[2\chi D\chi \Phi_n \widetilde{\Phi}_n^{2(t-1)} + \chi^2 \widetilde{\Phi}_n^{2(t-1)} D\Phi_n\right.\\
&\left.+2(t-1)\chi^2 \Phi_n \widetilde{\Phi}_n^{2(t-1)-1} D\widetilde{\Phi}_n\right] dx\\
&= \int_{\mathbb{R}^N} \chi^2 |D\Phi_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx + 2(t-1)\int_{\mathbb{R}^N} \chi^2 |D\widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx\\
&+2\int_{\mathbb{R}^N} \chi \Phi_n \widetilde{\Phi}_n^{2(t-1)} D\chi D\Phi_n dx\\
&\le \sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right)\chi^2 \Phi_n \widetilde{\Phi}_n^{2(t-1)} dx.
\end{aligned}$$

By the Young's inequality, we have

$$|(\chi D\Phi_n)\cdot(\Phi_n D\chi)| \le |D\chi|^2 \Phi_n^2 + \frac{1}{4}\chi^2 |D\Phi_n|^2.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |D\Phi_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx + 2(t-1)\int_{\mathbb{R}^N} \chi^2 |D\widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx\\
&\le \sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right)\chi^2 \Phi_n \widetilde{\Phi}_n^{2(t-1)} dx\\
&+2\int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(t-1)} \left[|D\chi|^2 \Phi_n^2 + \frac{1}{4}\chi^2 |D\Phi_n|^2\right] dx,
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |D\Phi_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx + 4(t-1)\int_{\mathbb{R}^N} \chi^2 |D\widetilde{\Phi}_n|^2 \widetilde{\Phi}_n^{2(t-1)} dx\\
&\le 2\sigma_n^{-1} \int_{\mathbb{R}^N} \left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right)\chi^2 \Phi_n \widetilde{\Phi}_n^{2(t-1)} dx\\
&+4\int_{\mathbb{R}^N} \widetilde{\Phi}_n^{2(t-1)} |D\chi|^2 \Phi_n^2 dx.
\end{aligned} \tag{11}$$

Now, we take $t = \frac{2^*}{2} > 1$. Consider $\Psi_M := \chi \Phi_n \widetilde{\Phi}_n^{t-1}$, we have

$$\begin{aligned}
D\Psi_M &= D\chi \Phi_n \widetilde{\Phi}_n^{(t-1)} + \chi D\Phi_n \widetilde{\Phi}_n^{(t-1)} + (t-1)\chi \Phi_n \widetilde{\Phi}_n^{(t-2)} D\widetilde{\Phi}_n\\
&= \chi D\Phi_n \widetilde{\Phi}_n^{(t-1)} + (t-1)\chi \widetilde{\Phi}_n^{(t-1)} D\widetilde{\Phi}_n + D\chi \Phi_n \widetilde{\Phi}_n^{(t-1)}
\end{aligned}$$

since $t > 1$,

$$|D\Psi_M|^2 \le C\big[\chi^2|D\Phi_n|^2\widetilde{\Phi}_n^{2(t-1)} + (t-1)^2\chi^2\widetilde{\Phi}_n^{2(t-1)}|D\widetilde{\Phi}_n|^2 + \widetilde{\Phi}_n^{2(t-1)}|D\chi|^2{\Phi_n}^2\big] \le Ct\big[\chi^2|D\Phi_n|^2\widetilde{\Phi}_n^{2(t-1)} + 4(t-1)\chi^2\widetilde{\Phi}_n^{2(t-1)}|D\widetilde{\Phi}_n|^2 + \widetilde{\Phi}_n^{2(t-1)}|D\chi|^2{\Phi_n}^2\big]. \tag{12}$$

Formula (11) and (12) yields that

$$\int_{\mathbb{R}^N} |D\Psi_M|^2 dx \le Ct\left[\sigma_n^{-1}\int_{\mathbb{R}^N}\left(\frac{\beta}{|x|^2}\Phi_n + 3\Phi_n^{2^*-1} + A\right)\chi^2\Phi_n\widetilde{\Phi}_n^{2(t-1)}dx + \int_{\mathbb{R}^N}\widetilde{\Phi}_n^{2(t-1)}|D\chi|^2\Phi_n^2 dx\right]$$

$$\le Ct\left[\sigma_n^{-1}\int_{\mathbb{R}^N}\frac{\beta}{|x|^2}\Phi_n^2\chi^2\widetilde{\Phi}_n^{2(t-1)}dx + \sigma_n^{-1}\int_{\mathbb{R}^N}\Phi_n^{2^*}\chi^2\widetilde{\Phi}_n^{2(t-1)}dx + A\sigma_n^{-1}\int_{\mathbb{R}^N}\chi^2\Phi_n\widetilde{\Phi}_n^{2(t-1)}dx + \int_{\mathbb{R}^N}\widetilde{\Phi}_n^{2(t-1)}|D\chi|^2\Phi_n^2 dx\right]. \tag{13}$$

By the Hardy and Sobolev embedding theorem, formula (13) yields that

$$\left(\int_{\mathbb{R}^N}(\Psi_M)^{2^*}dx\right)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N}|D\Psi_M|^2dx \le \sigma_n^{-1}C(\varepsilon)\int_{\mathbb{R}^N}\Phi_n^2\chi^2\widetilde{\Phi}_n^{2(t-1)}dx + \sigma_n^{-1}\int_{\mathbb{R}^N}\Phi_n^{2^*}\chi^2\widetilde{\Phi}_n^{2(t-1)}dx + A\sigma_n^{-1}\int_{\mathbb{R}^N}\chi^2\Phi_n\widetilde{\Phi}_n^{2(t-1)}dx + \int_{\mathbb{R}^N}\widetilde{\Phi}_n^{2(t-1)}|D\chi|^2\Phi_n^2dx. \tag{14}$$

By the Hölder inequality,

$$\int_{\mathbb{R}^N}\Phi_n^{2^*}\chi^2\widetilde{\Phi}_n^{2(t-1)}dx = \int_{\mathbb{R}^N}\Phi_n^{2^*-2}\chi^2\Phi_n^2\widetilde{\Phi}_n^{2(t-1)}dx \le \left[\int_{B(z_n,R)}\Phi_n^{2^*}dx\right]^{\frac{2^*-2}{2^*}}\left[\int_{\mathbb{R}^N}(\Psi_M)^{2^*}dx\right]^{\frac{2}{2^*}}.$$

Since $\widetilde{\Phi}_n \le \Phi_n$, we have $\int_{\mathbb{R}^N}\widetilde{\Phi}_n^{2(t-1)}|D\chi|^2\Phi_n^2dx \le \int_{\mathbb{R}^N}\Phi_n^{2^*}|D\chi|^2dx$,

$$\int_{\mathbb{R}^N}\chi^2\Phi_n^2\widetilde{\Phi}_n^{2(t-1)}dx \le \int_{\mathbb{R}^N}\chi^2\Phi_n^{2^*}dx \le \int_{B(z_n,R)}\Phi_n^{2^*}dx,$$

and

$$\int_{\mathbb{R}^N}\chi^2\Phi_n\widetilde{\Phi}_n^{2(t-1)}dx \le \int_{\mathbb{R}^N}\chi^2\Phi_n^{2^*-1}dx \le \int_{B(z_n,R)}\Phi_n^{2^*-1}dx.$$

So

$$\left(\int_{\mathbb{R}^N}(\Psi_M)^{2^*}dx\right)^{\frac{2}{2^*}} \le C\sigma_n^{-1}\left[\int_{B(z_n,R)}\Phi_n^{2^*}dx\right]^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}(\Psi_M)^{2^*}dx\right)^{\frac{2}{2^*}} + C\sigma_n^{-1}\int_{\mathbb{R}^N}\chi^2\Phi_n^{2^*}dx + C\sigma_n^{-1}\int_{\mathbb{R}^N}\chi^2\Phi_n^{2^*-1}dx + C\int_{\mathbb{R}^N}\Phi_n^{2^*}|D\chi|^2dx.$$

Since $B\left(y,\sigma_n^{-1/2}\right) \subset \mathcal{A}_n^1$, and the region $\mathcal{A}_n^1$ contains no concentration point of Φ_n, we can conclude that

$$\sigma_n^{-1}\left[\int_{B(z_n,R)}\Phi_n^{2^*}dx\right]^{\frac{2}{N}} \le \left[\int_{B\left(y,\sigma_n^{-1/2}\right)}|w_n|^{2^*}dx\right]^{\frac{2}{N}} \to 0$$

as $n \to +\infty$. This implies that

$$\left(\int_{\mathbb{R}^N} (\Psi_M)^{2^*} dx\right)^{\frac{2}{2^*}} \leq C \int_{\mathbb{R}^N} \chi^2 \Phi_n^{2^*} dx + C \int_{\mathbb{R}^N} \chi^2 \Phi_n^{2^*-1} dx + C \int_{\mathbb{R}^N} \Phi_n^{2^*} |D\chi|^2.$$

By taking the limit $M \to \infty$, we deduce that

$$\left(\int_{B(z_n,r)} \Phi_n^{2^*\beta} dx\right)^{\frac{2}{2^*}} \leq C \int_{\mathbb{R}^N} \chi^2 \Phi_n^{2^*} dx + C \int_{\mathbb{R}^N} \chi^2 \Phi_n^{2^*-1} dx + C \int_{\mathbb{R}^N} \Phi_n^{2^*} |D\chi|^2 dx. \tag{15}$$

Combining (15) with Young's inequality, we obtain that

$$\left(\int_{B(z_n,r)} \Phi_n^{2^* t} dx\right)^{\frac{1}{t2^*}} \leq C \left[\frac{1}{(R-r)^{\frac{1}{t}}} + 1\right] \left[\int_{B(z_n,R)} \Phi_n^{2^*} dx\right]^{\frac{1}{2^*}} + C < \infty. \tag{16}$$

Applying interpolation and Young's inequalities yields

$$\begin{aligned} &\left(\int_{B(z_n,r)} \Phi_n^{2^* t} dx\right)^{\frac{1}{2^* t}} \\ &\leq C \left[\frac{1}{(R-r)^{\frac{1}{t}}} + 1\right] \|\Phi_n\|^k_{L^1(B(z_n,R))} \times \|\Phi_n\|^{1-k}_{L^{2^* t}(B(z_n,R))} + C \\ &\leq \frac{1}{2} \|\Phi_n\|_{L^{2^* t}(B(z_n,R))} + C \left[\frac{1}{(R-r)^{\frac{1}{t}}} + 1\right] \|\Phi_n\|_{L^1(B(z_n,R))} + C, \end{aligned} \tag{17}$$

where the constant k is defined by the relation

$$\frac{1}{2^*} = k + \frac{1-k}{t2^*}.$$

To apply an iteration argument, we define a sequence $\{r_i\}_{i\in\mathbb{N}}$ by

$$r_i := R - \frac{R-r}{i+1}, \text{ for } i \in \mathbb{N}.$$

Using this construction, we obtain

$$\begin{aligned} &\left(\int_{B(z_n,r_0)} \Phi_n^{2^* t} dx\right)^{\frac{1}{2^* t}} \leq \frac{1}{2^i} \|\Phi_n\|_{L^{2^* t}(B(z_n,r_i))} \\ &+ C \sum_{j=1}^{i} \frac{1}{2^{j-1}} \left[\frac{1}{(r_j - r_{j-1})^{\frac{1}{t}}} + 1\right] \|\Phi_n\|_{L^1(B(z_n,r_j))} + C \sum_{j=1}^{i} \frac{1}{2^{j-1}}. \end{aligned} \tag{18}$$

A direct computation shows that

$$\sum_{i=1}^{+\infty} \frac{1}{2^{i-1}} \left[\frac{1}{(r_i - r_{i-1})^{\frac{1}{t}}} + 1\right] < \infty \text{ and } \sum_{j=1}^{+\infty} \frac{1}{2^{j-1}} < \infty.$$

Passing to the limit $i \to \infty$ in (18), it follows that

$$\left(\int_{B(z_n,r)} \Phi_n^{2^* t} dx\right)^{\frac{1}{2^* t}} \leq C \|\Phi_n\|_{L^1(B(z_n,1))} + C. \tag{19}$$

Letting $r \to \frac{1}{2}$ in (19), we infer that

$$\|\Phi_n\|_{L^{2t^2}\left(B\left(z_n,\frac{1}{2}\right)\right)} \leq C \|\Phi_n\|_{L^1(B(z_n,1))} + C. \tag{20}$$

Moreover, from $B\left(y,\sigma_n^{-1/2}\right)\subset\mathcal{A}_n^1$ and Lemma 2.4, it follows that

$$\int_{B(z_n,1)}\Phi_n dx \quad=\sigma_n^{\frac{N}{2}}\int_{B\left(y,\sigma_n^{-1/2}\right)}w_n dx\le C\sigma_n^{\frac{N}{2p_1}}.$$

This, together with (20) and the definition of Φ_n, leads to the desired conclusion.

Building on the previous lemma, we derive the following estimates, which are key to the proof of Proposition 2.8.

Lemma 2.6: *Consider a controlled sequence* $\left(w_{1,n},w_{2,n}\right)_{n\in\mathbb{N}}$. *For every* $s\le 2^*$, *there exists a constant* $C>0$ *such that the following holds for all* n

(i) $\int_{\mathcal{A}_n^2}|w_n|^s dx\le C\sigma_n^{-\frac{N}{2}+\frac{Ns}{2p_1}}$.

(ii) $\int_{\mathcal{A}_n^3}|Dw_n|^2 dx\le C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}$.

Proof:

(i) Applying Hölder's inequality together with Proposition 2.5, we deduce that for any $s\le 2^*$,

$$\int_{\mathcal{A}_n^2}|w_n|^s dx\quad\le C\left(\int_{\mathcal{A}_n^2}|w_n|^{2t^2}\right)^{\frac{s}{2t^2}}\sigma_n^{-\frac{N}{2}\left(1-\frac{s}{2t^2}\right)}\le C\sigma_n^{-\frac{N}{2}+\frac{Ns}{2p_1}}.$$

(ii) For each $n\in\mathbb{N}$, let $\phi_n\in C_0^2(\mathcal{A}_n^2)$ be a function satisfying $\phi_n=1$ on $\mathcal{A}_n^3$, $0\le\phi_n\le 1$, and $|D\phi_n|\le C\sigma_n^{1/2}$. We have

$$\int_\Omega Dw_n D(\phi_n^2 w_n)\le C\int_\Omega\left(\frac{\beta}{|x|^2}|w_n|+|w_n|^{2^*-1}+A\right)\phi_n^2|w_n|. \tag{21}$$

Alternatively, statement (i) implies that

$$\begin{aligned}\int_{A_n^3}\frac{\phi_n^2|w_n|^2}{|x|^2}dx\quad&\le\left(\int_{A_n^3}\phi_n^2|w_n|^{2^*}dx\right)^{\frac{2}{2^*}}\left(\int_{A_n^3}\frac{\phi_n^2}{|x|^{\frac{22^*}{2^*-2}}}dx\right)^{1-\frac{2}{2^*}}\\&\le C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}},\end{aligned} \tag{22}$$

Sinse $p_1>2^*$, we see $\frac{-N}{2}+\frac{2^*N}{2p_1}<\frac{2-N}{2}+\frac{N}{p_1}$ and $\frac{-N}{2}+\frac{N}{2p_1}<\frac{2-N}{2}+\frac{N}{p_1}$. Thus, from (i) and (21), we obtain that

$$\int_{A_n^3}|Dw_n|^2\le C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}+C\sigma_n^{\frac{-N}{2}+\frac{2^*N}{2p_1}}+C\sigma_n^{\frac{-N}{2}+\frac{N}{2p_1}}\le C\sigma_n^{\frac{2-N}{2}+\frac{N}{p_1}}.$$

This completes the proof of Lemma 2.6.

Lemma 2.7: *Let* B *be any bounded domain in* $\mathbb{R}^N$, *and let* $x_0\in\mathbb{R}^N$. *Then, the following local Pohozaev identity holds on* B *for any solution* (w_1,w_2) *of problem (4):*

$$
\begin{aligned}
&2N\left(\frac{1}{2^*-2\varepsilon}-\frac{1}{2^*}\right)\int_B |w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}dx\\
&+\beta\int_B \frac{(\varepsilon+x\cdot x_0)(|w_1|^2+|w_2|^2)}{(|x|^2+\varepsilon)^2}dx\\
&+2N\left(\frac{1}{\alpha_2+\beta_2}-\frac{1}{2^*}\right)\int_B |w_1|^{\alpha_2}|w_2|^{\beta_2}dx
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{2}{\alpha_1+\beta_1-2\varepsilon}\int_{\partial B} |w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}(x-x_0)\cdot\nu d\sigma\\
&+\frac{2}{\alpha_2+\beta_2}\int_{\partial B} |w_1|^{\alpha_2}|w_2|^{\beta_2}(x-x_0)\cdot\nu d\sigma\\
&+\int_{\partial B}\Big(\big(Dw_1\cdot(x-x_0)\big)(Dw_1\cdot\nu)+\big(Dw_2\cdot(x-x_0)\big)(Dw_2\cdot\nu)\Big)d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\beta}{2}\int_{\partial B}\frac{|w_1|^2+|w_2|^2}{|x|^2+\varepsilon}(x-x_0)\cdot\nu d\sigma\\
&-\frac{1}{2}\int_{\partial B}(|Dw_1|^2+|Dw_2|^2)(x-x_0)\cdot\nu d\sigma\\
&+\frac{N}{2^*}\int_{\partial B}((Dw_1\cdot\nu)w_1+(Dw_2\cdot\nu)w_2)d\sigma
\end{aligned}
$$

where ν is the outward normal to ∂B.

Proof: Set

$$
\begin{aligned}
f(x,w_1,w_2) &= \beta\frac{w_1}{|x|^2+\varepsilon}+\frac{2(\alpha_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon}|w_1|^{\alpha_1-2-\varepsilon}w_1|w_2|^{\beta_1-\varepsilon}\\
&\quad+\frac{2\alpha_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2-2}w_1|w_2|^{\beta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
g(x,w_1,w_2) &= \beta\frac{w_2}{|x|^2+\varepsilon}+\frac{2(\beta_1-\varepsilon)}{\alpha_1+\beta_1-2\varepsilon}|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-2-\varepsilon}w_2\\
&\quad+\frac{2\beta_2}{\alpha_2+\beta_2}|w_1|^{\alpha_2}|w_2|^{\beta_2-2}w_2.
\end{aligned}
$$

Then from (4), we have

$$
\begin{aligned}
\int_B (|Du|^2+|Dv|^2)dx &= \int_B (f(x,w_1,w_2)w_1+g(x,w_1,w_2)w_2)dx\\
&\quad+\int_{\partial B}((Dw_1.\nu)w_1+(Dw_2.\nu)w_2)d\sigma.
\end{aligned}
\tag{23}
$$

By applying the divergence theorem, we obtain, for any $x_0=(x_{01},x_{02},\dots,x_{0N})\in\mathbb{R}^N$,

$$
\begin{aligned}
&-\int_B Dw_1\big(Dw_1\cdot(x-x_0)\big)dx=-\int_{\partial B}(Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)d\sigma\\
&\quad+\int_B Dw_1\cdot D\big(Dw_1\cdot(x-x_0)\big)dx\\
&=-\int_{\partial B}(Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)d\sigma\\
&\quad+\sum_{i,j=1}^N\int_B\frac{\partial w_1}{\partial x_j}\cdot\frac{\partial}{\partial x_j}\left((x_i-x_{0i})\frac{\partial w_1}{\partial x_i}\right)dx\\
&=-\int_{\partial B}(Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{N}\int_B \frac{\partial w_1}{\partial x_j}\cdot\left(\delta_{ij}\frac{\partial w_1}{\partial x_i}+(x_i-x_{0i})\frac{\partial^2 w_1}{\partial x_i\,\partial x_j}\right)dx\\
&=-\int_{\partial B}(Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)d\sigma+\int_B|Dw_1|^2dx\\
&+\frac{1}{2}\sum_{i=1}^{N}\int_B(x_i-x_{0i})\frac{\partial}{\partial x_i}(|Dw_1|^2)dx\\
&=-\int_{\partial B}(Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)d\sigma+\frac{1}{2}\int_{\partial B}|Dw_1|^2\big(\nu\cdot(x-x_0)\big)d\sigma\\
&-\frac{N-2}{2}\int_B|Dw_1|^2dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
-\quad&\int_B\Delta w_1\big(Dw_1\cdot(x-x_0)\big)dx-\int_B\Delta w_2\big(Dw_2\cdot(x-x_0)\big)dx\\
=\quad&-\int_{\partial B}\Big((Dw_1.\nu)\big(Dw_1\cdot(x-x_0)\big)+(Dw_2.\nu)\big(Dw_2\cdot(x-x_0)\big)\Big)d\sigma\\
&+\frac{1}{2}\int_{\partial B}(|Dw_1|^2+|Dw_2|^2)\big(\nu\cdot(x-x_0)\big)d\sigma\\
&-\frac{N-2}{2}\int_B(|Dw_1|^2+|Dw_2|^2)dx.
\end{aligned}
\tag{24}
$$

Alternatively,

$$
\begin{aligned}
&-\int_B\Delta w_1\big(Dw_1\cdot(x-x_0)\big)dx-\int_B\Delta w_2\big(Dw_2\cdot(x-x_0)\big)dx\\
&=\int_B f(x,w_1,w_2)\big(Dw_1\cdot(x-x_0)\big)dx\\
&+\int_B g(x,w_1,w_2)\big(Dw_2\cdot(x-x_0)\big)dx\\
&=\frac{2}{\alpha_1+\beta_1-2\varepsilon}\int_B D\big(|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}\big)\cdot(x-x_0)dx\\
&+\frac{2}{\alpha_2+\beta_2}\int_B D\big(|w_1|^{\alpha_2}|w_2|^{\beta_2}\big)\cdot(x-x_0)dx\\
&+\frac{\beta}{2}\int_B\frac{(x-x_0)\cdot D\left(|w_1|^2+|w_2|^2\right)}{|x|^2+\varepsilon}dx\\
&=\frac{2}{\alpha_1+\beta_1-2\varepsilon}\int_{\partial B}|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}\big(\nu.(x-x_0)\big)d\sigma\\
&+\frac{2}{\alpha_2+\beta_2}\int_{\partial B}|w_1|^{\alpha_2}|w_2|^{\beta_2}\big(\nu.(x-x_0)\big)d\sigma\\
&+\frac{\beta}{2}\int_{\partial B}\frac{(|w_1|^2+|w_2|^2)(w_2\cdot(x-x_0))}{|x|^2+\varepsilon}d\sigma\\
&-\frac{2N}{\alpha_1+\beta_1-2\varepsilon}\int_B|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}dx-\frac{2N}{\alpha_2+\beta_2}\int_B|w_1|^{\alpha_2}|w_2|^{\beta_2}dx\\
&-\frac{\beta}{2}\int_B(|w_1|^2+|w_2|^2)\left(\frac{N-2}{|x|^2+\varepsilon}+\frac{2(\varepsilon+x\cdot x_0)}{(|x|^2+\varepsilon)^2}\right)dx.
\end{aligned}
\tag{25}
$$

It follows from (23)-(25) that the proof of Lemma 2.7 is concluded.

Proposition 2.8: Assume that $\left(w_{1,n},w_{2,n}\right)_{n\in\mathbb{N}}$ is a balanced sequence with $\left\|(w_{1,n},w_{2,n})\right\|\le C$ for some constant $C>0$ independent of n. Then $\left(w_{1,n},w_{2,n}\right)$ converges strongly in $H^1(\Omega)$.

Proof: From Lemma 2.6, It is possible to choose $t_n\in[\bar{C}+2,\bar{C}+3]$ such that

$$\begin{aligned}\int_{\Omega\cap\partial B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n)} & \left(|w_{1,n}|^{2^*}+|w_{2,n}|^{2^*}\right)dS \\ &\le C\sigma_n^{\frac{1}{2}}\int_{\mathcal{A}_n^2}\left(|w_{1,n}|+|w_{2,n}|\right)^{2^*}dx \\ &\le C\sigma_n^{\frac{1}{2}-\frac{N}{2}+\frac{N2^*}{2p_1}}.\end{aligned} \tag{26}$$

and

$$\begin{aligned}\int_{\Omega\cap\partial B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n)} & \left(|Dw_{1,n}|^2+|Dw_{2,n}|^2+\frac{w_{1,n}^2+w_{2,n}^2}{|x|^2}\right)dS \\ &\le C\sigma_n^{\frac{1}{2}}\int_{\mathcal{A}_n^3}\left(|Dw_{1,n}|^2+|Dw_{2,n}|^2+\frac{w_{1,n}^2+w_{2,n}^2}{|x|^2}\right)dx \\ &\le C\sigma_n^{\frac{1}{2}-\frac{N}{2^*}+\frac{N}{p_1}}.\end{aligned} \tag{27}$$

According to Lemma 2.7, the balanced concentrating sequence $\{(w_{1,n},w_{2,n})\}$ corresponding to problem (4) with $\varepsilon=\varepsilon_n\to 0$ satisfies the local Pohozaev identity on $D_n=B_{t_n\sigma_n^{-1/2}}(x_n)\cap\Omega$.

$$\begin{aligned}2N & \left(\frac{1}{2^*-2\varepsilon_n}-\frac{1}{2^*}\right)\int_{D_n}|w_{1,n}|^{\alpha_1-\varepsilon_n}|w_{2,n}|^{\beta_1-\varepsilon_n}dx \\ &+\beta\int_{D_n}\frac{(\varepsilon_n+x\cdot x_0)(|w_{1,n}|^2+|w_{2,n}|^2)}{(|x|^2+\varepsilon_n)^2}dx \\ &+2N\left(\frac{1}{\alpha_2+\beta_2}-\frac{1}{2^*}\right)\int_{D_n}|w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2}dx \\ &=\frac{2}{\alpha_1+\beta_1-2\varepsilon_n}\int_{\partial D_n}|w_{1,n}|^{\alpha_1-\varepsilon_n}|w_{2,n}|^{\beta_1-\varepsilon_n}(x-x_0)\cdot\nu d\sigma \\ &+\frac{2}{\alpha_2+\beta_2}\int_{\partial D_n}|w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2}(x-x_0)\cdot\nu d\sigma \\ &+\int_{\partial D_n}\Big(\big(Dw_{1,n}\cdot(x-x_0)\big)\big(Dw_{1,n}\cdot\nu\big) \\ &+\big(Dw_{2,n}\cdot(x-x_0)\big)\big(Dw_{2,n}\cdot\nu\big)\Big)d\sigma \\ &+\frac{\beta}{2}\int_{\partial D_n}\frac{|w_{1,n}|^2+|w_{2,n}|^2}{|x|^2+\varepsilon_n}(\nu\cdot x)d\sigma \\ &-\frac{1}{2}\int_{\partial D_n}\left(|Dw_{1,n}|^2+|Dw_{2,n}|^2\right)(x-x_0)\cdot\nu d\sigma \\ &+\frac{N}{2^*}\int_{\partial D_n}\Big(\big(Dw_{1,n}\cdot\nu\big)w_{1,n}+\big(Dw_{2,n}\cdot\nu\big)w_{2,n}\Big)d\sigma,\end{aligned} \tag{28}$$

where ν is the unit outward normal vector on ∂D_n. Set $\partial D_n=\partial_i D_n\cup\ \partial_e D_n$, where $\partial_i D_n=\partial B_n\cap\ \Omega$, $\partial_e D_n=\partial\Omega\cap\ \bar{B}_n$, $B_n=B_{t_n\sigma_n^{-\frac{1}{2}}}(x_n)$.

If $\lim_{n\to\infty}dist(x_n,\partial\Omega)>0$. Then $\partial_e D_n=\emptyset, D_n=B_n\subset\Omega$ and $\partial D_n=\partial_i D_n=\partial B_n$ for large n. We take x_0 in (28) equal to $\delta(x_n)$, where $\delta(x_n)=x_n$, if the concentration point satisfies $x_n\nrightarrow 0$ as $n\to\infty$, and $\delta(x_n)=0$ otherwise. Therefore, combining (26)-(28), we conclude that

$$2N \quad \left(\frac{1}{\alpha_2+\beta_2}-\frac{1}{2^*}\right)\int_{B_n}|w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2}dx + \beta\int_{B_n}\frac{x\cdot\delta(x_n)(|w_{1,n}|^2+|w_{2,n}|^2)}{(|x|^2+\varepsilon_n)^2}dx \leq \tilde{C}\sigma_n^{-\frac{N}{2^*}+\frac{N}{p_1}} \text{for} n \text{large enough.} \tag{29}$$

Let $w_{1,n}=w_{1,n}^0+w_{1,n}^1+w_{1,n}^2, w_{2,n}=w_{2,n}^0+w_{2,n}^1+w_{2,n}^2$, where

$w_{1,n}^1=w_{1,\infty}, w_{1,n}^0=w_{1,n}-w_{1,n}^1-w_{1,n}^2,$

$w_{2,n}^1=w_{2,\infty}, w_{2,n}^0=w_{2,n}-w_{2,n}^1-w_{2,n}^2,$

$w_{1,n}^2=\sum_{i=1}^{k_1+k_2}(\sigma_n^i)^{\frac{N-2}{2}}W_{1,i}\left(\sigma_n^i(x-x_n^i)\right)+\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3}(\sigma_n^i)^{\frac{N-2}{2}}W_{1,i}(\sigma_n^i x),$

$w_{2,n}^2=\sum_{i=1}^{k_1+k_2}(\sigma_n^i)^{\frac{N-2}{2}}W_{2,i}\left(\sigma_n^i(x-x_n^i)\right)+\sum_{i=k_1+k_2+1}^{k_1+k_2+k_3}(\sigma_n^i)^{\frac{N-2}{2}}W_{2,i}(\sigma_n^i x).$

It is a classical result that the following inequality holds:

$$|a+b|^p|c+d|^q \geq \frac{1}{4}|a|^p|c|^q - C(|a|^p|d|^q+|b|^p|c|^q+|b|^p|d|^q)\ \forall a,b,c,d\in\mathbb{R}$$

with $p,q>0$ and some constant $C>0$. Thus, for large n, we have

$$\begin{aligned}&\int_{B_n}|w_{1,n}|^p|w_{2,n}|^q dx=\int_{B_n}|w_{1,n}^2+(w_{1,n}^1+w_{1,n}^0)|^p|w_{2,n}^2\\&+(w_{2,n}^1+w_{2,n}^0)|^q dx\\&\geq\frac{1}{4}\int_{B_n}|w_{1,n}^2|^p|w_{2,n}^2|^q dx-C\int_{B_n}\left[|w_{1,n}^2|^p|w_{2,n}^0|^q+|w_{1,n}^2|^p|w_{2,n}^1|^q\right.\\&+|w_{1,n}^0|^p|w_{2,n}^2|^q+|w_{1,n}^0|^p|w_{2,n}^0|^q+|w_{1,n}^0|^p|w_{2,n}^1|^q\\&\left.+|w_{1,n}^1|^p|w_{2,n}^2|^q+|w_{1,n}^1|^p|w_{2,n}^0|^q+|w_{1,n}^1|^p|w_{2,n}^1|^q\right]dx.\end{aligned} \tag{30}$$

Next, we provide an estimate for each term appearing on the right-hand side of (30). Therefore, after a direct calculation, one has

$$\int_{B_n}|w_{1,n}^2|^{\alpha_2}|w_{2,n}^2|^{\beta_2}dx\geq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N}. \tag{31}$$

Alternatively, let $B_n'=B_{\sigma_n^{-1}}(x_n)\cap\Omega$, we have

$$\int_{B_n'}|w_{1,n}^0|^{\alpha_2}|w_{2,n}^0|^{\beta_2}\leq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N},$$

$$\begin{aligned}\int_{B_n'}|w_{1,n}^0|^{\alpha_2}|w_{2,n}^2|^{\beta_2}&\leq\left(\int_{B_n}|w_{2,n}^2|^{2^*}\right)^{\frac{\beta_2}{2^*}}|B_n'|^{1-\frac{\beta_2}{2^*}}\\&=C\sigma_n^{\frac{\beta_2 N}{2^*}-N}\leq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N},\end{aligned}$$

$$\begin{aligned}\int_{B_n'}|w_{1,n}^0|^{\alpha_2}|w_{2,n}^1|^{\beta_2}&\leq\left(\int_{B_n'}|w_{2,n}^1|^{2^*}\right)^{\frac{\beta_2}{2^*}}|B_n'|^{1-\frac{\beta_2}{2^*}}\\&=C\sigma_n^{\frac{\beta_2 N}{2^*}-N}\leq\sigma_n^{\frac{N-2}{2}(\alpha_2+\beta_2)-N},\end{aligned}$$

$$\begin{aligned}\int_{B_n'}|w_{1,n}^2|^{\alpha_2}|w_{2,n}^1|^{\beta_2}&\leq\left(\int_{B_n'}|w_{1,n}^2|^{2^*}\right)^{\frac{\alpha_2}{2^*}}\left(\int_{B_n'}|w_{2,n}^1|^{2^*}\right)^{\frac{\beta_2}{2^*}}|B_n'|^{1-\frac{\alpha_2+\beta_2}{2^*}}\\&\leq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N},\end{aligned}$$

and similarly,

$$\int_{B_n'} \left[|w_{1,n}^2|^{\alpha_2}|w_{2,n}^0|^{\beta_2} + |w_{1,n}^1|^{\alpha_2}|w_{2,n}^2|^{\beta_2} + |w_{1,n}^1|^{\alpha_2}|w_{2,n}^0|^{\beta_2} + |w_{1,n}^1|^{\alpha_2}|w_{2,n}^1|^{\beta_2}\right] \leq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N}.$$

By substituting the above estimates into (30), we deduce the existence of a constant $c' > 0$ such that

$$\int_{B_n} |w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2} \geq c'\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N}. \tag{32}$$

In addition,

$$\left|\int_{B_n} \frac{x\cdot\delta(x_n)(|w_{1,n}|^2+|w_{2,n}|^2)}{(|x|^2+\varepsilon_n)^2}dx\right| \leq C\sigma_n^{-\frac{N}{2}+\frac{N}{p_1}}. \tag{33}$$

Combining (29), (32) and (33), It has been shown that

$$\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N} \leq C\sigma_n^{-\frac{N}{2^*}+\frac{N}{p_1}}, \tag{34}$$

where $p_1 > 2^*$ is an arbitrary constant such that $p_1 < \frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}-\sqrt{\bar{\beta}-\beta}}$.

Choose $p_1 = \frac{2N}{(N-2)(\alpha_2+\beta_2+1)-2N} + \delta$, with $p_1 < \frac{2^*\sqrt{\bar{\beta}}}{\sqrt{\bar{\beta}}-\sqrt{\bar{\beta}-\beta}}$, where $\delta > 0$ is a small constant. This choice is possible provided that $\beta < \bar{\beta} - 4$. With this value of p_1, we have $\frac{(2-N)(\alpha_2+\beta_2)}{2} + N < \frac{N}{2^*} - \frac{N}{p_1}$. So, we obtain a contradiction from (34).

If $\lim_{n\to\infty} dist(x_n, \partial\Omega) = 0$, then $\partial_e D_n \neq \emptyset$ for sufficiently large n. We choose a point x_0 outside Ω such that

$$d(x_0, x_n) \leq 2t_n\sigma_n^{-\frac{1}{2}} \text{ and } \forall x \in \partial_e B_n, \nu\cdot(x - x_0) < 0. \tag{35}$$

Recall that $\frac{\partial w_{1,n}}{\partial \nu} = \frac{\partial w_{2,n}}{\partial \nu} = 0$ on $\partial_e D_n = \partial\Omega \cap \overline{B_n}$. From (26)-(28) and (35), It can be concluded that

$$\begin{aligned} &2N\left(\frac{1}{\alpha_2+\beta_2} - \frac{1}{2^*}\right)\int_{D_n} |w_{1,n}|^{\alpha_2}|w_{2,n}|^{\beta_2}dx \\ &+\beta\int_{D_n} \frac{x\cdot x_0(|w_{1,n}|^2+|w_{2,n}|^2)}{(|x|^2+\varepsilon_n)^2}dx \\ &\leq \bar{C}\sigma_n^{-\frac{N}{2^*}+\frac{N}{p_1}} \text{for} n \text{large} \text{enough}. \end{aligned} \tag{36}$$

In addition,

$$\int_{D_n} |w_{1,n}^2|^{\alpha_2}|w_{2,n}^2|^{\beta_2}dx \geq C\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N}. \tag{37}$$

Noting that $0 \notin \partial\Omega$, it follows for sufficiently large n that

$$\begin{aligned} \left|\int_{D_n} \frac{x\cdot x_0(|w_{1,n}|^2+|w_{2,n}|^2)}{(|x|^2+\varepsilon_n)^2}dx\right| &\leq C\int_{D_n}\left(|w_{1,n}|^2 + |w_{2,n}|^2\right)dx \\ &\leq C\sigma_n^{-\frac{N}{2}+\frac{N}{p_1}}. \end{aligned} \tag{38}$$

Combining (36), (37) and (38), It has been proven that

$$\sigma_n^{\frac{(N-2)(\alpha_2+\beta_2)}{2}-N} \leq C\sigma_n^{-\frac{N}{2^*}+\frac{N}{p_1}}. \tag{39}$$

So, we obtain a contradiction from (39) as in (34). From the above arguments, we complete the proof of Proposition 2.8.

3. The proof of Theorem 1.1.

In this section, we begin by introducing some notations (see [15, 18, 19]). Let $\{\lambda_i\}_{i=1}^{\infty}$ denote the eigenvalues of the operator $-\Delta + I$ in $H^1(\Omega)$ with homogeneous Neumann boundary conditions, ordered such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and let $\{e_i(x)\}_{i=1}^{\infty}$ be the corresponding eigenfunctions. Then, $\{e_i\}_{i=1}^{\infty}$ forms an orthogonal basis of $H^1(\Omega)$.

It is well known that nontrivial solutions of problem (4) correspond to the nonzero critical points of the following energy functional defined on $H^1(\Omega) \times H^1(\Omega)$:

$$\begin{aligned} I^{\varepsilon}(w_1, w_2) \quad &= \frac{1}{2}\int_{\Omega}\left(|Dw_1|^2 + |Dw_2|^2 - \beta\frac{|w_1|^2+|w_2|^2}{|x|^2+\varepsilon}\right)dx \\ &- \frac{2}{\alpha_1+\beta_1-2\varepsilon}\int_{\Omega}|w_1|^{\alpha_1-\varepsilon}|w_2|^{\beta_1-\varepsilon}dx \\ &- \frac{2}{\alpha_2+\beta_2}\int_{\Omega}|w_1|^{\alpha_2}|w_2|^{\beta_2}dx. \end{aligned}$$

Set $\tilde{Y}_k := \oplus_{j=1}^{k} e_j, \tilde{Z}_k := \overline{\oplus_{j=k}^{\infty} e_j}$; $Y_k := \tilde{Y}_k \times \tilde{Y}_k, Z_k := \tilde{Z}_k \times \tilde{Z}_k$,

$$B_k := \{(w_1, w_2) \in Y_k \,/\, \mathrm{P}(w_1, w_2)\mathrm{P} \leq \rho_k\},$$
$$N_k := \{(w_1, w_2) \in Z_k \,/\, \mathrm{P}(w_1, w_2)\mathrm{P} = r_k\}.$$

where $\rho_k > r_k > 0$.

Define

$$\Gamma_k := \left\{\gamma \in C(B_k, H^1(\Omega) \times H^1(\Omega))/\gamma|_{\partial B_k} = id\right\},$$
$$c_k := \inf_{\gamma\in\Gamma_k}\max_{(w_1,w_2)\in B_k} I(\gamma(w_1, w_2)), c_k^n := \inf_{\gamma\in\Gamma_k}\max_{(w_1,w_2)\in B_k} I^{\varepsilon_n}(\gamma(w_1, w_2)),$$
$$b_k^n := \inf_{(w_1,w_2)\in N_k} I^{\varepsilon_n}(w_1, w_2),$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. To simplify notation, we define $p_n := 2^* - 2\varepsilon_n \in (2, 2^*)$.

The proof of (i) of Theorem 1.1: Following the approach of Theorem 3.20 in [19], one can show that for each $k \geq k_0$, there exist radii $\rho_k > r_k > 0$ satisfying $\rho_k \to 0$ as $k \to +\infty$, and such that for all $n \in \mathbb{N}$,

$$a_k^n := \inf_{\substack{(w_1,w_2)\in Z_k \\ \|(w_1,w_2)\|=\rho_k}} I^{\varepsilon_n}(w_1, w_2) \geq 0, b_k^n := \max_{\substack{(w_1,w_2)\in Y_k \\ \|(w_1,w_2)\|=r_k}} I^{\varepsilon_n}(w_1, w_2) < 0,$$
$$b_k := \max_{\substack{(w_1,w_2)\in Y_k \\ \|(w_1,w_2)\|=r_k}} I(w_1, w_2) < 0, d_k^n := \inf_{\substack{(w_1,w_2)\in Z_k \\ \|(w_1,w_2)\|\leq\rho_k}} I^{\varepsilon_n}(w_1, w_2) \to 0$$

as $k \to +\infty$, and $(\varepsilon_n)_n$ is a decreasing sequence such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Therefore, by Theorem 3.18 in [19] (the Dual Fountain Theorem), the functional I^{ε_n} admits a sequence of critical points, denoted by $\left(\widetilde{w}_{1,k}^n, \widetilde{w}_{2,k}^n\right)$, moreover

$$I^{\varepsilon_n}(\widetilde{w}_{1,k}^n, \widetilde{w}_{2,k}^n) = c_k^n \in [d_k^n, b_k^n].$$

Observing that c_k^n is negative, we can easily deduce that

$$\left(\frac{1}{2} - \frac{1}{2^*-2\varepsilon_n}\right) \left\|\left(\widetilde{w}_{1,k}^n, \widetilde{w}_{2,k}^n\right)\right\|^2 < C\left(\frac{1}{\alpha_2+\beta_2} - \frac{1}{2^*-2\varepsilon_n}\right) \int_\Omega |\widetilde{w}_{1,k}^n|^{\alpha_2} |\widetilde{w}_{2,k}^n|^{\beta_2} dx.$$

By applying Young's inequality and Sobolev's embedding theorem, we deduce that $(\widetilde{w}_{1,k}^n, \widetilde{w}_{2,k}^n)$ is bounded in $H^1(\Omega) \times H^1(\Omega)$. Using Proposition 2.8, we can extract a subsequence of $(\widetilde{w}_{1,k}^n, \widetilde{w}_{2,k}^n)$ that converges strongly to a solution $(\widetilde{w}_{1,k}, \widetilde{w}_{2,k})$ of problem (1) at the energy level

$$c_k := \lim_{n\to\infty} c_k^n.$$

We first claim that $c_k < 0$ for all $k \geq k_0$. Indeed, since ∂B_k is compact and the functionals $(I^{\varepsilon_n})_n$ are equicontinuous, it follows that $b_k^n \to b_k$ as $n \to \infty$. Consequently, we obtain

$$c_k \leq \lim_{n\to\infty} b_k^n = b_k < 0.$$

Next, we claim that $\lim_{k\to\infty} c_k = 0$. To see this, for each $k \geq k_0$, there exists an index $n_k > k$ such that

$$|c_k^{n_k} - c_k| < \frac{1}{k}.$$

Define

$$\alpha_k := \inf_{w\in\tilde{Z}_k, \|w\|_{p_{n_k}}=1} \int_\Omega (|Dw|^2 + |w|^2) dx, \tag{40}$$

where $p_{n_k} = 2^* - 2\varepsilon_{n_k} \in (2, 2^*)$. We aim to show that, up to a subsequence, $\alpha_k \to +\infty$ as $k \to \infty$. Since $p_{n_k} < 2^*$, the scalar α_k is attained by some function $w_k \in \tilde{Z}_k$, which satisfies

$$-\Delta w_k + w_k = \alpha_k |w_k|^{p_{n_k}-2} w_k \text{ in } \Omega, \text{and } \frac{\partial w_k}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

If $\alpha_k \nrightarrow \infty$ as $k \to \infty$, then there exists a constant $C > 0$ such that

$$\int_\Omega (|Dw_k|^2 + |w_k|^2)\, dx \leq C.$$

By Proposition 2.8, it follows that $(w_k)_k$ converges strongly in $H^1(\Omega)$. Since $w_k \in \tilde{Z}_k$, we may, up to a subsequence, assume that

$$w_k \to 0 \text{ in } H^1(\Omega).$$

Applying Hölder's inequality, we obtain

$$\lim_{k\to\infty} \int_\Omega |w_k|^{p_{n_k}}\, dx = 0,$$

which contradicts the normalization condition $\int_\Omega |w_k|^{p_{n_k}}\, dx = 1$. Therefore, we conclude that $\alpha_k \to \infty$ as $k \to \infty$. On the other hand, assume that $(w_1, w_2) \in Z_k$

with $\|(w_1, w_2)\| \leq \rho_k$, and k large enough. Then, using Young's inequality, we obtain for some $\varepsilon > 0$ with $\varepsilon < \frac{\beta}{\beta} - 1$

$$\begin{aligned} I^{\varepsilon_k}(w_1, w_2) &= \frac{1}{2}\int_\Omega \left(|Dw_1|^2 + |Dw_2|^2 - \beta \frac{|w_1|^2+|w_2|^2}{|x|^2+\varepsilon_k}\right) dx \\ &- \frac{2}{\alpha_1+\beta_1-2\varepsilon_k}\int_\Omega |w_1|^{\alpha_1-\varepsilon_k}|w_2|^{\beta_1-\varepsilon_k} dx \\ &- \frac{2}{\alpha_2+\beta_2}\int_\Omega |w_1|^{\alpha_2}|w_2|^{\beta_2} dx \\ &\geq C_1\|(w_1, w_2)\|^2 - C_2\alpha_k^{-\frac{p_{n_k}}{2}}\|(w_1, w_2)\|^{p_{n_k}} \\ &- C_3\|(w_1, w_2)\|^{\alpha_2+\beta_2} \\ &\geq h(\mathrm{P}(w_1, w_2)\mathrm{P}) - C_2\alpha_k^{-\frac{p_{n_k}}{2}}\mathrm{P}(w_1, w_2)\mathrm{P}^{p_{n_k}}, \end{aligned}$$

where $h(t) = C_1t^2 - C_3t^{\alpha_2+\beta_2}$. Since the function h is decreasing on the interval $[0, \rho_k]$ for sufficiently large k, it follows that

$$I^{\varepsilon_k}(w_{1,k}^{n_k}, w_{2,k}^{n_k}) \geq C_1\rho_k^2 - C_2\alpha_k^{-\frac{p_{n_k}}{2}}\rho_k^{p_{n_k}} - C_3\rho_k^{\alpha_2+\beta_2}.$$

Hence we get $c_k^{n_k} \geq d_k^{n_k} \geq C_1\rho_k^2 - C_2\alpha_k^{-\frac{p_{n_k}}{2}}\rho_k^{p_{n_k}} - C_3\rho_k^{\alpha_2+\beta_2}$. Since $\rho_k \to 0$ and $\alpha_k \to +\infty$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_k^{n_k} = 0.$$

This completes the proof of part (i) of Theorem 1.1.

The proof of (ii) of Theorem 1.1: By employing the techniques presented in Theorem 3.7 of [19], one can show that for each k, there exist positive numbers ρ_k and τ_k with $\rho_k > \tau_k$, such that ρ_k tends to infinity as $k \to +\infty$, and

$$\mathrm{a}_k^n := \max_{\substack{(w_1,w_2)\in Y_k \\ \|(w_1,w_2)\|=\rho_k}} I^{\varepsilon_n}(w_1, w_2) \leq 0, \mathrm{b}_k^n := \inf_{\substack{(w_1,w_2)\in Z_k \\ \|(w_1,w_2)\|=r_k}} I^{\varepsilon_n}(w_1, w_2) \to \infty$$

as $k \to +\infty$, where

$$\mathrm{B}_k := \{(w_1, w_2) \in Y_k : \|(w_1, w_2)\| \leq \rho_k\},$$
$$\mathrm{N}_k := \{(w_1, w_2) \in Z_k : \|(w_1, w_2)\| = \tau_k\},$$

and the sequence $(\varepsilon_n)_n$ is decreasing with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Therefore, by applying Theorem 3.6 in [19] (the Fountain theorem), we deduce that the functional I^{ε_n} admits a sequence of critical points $\{(w_{1,k}^n, w_{2,k}^n)\}_n$. Furthermore, the associated critical values $c_k^n = I^{\varepsilon_n}(w_{1,k}^n, w_{2,k}^n)$ satisfy

$$c_k^n := \inf_{\gamma\in\Gamma_k} \max_{(w_1,w_2)\in \mathrm{B}_k} I^{\varepsilon_n}(\gamma(w_1, w_2)),$$

and

$$\Gamma_k := \left\{\gamma \in \mathcal{C}(\mathrm{B}_k, H^1(\Omega) \times H^1(\Omega)) : \gamma_{|\partial \mathrm{B}_k} = id\right\}.$$

We assert that for every $k \in \mathbb{N}$,

$$\mathfrak{c}_k^n \to \mathfrak{c}_k := \inf_{\gamma\in\Gamma_k}\max_{(w_1,w_2)\in B_k} I\big(\gamma(w_1,w_2)\big) \text{ as } n\to+\infty.$$

For any $(w_1,w_2)\in H^1(\Omega)\times H^1(\Omega)$, $I(w_1,w_2)=I^{\varepsilon_n}(w_1,w_2)+G_n(w_1,w_2)$, where

$$\begin{aligned} G_n(w_1,w_2) &= \frac{\beta}{2}\int_\Omega\left(\frac{1}{|x|^2+\varepsilon_n}-\frac{1}{|x|^2}\right)(|w_1|^2+|w_2|^2)dx \\ &+\frac{2}{\alpha_1+\beta_1-2\varepsilon_n}\int_\Omega |w_1|^{\alpha_1-\varepsilon_n}|w_2|^{\beta_1-\varepsilon_n}dx \\ &-\frac{2}{\alpha_1+\beta_1}\int_\Omega |w_1|^{\alpha_1}|w_2|^{\beta_1}dx. \end{aligned}$$

Since B_k is compact and the family of functionals $\{\pm G_n\}$ is equicontinuous on B_k, it follows that

$$\lim_{n\to\infty}\sup_{(w_1,w_2)\in B_k}|G_n(w_1,w_2)|=0.$$

Moreover, since the identity map id belongs to Γ_k for every $k\in\mathbb{N}^*$, we conclude that

$$\begin{aligned} \mathfrak{c}_k &= \inf_{\gamma\in\Gamma_k}\sup_{(w_1,w_2)\in B_k} I(\gamma(w_1,w_2)) \\ &\leqslant \overline{\lim_{n\to\infty}}\inf_{\gamma\in\Gamma_k}\sup_{(w_1,w_2)\in B_k} I^{\varepsilon_n}(\gamma(w_1,w_2)) \\ &+\overline{\lim_{n\to\infty}}\sup_{(w_1,w_2)\in B_k}(G_n(w_1,w_2)) = \overline{\lim_{n\to\infty}}\mathfrak{c}_k^n. \end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned} \overline{\lim_{n\to\infty}}\mathfrak{c}_k^n &= \overline{\lim_{n\to\infty}}\inf_{\gamma\in\Gamma_k}\sup_{(w_1,w_2)\in B_k} I^{\varepsilon_n}(\gamma(w_1,w_2)) \\ &\leqslant \overline{\lim_{n\to\infty}}\inf_{\gamma\in\Gamma_k}\sup_{(w_1,w_2)\in B_k} I(\gamma(w_1,w_2)) \\ &+\overline{\lim_{n\to\infty}}\sup_{(w_1,w_2)\in B_k}(-G_n(w_1,w_2)) = \mathfrak{c}_k. \end{aligned} \tag{42}$$

Combining (41) with (42), we infer that

$$\lim_{n\to\infty}\mathfrak{c}_k^n=\mathfrak{c}_k. \tag{43}$$

We have that $I^{\varepsilon_n}(w_{1,k}^n,w_{2,k}^n)=\mathfrak{c}_k^n$ and $I^{\varepsilon_n\prime}(w_{1,k}^n,w_{2,k}^n)(w_{1,k}^n,w_{2,k}^n)=0$. From these relations, it follows that

$$\begin{aligned} \left(\frac{1}{2}-\frac{1}{p_n}\right)\left\|(w_{1,k}^n,w_{2,k}^n)\right\|^2 &-C\left(\frac{1}{\alpha_2+\beta_2}-\frac{1}{p_n}\right)\int_\Omega |w_{1,k}^n|^{\alpha_2}|w_{2,k}^n|^{\beta_2}dx \\ &<\mathfrak{c}_k^n. \end{aligned}$$

Since the sequence $(\mathfrak{c}_k^n)_n$ is bounded, it follows that for all n,

$$\begin{aligned} \left(\tfrac{1}{2}-\tfrac{1}{p_n}\right)\left\|(w_{1,k}^n,w_{2,k}^n)\right\|^2 &< C\left(\tfrac{1}{\alpha_2+\beta_2}-\tfrac{1}{p_n}\right)\int_\Omega |w_{1,k}^n|^{\alpha_2}|w_{2,k}^n|^{\beta_2}dx \\ &+C. \end{aligned}$$

By Young's inequality, Sobolev's embedding, and the assumption that $\alpha_2+\beta_2<2$, we deduce that the sequence $\left(w_{1,k}^n,w_{2,k}^n\right)_n$ is bounded in $H^1(\Omega)\times$

$H^1(\Omega)$. Applying Proposition 2.8, we can extract a subsequence, still denoted by $\left(w_{1,k}^n, w_{2,k}^n\right)_n$, such that

$$\left(w_{1,k}^n, w_{2,k}^n\right) \to \left(w_{1,k}, w_{2,k}\right) \text{ strongly in } H^1(\Omega) \times H^1(\Omega),$$

for some $(w_{1,k}, w_{2,k}) \in H^1(\Omega) \times H^1(\Omega)$, and satisfying

$$I(w_{1,k}, w_{2,k}) = \mathfrak{c}_k.$$

Therefore, $(w_{1,k}, w_{2,k})$ is a solution to problem (1). We now claim that

$$\lim_{k\to\infty} \mathfrak{c}_k = +\infty.$$

Using similar arguments as in the proof of (i), we observe that the sequence $\left(w_{1,k}^n, w_{2,k}^n\right)_n$ is bounded in $H^1(\Omega) \times H^1(\Omega)$. Therefore, we can extract a subsequence (still denoted by $\left(w_{1,k}^n, w_{2,k}^n\right)_n$ which converges strongly to a solution $(w_{1,k}, w_{2,k})$ of problem (1) at the level $\mathfrak{c}_k$. It then follows that for every $k \in \mathbb{N}$, there exists $n_k > k$ such that

$$|\mathfrak{c}_k^{n_k} - \mathfrak{c}_k| < \frac{1}{k}.$$

Applying Young's inequality, we obtain

$$I^{\varepsilon_n}(w_1, w_2) \geq C_1 \|(w_1, w_2)\|^2 - C_2 \alpha_k^{-\frac{p_{n_k}}{2}} \|(w_1, w_2)\|^{p_{n_k}} - C_3.$$

where α_k is defined in (40). Choosing

$$\tau_k = \left(\frac{2C_1 \alpha_k^{\frac{p_{n_k}}{2}}}{C_2 p_{n_k}} \right)^{\frac{1}{p_{n_k}-2}},$$

we find that if $(w_1, w_2) \in Z_k$ with $\|(w_1, w_2)\| = \tau_k$, then

$$I^{\varepsilon_n}(w_1, w_2) \geq C_1 \left(1 - \frac{2}{p_{n_k}}\right) \left(\frac{2C_1 \alpha_k^{\frac{p_{n_k}}{2}}}{C_2 p_{n_k}} \right)^{\frac{2}{p_{n_k}-2}} - C_3.$$

Since $\alpha_k \to \infty$ as $k \to \infty$, it follows that $b_k^{n_k} \to \infty$ as $k \to \infty$. By Theorem 3.5 in [19], we know that $\mathfrak{c}_k^{n_k} \geq b_k^{n_k}$, and thus

$$\lim_{k\to\infty} \mathfrak{c}_k = \lim_{k\to\infty} \mathfrak{c}_k^{n_k} = +\infty.$$

This completes the proof of part (ii).

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Conflict of interest

I have nothing to declare.

Data availability statements

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