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Green's function in PREM model of the Earth for investigation of seismic waves

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Abstract

Model of point-like source of earthquake is suggested in the framework of PREM model of the Earth. The Earth is considered to be composed of two liquid (ocean and the outer core) and six solid layers. General solution of differential equation for each layer is solved and coupling conditions at all inter-layer boundaries are satisfied. The Green function is constructed for the whole Earth. Synthetic seismograms are calculated and compared with the real seismograms of two earthquakes.

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1. Introduction

1.1 *General notes*

The calculation of synthetic seismograms is one of the most important topics in seismology (see, e.g., [1, 2, 9, 16]). This result can be achieved by different methods. The first type is numerical methods, e.g., the finite element method [3]. The second type is asymptotic method, e.g., the generalized ray method [8]. The third type is semi-analytical method, e.g., the normal-mode summation method

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[4] for spherically symmetric Earth model. The normal-mode summation is mostly used for point sources [7, 10, 14, 17]. In the present paper, we use the normal-mode summation to construct the model. It is well-known that mathematical model gives one useful tool in geophysical problems (see, e.g., [4, 12, 13]). The rest of Introduction is devoted to the description of the model background. In the next section, we construct the model. To verify the model, we calculate synthetic seismograms for two real earthquakes and compare the results with the real data. Good correlation confirms that the model is appropriate.

1.2 *Linearization*

We will assume that the Earth is at rest before the earthquake occurs. The appearance of the earthquake source will only cause deformations that are small in amplitude. Since seismic deformations will have small amplitudes, we will be able to linearize the equations of motion so that they take into account changes in parameters relative to the magnitude of the displacement with an accuracy of the first order. The linearized equations we end up with will be much easier to analyze. This simplification is not very different from reality, but it will help us greatly simplify future calculations.

The position of the particles will be described in terms of Lagrangian variables, that is, relative to their initial position x and displacement relative to this position at time t

$$\vec{r}(\vec{x},t) = \vec{x} + \vec{s}(\vec{x},t).$$

When linearizing the equation of motion, we will ignore all quantities of the order of x $|s|^2$. It should be noted that in this case the difference between the methods of describing the motion of particles disappears. Consider the perturbation of q relative to the initial equilibrium position q^0 at the time instant $t = 0$.

$$q^E(\vec{r},t) = q^0(\vec{r}) + q^{E1}(\vec{r},t), q^L(\vec{r},t) = q^0(\vec{x}) + q^{L1}(\vec{r},t).$$

Here and below the superscript E is a notation of the Euler variable and superscript L is a notation of the Lagrange variable. For sufficiently small deformations, we can assume that, in the first order approximation in respect to $|s|$ one has

$$q^{E1}(\vec{r},t) = q^{E1}(\vec{x},t), q^{L1}(\vec{r},t) = q^{L1}(\vec{x},t).$$

The motion of matter is determined by the momentum conservation equation. For the Earth without rotation, this equation has the form

$$\rho^E D_t \vec{u}^E = \nabla \cdot T^E + \rho^E g^E, \tag{1}$$

where

$$D_t q^E = \partial_t q^E + \vec{u}^L \cdot \nabla q^E.$$

The left hand side of the equation is the total momentum change in a small volume. The right hand side consists of the sum of changes in momentum, respectively, associated with the flow of matter through the walls and associated with the actions of some external forces, in our case it is gravity. As a result of linearisation, we obtain the equation of motion for a static Earth in the form

$$\rho^0\,\partial_t^2\vec{s} = \nabla\cdot T^{L1} - \nabla(\rho^0\vec{s}\cdot\nabla\phi^0) - \rho^0\nabla\phi^{E1} - \rho^{E1}\nabla\phi^0 \tag{2}$$

Here T is the Cauchy stress tensor, ρ is the density, ϕ is the gravitational potential. It is for this differential equation that we will construct the Green's function.

1.3 *Free vibrations*

To simplify the work, we can assume that our planet does not change over time, since in reality the changes are very insignificant relative to the time intervals under consideration. Then, to describe free oscillations, we can use the method of separation of variables, then the solutions will be in the form

$$\vec{s}(\vec{x},t) = \vec{s}(\vec{x})e^{i\omega t}, \tag{3}$$

where the quantity ω is called the natural frequency. Equivalent to finding a solution in the form (3) would be to make the Fourier transform for all equations and conditions on the boundaries and look for solutions in terms of frequency, not time. The Fourier transform in our case looks like this

$$\vec{s}(\vec{x},\omega) = \int_{-\infty}^{\infty}\vec{s}(\vec{x},t)e^{-i\omega t}dt \tag{4}$$

In the frequency domain, equation (2) looks like this

$$-\omega^2\rho^0\vec{s} - \nabla\cdot T^{L1} + \nabla(\rho^0\vec{s}\cdot\nabla\phi^0) + \rho^0\nabla\phi^{E1} + \rho^{E1}\nabla\phi^0 = 0. \tag{5}$$

1.4 *Source of earthquakes*

If no external forces act on the Earth, then provided that in the initial condition it was at rest, solution (1) will be trivial: the Earth is in the state of constant rest $s = 0$. We do not consider various tectonic processes (see, e.g., [15]). From this, we can conclude that our planet is affected by certain forces that we have not yet taken into account. In order to take into account their influence, it is enough to add the density of volume forces to the right hand side of equation (1).

The presence of these forces can be interpreted as a violation of Hooke's empirical law in some local zone of a continuous uniform medium. Then the excess stress can be defined as the difference between the stress tensor, which is dictated by Hooke's law, and the true physical tensor, for which equation (2) is written: $S = T^{L1}_{model} - T^{L1}_{true}$. From this equation, we can conclude that $S^T = S$. Then we can define the density of body forces in terms of S as $\vec{f} = -\nabla\cdot S$.

It is important to note that in relation to the size of our planet and the length of seismic waves, the sources of earthquakes are extremely small. The time at which the excess stress described above occurs can also be considered extremely short. Based on this, we can consider that the source of the earthquake in our model of the earth, we can describe as a point. Then our source can be described by the equation

$$\partial_t S = M\delta(\vec{x} - \vec{x}_s)\delta(t - t_s),$$

where the value of M is called the tensor of the seismic moment of the source which is one of the main in the description of the source. It is important to note that this tensor is symmetric ($M^T = M$). As a result, we can write down the density of distribution of volumetric forces in the form

$$\vec{f} = -\nabla \cdot (M\delta(\vec{x} - \vec{x}_s))H(t - t_s),$$

where H is the stepped Heaviside function.

If we go into terms of frequency, then the expression for the density of forces is transformed into the following one

$$\vec{f} = -(\pi\delta(\omega) + \frac{1}{i\omega})\nabla \cdot (M\delta(\vec{x} - \vec{x}_s)) \tag{6}$$

1.5 *Earth model*

The most balanced model between complexity and accuracy is the SNREI model. This model has the properties of spherical symmetry, perfect elasticity, isotropy and no rotation. This earth model is specified by three parameters. The values of these parameters, due to symmetry and uniformity, depend only on the distance from the center of the earth. The first parameter is the density p. The second and third parameters are the bulk modulus of elasticity k and the shear modulus μ. Usually, one works with SNREI models not in terms of μ and k, but in terms of a longitudinal wave α and a shear wave β. The velocities of these waves are given by the relations

$$\alpha = \sqrt{\frac{k + \frac{4}{3}\mu}{\phi}}, \quad \text{and} \quad \beta = \sqrt{\frac{\mu}{\phi}}.$$

In our case, we will use a specific PREM Earth model [5, 15] as the SNREI model. Its features and values of α and β waves and the density will be described later.

1.6 *Decomposition into spherical harmonics*

Since our earth model has the property of spherical symmetry, it is convenient for us to consider our physical system in spherical coordinates. Consider the Laplace equation $\nabla^2 u = 0$. After separation of radial variable, the Laplace operator turns to the Laplace-Beltrami operator on the unit sphere. The solutions of the corresponding equation, $Y_{lm}, l = 0,1,2,\dots, m = -l, -l + 1, \dots, l,$ are called spherical functions (harmonics). It is important that the spherical harmonics form a complete orthogonal and normalized system of functions in the space of square integrable functions on the unit sphere:

$$\langle Y_{lm} | Y_{l'm'} \rangle = \delta_{ll'}\delta_{mm'}$$

where the scalar product is defined as

$$\langle f | g \rangle = \int_0^{2\pi} \int_0^{\pi} f\bar{g} \sin\theta \, d\theta \, d\varphi.$$

We can decompose any square integrable function in R^3 into the harmonics

$$U(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_{lm}(r) Y_{lm}(\theta, \varphi).$$

Consider vector fields. Any vector field on a sphere can be expressed as

$$\vec{u} = \hat{r}U + \nabla_l V - \hat{r} \times \nabla_l W,$$

where $\nabla_l = \theta\, \partial_\theta + \hat{\varphi}(\sin\theta)^{-1}\, \partial_\varphi$ is a surface gradient, which is called the spheroidal part, and $-\hat{r} \times \nabla_l W$ is called the toroidal part.

One can expand a vector-field $\vec{u} = (U, V, W)$ in the spherical harmonics

$$\vec{u} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (U_{lm}\vec{P}_{lm} + V_{lm}\vec{B}_{lm} + W_{lm}\vec{C}_{lm}), \tag{7}$$

where

$$\vec{P}_{lm} = \hat{r}Y_{lm}, \quad \vec{B}_{lm} = \frac{1}{\sqrt{l(l+1)}}\nabla_l Y_{lm}, \quad \overrightarrow{C_{lm}} = \frac{1}{\sqrt{l(l+1)}}(\hat{r} \times \nabla_l)Y_{lm}$$

(these vectors are pairwise orthogonal). For using it below, we recall the expressions for the spherical function:

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} P_{l|m|}(\cos\theta) \times \begin{pmatrix} \sqrt{2}\cos m\varphi & , -l \le m < 0 \\ 1 & , m = 0 \\ \sqrt{2}\sin m\varphi & , 0 < m \le l \end{pmatrix}$$

where

$$P_{lm}(\mu) = \frac{1}{2^l l!}(1 - \mu^2)^{\frac{m}{2}}(\frac{d}{d\mu})^{l+m}(\mu^2 - 1)^l.$$

There are also such recurrent relations:

$$(1 - \mu^2)P'_{lm}(\mu) = (l + m)P_{l-1m}(\mu) - l\mu P_{lm}(\mu),$$

$$P_{ll}(\mu) = (-1)^l(2l - 1)!!\,(1 - \mu^2)^{\frac{l}{m}}.$$

1.7 *PREM model of the Earth*

In PREM, the earth model is usually divided into eight layers, each of which has its own density and the speed of S and P waves. There are 2 liquid and 6 solid layers. The ocean and the outer core are considered to be liquid; S waves are completely absent in them. Typically, PREM ground within each layer has different parameter values depending on the distance to the center of the Earth. In our case, we assume that the medium is homogeneous inside each layer, and therefore the density and speed of S and P waves are constant. At each boundary between the layers, there is a sharp transition of some parameters. For each layer, we take the average parameters of the usual PREM model [4].

2. Model construction

We will construct Green's functions analogously to [6] but in that paper, there is no explicit division into layers. We will do this. The Green's Function Plan for our model is as follows. We work in the frequency domain, since this greatly simplifies all calculations in spherical coordinates. First, we take equation of motion (5) and add to it the point source described in equation (6). Since our model consists of layers, for each layer we will explicitly write down its linearly independent solutions. Next, let's add conditions at the boundaries of the layers. Combining all boundary conditions and solutions for all layers into a system of linear equations, we obtain a solution for the entire ball. To get the results, it will be enough for us to do the inverse Fourier transform to return from the frequency domain to the time domain.

2.1 *Decomposition into spherical harmonics*

We need to expand the equation

$$-\omega^2\rho^0 - \nabla \cdot T^{L1} + \nabla(\rho^0\vec{s} \cdot \nabla\phi^0) + \rho^0\nabla\phi^{E1} + \rho^{E1}\nabla\phi^0 = -\vec{f}, \tag{8}$$

to spherical harmonics, where

$$\nabla^2\phi^{E1} = -4\pi G(\rho^0\nabla \cdot \vec{s} + \dot{\rho}s_r). (s_r = \vec{s} \cdot \hat{r}) \tag{9}$$

For the case when the right-hand side is equal to zero, the solution (9) is described in [4]. We omit details of calculations which coincide with that in [6] and present only actions that differ and the result obtained.

The first step is the expansion into spherical harmonics of the displacement field,

$$\vec{s} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_{lm}\vec{P}_{lm} + V_{lm}\vec{B}_{lm} + W_{lm}\vec{C}_{lm},$$

the force field,

$$\vec{f} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} G_{lm}\vec{P}_{lm} + H_{lm}\vec{B}_{lm} + K_{lm}\vec{C}_{lm},$$

and the perturbation of the gravitational potential,

$$\phi^{E1} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm}Y_{lm}.$$

One substitutes these expansions into equation (8). One can solve this equation separately for each degree l. When one calculates the derivatives, each of the equations is divided into three more solutions, one for each of the harmonics. As a result, one comes to a system of three equations with unknown harmonic coefficients. Our equation differs from the homogeneous equation from [4] by the presence of functions U_{lm}, H_{lm} and K_{lm} in the right-hand side of the equation.

Let's do the same with equation (9). We obtain an equation that coincides with the homogeneous analogue from [4].

In total, we obtained four unknowns U_{lm}, V_{lm}, W_{lm} and P_{lm} and four equations. One can notice that three equations contain only U_{lm}, V_{lm}, P_{lm} (spherical waves) and one contains only W_{lm} (toroidal waves). For this reason, we can solve these equations separately, and then glue the solutions together.

Since the initial system is spherically symmetric, there is no dependence on the parameter m, but after adding a point source, the symmetry will be broken, and we will start to consider the dependence on it again.

2.2 *Solutions of equations for spherical waves*

Since we have three second-order equations for spherical waves, it makes sense to add three more variables R_{lm}, S_{lm} and B_{lm} and bring them to the first-order equations. Consider the resulting system of six equations

$$
\begin{aligned}
\dot{U}_{lm} &= -2\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}\left(\kappa-\tfrac{2}{3}\mu\right)r^{-1}U_{lm} \\
&\quad +k\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}\left(\kappa-\tfrac{2}{3}\mu\right)r^{-1}V_{lm}+\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}R_{lm} \\
\dot{V}_{lm} &= -kr^{-1}U_{lm}+r^{-1}V_{lm}+\mu^{-1}S_{lm} \\
\dot{P}_{lm} &= -4\pi G\rho U_{lm}-(l+1)r^{-1}P_{lm}+B_{lm} \\
\dot{R}_{lm} &= \left[-\omega^2\rho-4\rho g r^{-1}+12\kappa\mu\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-2}\right]U_{lm} \\
&\quad +\left[k\rho g r^{-1}-6k\kappa\mu\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-2}\right]V_{lm} \\
&\quad -4\mu\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-1}R_{lm}+kr^{-1}S_{lm} \\
&\quad -(l+1)\rho r^{-1}P_{lm}+\rho B_{lm}-G_{lm} \\
\dot{S}_{lm} &= \left[k\rho g r^{-1}-6k\kappa\mu\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-2}\right]U_{lm} \\
&\quad -\left[\omega^2\rho-2\mu r^{-2}+4k^2\mu\left(\kappa+\tfrac{1}{3}\mu\right)\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-2}\right]V_{lm} \\
&\quad -k\left(\kappa-\tfrac{2}{3}\mu\right)\left(\kappa+\tfrac{4}{3}\mu\right)^{-1}r^{-2}R_{lm} \\
&\quad -3r^{-1}S_{lm}+k\rho r^{-1}P_{lm}-H_{lm} \\
\dot{B}_{lm} &= -4\pi G(l+1)\rho r^{-1}U_{lm}+4\pi Gk\rho r^{-1}V_{lm}+(l-1)r^{-1}B_{lm}
\end{aligned} \tag{10}
$$

where $k = l^{\frac{1}{2}} + l$.

We have homogeneous layers, so this system has exactly six linearly independent solutions. We need to solve this system. Two solutions from the book [4] should be corrected. One can find three solutions in [11]. Thus, we obtain:

$$
\begin{aligned}
U_{lm}(r) &= l\xi r^{-1}j_l(\gamma r)-\zeta\gamma j_{l+1}(\gamma r) \\
V_{lm}(r) &= k\xi r^{-1}j_l(\gamma r)+k\gamma j_{l+1}(\gamma r) \\
P_{lm}(r) &= -4\pi G\rho\zeta j_l(\gamma r) \\
R_{lm}(r) &= -\left(\left(\kappa+\tfrac{4}{3}\mu\right)\zeta\gamma^2-2l(l-1)\mu\xi r^{-2}\right)j_l(\gamma r) \\
&\quad +2\mu(2\zeta+k^2)\gamma r^{-1}j_{l+1}(\gamma r) \\
S_{lm}(r) &= k\mu(\gamma^2+2(l-1)\xi r^{-2})j_l(\gamma r)-2k\mu(\zeta+1)\gamma r^{-1}j_{l+1}(\gamma r) \\
B_{lm}(r) &= -4\pi G\rho r^{-1}(k^2+(l+1)\zeta)j_l(\gamma r)
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\gamma^2 &= \frac{\omega^2}{2\beta^2}+\frac{\omega^2+\frac{16}{3}\pi G\rho}{2\alpha^2} \\
&\quad \pm\left[\left(\frac{\omega^2}{\beta^2}-\frac{\omega^2+\frac{16}{3}\pi G\rho}{\alpha^2}\right)^2+\left(\frac{8\pi Gk\rho}{3\alpha\beta}\right)\right]
\end{aligned}
$$

$$\zeta = \frac{3}{4}(\pi G\rho)^{-1}\beta^2\left(\gamma^2 - \frac{\omega^2}{\beta 62}\right)$$
$$\xi = \zeta - (l+1).$$

Thus, we obtain two solutions, corresponding to different signs of γ.
Another solution is as follow

$$\begin{aligned}
U_{lm}(r) &= lr^{l-1}\\
V_{lm}(r) &= kr^{l-1}\\
P_{lm}(r) &= \left(\omega^2 - \frac{4}{3}\pi G\rho l\right)r^l\\
R_{lm}(r) &= 2l(l-1)\mu r^{l-2}\\
S_{lm}(r) &= 2k(l-1)\mu r^{l-2}\\
B_{lm}(r) &= \left[(2l+1)\omega^2 - \frac{8}{3}\pi Gl(l-1)\rho\right]r^{l-1}.
\end{aligned} \tag{12}$$

We have three solutions left to obtain. For this, we can use work [11]. One can note that the spherical Bessel functions $j_l(x)$ in the two solutions are eigenfunctions of the operator

$$Hy = y'' + \frac{2}{x}y' - n(n+1)\frac{1}{x^2}y.$$

As we know, the solution to equation $Hy = \lambda y$ is the Bessel functions of the first and the second kind $j_n(x), y_n(x)$. In [11], functions $y_n(x)$ were discarded, since they had singularities at zero. If we change $j_l(\gamma)$ in solution (11) by $y_l(\gamma)$, then the functions will be linearly independent to the previous solutions, and by substituting these solutions into equation (10) we can make sure that they really are solutions.

$$\begin{aligned}
U_{lm}(r) &= l\xi r^{-1}y_l(\gamma r) - \zeta\gamma y_{l+1}(\gamma r)\\
V_{lm}(r) &= k\xi r^{-1}y_l(\gamma r) + k\gamma y_{l+1}(\gamma r)\\
P_{lm}(r) &= -4\pi G\rho\zeta y_l(\gamma r)\\
R_{lm}(r) &= -((\kappa + \frac{4}{3}\mu)\zeta\gamma^2 - 2l(l-1)\mu\xi r^{-2})y_l(\gamma r) +\\
&\qquad 2\mu(2\zeta + k^2)\gamma r^{-1}y_{l+1}(\gamma r)\\
S_{lm}(r) &= k\mu(\gamma^2 + 2(l-1)\xi r^{-2})y_l(\gamma r) - 2k\mu(\zeta+1)\gamma r^{-1}y_{l+1}(\gamma r)\\
B_{lm}(r) &= -4\pi G\rho r^{-1}(k^2 + (l+1)\zeta)y_l(\gamma r)
\end{aligned} \tag{13}$$

We have one solution left to find. To do this, note that r^l in (12) appeared due to the fact that they are a solution to the equation

$$Hy = 0$$

Another solution to this equation is the function r^{-l-1}. Previously, it was discarded because it had singularities at zero. Replace l in solution (12) by $-l-1$. We can say for sure that these equations will be linearly independent of the five previous solutions, but they will not yet be a solution. In order for these functions to become a solution, some coefficients must be corrected. After all the corrections, the last solution will look like

$$\begin{aligned}
U_{lm}(r) &= -(l+1)r^{-l-2}\\
V_{lm}(r) &= kr^{-l-2}\\
P_{lm}(r) &= \left(\omega^2 + \tfrac{4}{3}\pi G\rho(l+1)\right)r^{-l-1}\\
R_{lm}(r) &= 2(l+1)(l+2)\mu r^{-l-3}\\
S_{lm}(r) &= -2k(l+2)\mu r^{-l-3}\\
B_{lm}(r) &= -4\pi G(l+1)\rho r^{-l-2}.
\end{aligned} \tag{14}$$

Thus, we obtained all six solutions for the original equation (10) for a homogeneous layer, but these solutions are correct only in solid layers. In the liquid layers, as mentioned earlier, there are no S-waves, and there are only four independent solutions.

Solutions (12) and (14) are suitable for a liquid layer, we do not change them. But to get two more solutions, we will change some notation

$$\begin{aligned}
\gamma^2 &= \frac{\omega^2 + \frac{16}{3}\pi G\rho}{\alpha^2} - \left(\frac{4\pi Gk\rho}{3\omega\alpha}\right)^2\\
\zeta &= -\tfrac{3}{4}(\pi G\rho)^{-1}\omega^2\\
\xi &= \zeta - (l+1).
\end{aligned}$$

Previously, we had two solutions depending on the choice of the sign in γ, now there is no choice of the sign. Thus, we have found all solutions for spherical waves in all layers of the earth.

2.3 *Solutions of equations for toroidal waves*

From equation (8), one can select the toroidal part. It consists of one equation

$$\ddot{W}_{lm} + \frac{2}{r}\dot{W}_{lm} + \left(\frac{\omega^2}{\beta^2} - l(l+1)r^{-2}\right)W_{lm} = -K_{lm} \tag{15}$$

For $K_{lm} = 0$, the solutions are the spherical Bessel functions of order l of the first and the second kind:

$$W_{lm}(r) = aj_l\left(\frac{\omega}{\beta}r\right) + by_l\left(\frac{\omega}{\beta}r\right).$$

Introducing a new variable $T_{lm} = 0$ for the transition from a second-order equation, one obtains a system of two first-order equations:

$$\begin{aligned}
\dot{W} &= r^{-1}W + \mu^{-1}T,\\
\dot{T} &= (-\omega^2\rho + (k^2-2)\mu r^{-2})W - 3r^{-1}T - K_{lm}.
\end{aligned}$$

If one considers the physical meaning of the new variable, then it means the cohesion associated with the toroidal displacement W.

Due to the fact that one has

$$T_{lm} = \mu(\dot{W}_{lm} - r^{-1}W_{lm}),$$

it is possible to obtain the following expression:

$$T_{lm}(r) = \mu\frac{a}{r}\left[(l-1)j_l\left(\frac{\omega}{\beta}r\right)-\left(\frac{\omega}{\beta}r\right)j_{l+1}\left(\frac{\omega}{\beta}r\right)\right]$$
$$+\mu\frac{b}{r}\left[(l-1)y_l\left(\frac{\omega}{\beta}r\right)-\left(\frac{\omega}{\beta}r\right)y_{l+1}\left(\frac{\omega}{\beta}r\right)\right].$$

2.4 *Adding an earthquake source*

In the previous section, we found solutions to equation (8) for a homogeneous layer. The solutions are reduced to a system of eight linear differential equations of the first order with unknown functions $U, V, P, R, S, B, , T$, which is divided into two independent systems. We can write the whole system in matrix form

$$\frac{d}{dr}y(r) = A(r)y(r) + z(r), \tag{16}$$

where

$$y(r) = (U_{lm}; V_{lm}; P_{lm}; R_{lm}; S_{lm}; B_{lm}; W_{lm}; T_{lm})^T$$

and $z(r)$ contains just an inhomogeneous component

$$z(r) = (0; 0; 0; -G_{lm}; -H_{lm}; 0; 0; -K_{lm})^T.$$

The components G_{lm}, H_{lm} and K_{lm} together describe some kind of point source of earthquakes.

Let M be the seismic moment tensor

$$M = \begin{pmatrix} M_{rr} & M_{r\theta} & M_{r\varphi} \\ M_{\theta r} & M_{\theta\theta} & M_{\theta\varphi} \\ M_{\varphi r} & M_{\varphi\theta} & M_{\varphi\varphi} \end{pmatrix}$$

This tensor is symmetric, then the distribution of forces has the form

$$\hat{f} - h(\omega)\nabla\cdot(M\delta(\vec{x}-\vec{x}_s)),$$

where

$$h(\omega) = \left(\pi\delta(\omega) + \frac{1}{i\omega}\right).$$

Then the coefficients of spherical harmonics are calculated as

$$G_{lm}(r) = \langle\vec{f}|\vec{P}_{lm}\rangle,$$
$$H_{lm}(r) = \langle\vec{f}|\vec{B}_{lm}\rangle,$$
$$K_{lm}(r) = \langle\vec{f}|\vec{C}_{lm}\rangle,$$

where

$$\langle\vec{f}|\vec{g}\rangle = \int_0\int_0 (\vec{f}\cdot\vec{g})\sin\theta\, d\theta\varphi.$$

As a result, one obtains

$$-\frac{1}{h(\omega)}G_{lm}(r) = \frac{\delta'(r-r_s)}{r^2}M_{rr}Y_{lm}$$
$$-\frac{\delta(r-r_s)}{r^3}\left[M_{\theta r}\,\partial_\theta Y_{lm} + \frac{M_{\varphi r}}{\sin\theta_s}\partial_\varphi Y_{lm} + (M_{\theta\theta}+M_{\varphi\varphi})Y_{lm}\right]$$

$$-\frac{\sqrt{l(l+1)}}{h(\omega)}H_{lm}(r) = \frac{\delta'(r-r_s)}{r^2}[M_{r\theta}\,\partial_\theta Y_{lm} + M_{r\varphi}\frac{\partial_\varphi Y_{lm}}{\sin\theta_s}]$$
$$-\frac{\delta(r-r_s)}{r^3}[M_{\theta\theta}\,\partial_\theta^2 Y_{lm} + M_{\theta\varphi}\frac{1}{\sin\theta_s}(\partial_\theta\,\partial_\varphi Y_{lm} - \cot\theta_s\,\partial_\varphi Y_{lm})$$
$$+M_{\varphi\theta}\frac{1}{\sin\theta_s}\partial_\varphi\,\partial_\theta Y_{lm} + M_{\varphi\varphi}\frac{1}{\sin^2\theta_s}\partial_\varphi^2 Y_{lm}$$
$$-(M_{\theta r} - \cot\theta_s M_{\varphi\varphi})\,\partial_\theta Y_{lm} - (M_{\varphi r} + \cot\theta_s M_{\varphi\theta})\frac{1}{\sin\theta_s}\partial_\varphi Y_{lm}]$$
$$-\frac{\sqrt{l(l+1)}}{h(\omega)}K_{lm}(r) = \frac{\delta'(r-r_s)}{r^2}[-M_{r\theta}\frac{\partial_\varphi Y_{lm}}{\sin\theta_s} + M_{r\varphi}\,\partial_\theta Y_{lm}]$$
$$-\frac{\delta(r-r_s)}{r^3}[-M_{\theta\theta}\frac{1}{\sin\theta_s}(\partial_\theta\,\partial_\varphi Y_{lm} - \cot\theta_s\theta_\varphi Y_{lm}) + M_{\theta\varphi}\,\partial_\theta^2 Y_{lm}$$
$$-M_{\varphi\theta}\frac{1}{\sin^2\theta_s}\partial_\varphi^2 Y_{lm} + M_{\varphi\varphi}\frac{1}{\sin\theta_s}\partial_\varphi\,\partial_\theta Y_{lm}$$
$$+(M_{\theta r} - \cot\theta_s M_{\varphi\varphi})\frac{\partial_\varphi Y_{lm}}{\sin\theta_s} - (M_{\varphi r} + \cot\theta_s M_{\varphi\theta})\,\partial_\theta Y_{lm}]$$

We have written down explicit expressions for the components. In general, they can be expressed in terms of the delta function and its derivative. Then

$$z(r) = z_1(r)\gamma(r-r_s) + z_2(r)\gamma'(r-r_s). \tag{17}$$

We will seek solutions for (16) by substituting (17) as an inhomogeneity in the form

$$y(r) = v(r) + z_2(r)\delta(r-r_s).$$

One obtains

$$\frac{d}{dr}v(r) = A(r)v(r) + (A(r_s)z_2(r_s) + z_1(r_s) - z_2'(r_s))\delta(r-r_s). \tag{18}$$

Then one can obtain the components y by solving this equation in the regions above and below the source, while imposing the condition $r = r_s$ as the boundary

$$v_+(r_s) - v_-(r_s) = A(r_s)z_2(r_s) + z_1(r_s) - z_2'(r_s). \tag{18}$$

2.5 *Boundary conditions imposition*

The model we have chosen has eight layers with different parameters. Each of them has its own values of density and speed of S and P waves. Therefore, boundary conditions must be imposed on the boundaries of each layer. In the last section, we found out that a point source divides one of the layers into two sub-layers and adds one more boundary condition (19). It turns out that our model has nine layers when specifying the source. Next, we will set the boundary conditions for different boundaries.

From physical considerations, the boundary conditions for the boundary between solid layers are given in the form

$$[U]_-^+ = [V]_-^+ = [P]_-^+ = [R]_-^+ = [S]_-^+ = [B]_-^+ = [W]_-^+ = [T]_-^+ = 0,$$

where we use the following notation for the jump of function at the border:

$$[X]_-^+ = X_+(r_{border}) - X_-(r_{border})$$

For the boundary between solid and liquid layers, the boundary conditions are as follows

$$[U]_-^+ = [P]_-^+ = [R]_-^+ = [S]_-^+ = [B]_-^+ = [T]_-^+ = 0.$$
$$S = T = 0.$$

For the surface of the Earth, the boundary conditions look like this

$$R = S = B = T = 0.$$

One more condition is that there should be no singularities at the center of the Earth.

2.6 *A solution for the whole Earth*

In the previous sections, we obtained solutions for each layer and defined boundary conditions for all boundaries. Each layer has several solutions, their number depends on the type of layer. Then the solution for each layer will be a linear combination of linearly independent solutions for this layer. The coefficients of each component are arbitrary initially. We will calculate them by writing down equations for all boundary conditions, which gives us a system of linear equations. Having solved it, we find unknown coefficients. After that, we can combine the solutions for each layer and obtain a solution for the whole Earth.

We can notice that in our case the matrix of the system is sparse. Indeed, each layer interacts only with its neighbors, which means that most of the matrix components will be zero. You can see that all nonzero components will be near the diagonal.

3. Modeling

3.1 Acquisition of seismograms

To estimate the quality of our model, we compare a few synthetic seismograms with the values of real earthquakes observed at the stations.

To construct seismograms, we must count all the spherical harmonics and sum them up. After that, we will need to move from the terms ω back to the terms t, i.e. from the frequency domain to the time domain. In computational experiments we cannot count the infinite sum of harmonics, have to break the series. Because of this, we have to use anti-aliasing, since we discard high-frequency harmonics, which lead to some "artifacts".

We use data from the TNS station of the German Regional Seismic Network (GRSN) as an example. The data are presented below (see table 1)

Table 1
Location, Date, Time of occurrence, epicentric distance in degrees and source depth in km for two earthquakes.

Location	Date	Start time	Epicentral distance	Depth
Hokkaido	26.11.1991	19:40:52	80	46,3
Mindanao	13.11.1991	11:12:19	101	42,7

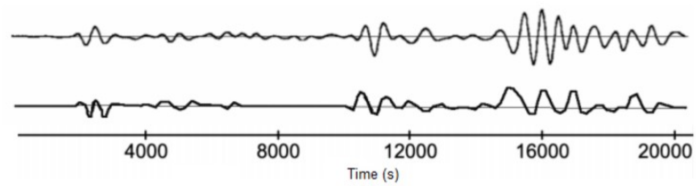

Figure 1
Comparison of synthetic (bottom) seismogram with data (top) observed at TNS station in Hokkaido

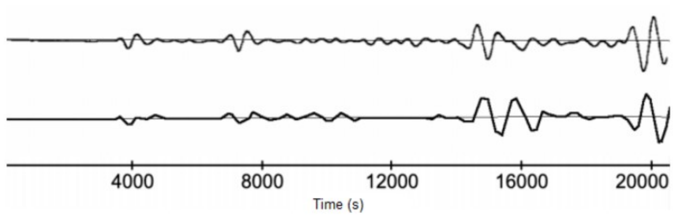

Figure 2
Comparison of synthetic (bottom) seismogram with data (top) observed at TNS station in Mindanao.

Conclusion

In this work, we constructed a model of the Earth, found Green's function for it, and tested this model on some earthquake data obtained at the station. We have presented a method that efficiently computes full synthetic gathers for a spherically symmetric Earth model with uniform layers, up to high frequencies. The obtained seismograms demonstrate the efficiency of the proposed method, which is confirmed by a good coincidence of the synthetic seismograms with real data (see Fig. 1 and Fig. 2).

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