Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
DOI : 10.47974/JIM-2443

Geometric flows along acceleration vector field of null curves

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Abstract

In this work, we study time evolution equations of null curves according to acceleration vector fields in Minkowski 3-space. First of all, we find a solution of the nonlinear system of time evolution equations for planar null curves. Also, we introduce a null binormal acceleration (NBA) flow $\partial^2\gamma/\partial t^2 = \kappa\mathbf{e}_3$ along a null curve γ. From this, we study soliton solutions of the NBA flow in Minkowski plane. Also, we give some soliton solutions of the NBA flows by using d'Alembert partial differential equation in Minkowski 3-space. Lastly, we construct Bäcklund transformations corresponding to the NBA flows, and we study soliton solutions of integrable system in the intrinsic invariants of the null curve by using hyperbolic functions.

Subject Classification: *35C08, 53C50, 53Z05.*

Keywords: *Integrable system, Vortex filament flow, D'Alembert equation, Bäcklund transformation.*

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1. Introduction

Many mathematician have been interested in studying relationships between integrable equations (soliton equations) and geometric flows of a curve in various spaces for a long time. In particular, time evolution equations of a curve are described by inextensible flows as geometric flows. From a physical standpoint, inextensible flows produce motions that do not generate strain energy. In addition, a variety of nonlinear processes in chemistry and biology can be characterized by the geometric evolution of shapes such as curves. Therefore, understanding the evolution equations of curves is essential in applications including computer vision and image analysis.[10].

Consider a geometric flow by velocity vector field as follows:

$$\frac{\partial \gamma}{\partial t} = \phi_1 \mathbf{e}_1 + \phi_2 \mathbf{e}_2 + \phi_3 \mathbf{e}_3, \tag{1.1}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is Frenet frame of a space curve $\gamma = \gamma(s)$ and $\phi_i (i = 1, 2, 3)$ are tangential, principal normal and binormal velocities. As applications to physics of the flow (1.1), these models are effective in representing the motion of vortex filament flows in an inviscid fluid. In these situations, the evolution of curves is dictated by a corresponding velocity vector field. In fact, for vortex filament flows, we take the velocities $\{\phi_1, \phi_2, \phi_3\} = \{0, 0, \kappa\}$, that is, (1.1) becomes

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{e}_3, \tag{1.2}$$

where κ is the curvature of the curve $\gamma = \gamma(s)$. Vortex filament flow has been the subject of extensive study by numerous experts and geometers [3, 5, 6, 11, 12, 14, 15, 16, 17] etc. The vortex filament flow is related to solutions of the cubic nonlinear Schrödinger equation. Ding and Inoguchi [4] studied the vortex filament flow for non-null curves in Minkowski 3-space, and they discussed nonlinear heat system and the nonlinear Schrödinger equation for spacelike and timelike curves, respectively. Gürbüz and Yoon [7] obtained the connection between the visco-Da Rios equation and nonlinear Schrödinger equation in Riemannain 3-manifold and gave the Bäcklund transformations of space curves with the visco-Da Rios equation. Anco and Asadi [2] derived the geometrical relationships between the nonlinear Schrödinger equation and the vortex filament equation in Hermitian symmetric spaces. Recently, Gürbüz, Yuzbasi and Yoon [8] proved that a soliton of the nonlinear Schrödinger equations of the non-null curves is a solution of a traveling wave. Also, they discussed the method to find the exact shape of the non-null curves from the vortex filament equation by solving the Frenet vectors of these curves.

In this paper, we consider a class of geometric flows which is the motion of a curve $\gamma = \gamma(s)$ satisfying the equation as follows:

$$\frac{\partial^2 \gamma}{\partial t^2} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3, \tag{1.3}$$

where f_1, f_2 and f_3 are smooth functions. The geometric flow (1.3) is introduced by Tsurumi et al. [13] in Euclidean 3-space. It is natural to investigate what changes occur under the replacement of (1.1) with (1.3). Geometric flows are important in physical applications, as their evolution equations can be interpreted as standard equations of motion for curves if we take (f_1, f_2, f_3) to be forces that act on a particle of unit mass. In [13] authors constructed evolution equations for the geometric flow (1.3). As a result, they studied special solution and also gave application to physics included the motion of an elastic thin rod, of currents in plasma and of lines of magnetic force in Euclidean space. Recently, in [1] authors generalized the results in [13] in Euclidean n-space. Therefore, we will study Lorentz version of the geometric flow (1.3) in Minkowski 3-space. Usually, results of the geometric flows (1.1) and (1.3) along non-null curves in Minkowski 3-space are similar to the results in Euclidean 3-space. In this sense, we only deal with the geometric flow (1.3) along null curves in Minkowski 3-space. As applications, we present the novel geometric models of the evolution equations for the curvature and the torsion derived from the fundamental equations for null curve flows in Minkowski 3- space, obtain the exact solutions for them, and illustrate the moving curve for these solutions. These answers lead us to two conclusions about solitary waves: shock and singular solitary waves. Figures of soliton solutions that is produced using hyperbolic functions are also displayed.

2. Preliminaries

Let $\mathbb{R}_1^3$ be the Minkowski 3-space with the Lorentz product $\langle \cdot,\cdot \rangle$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_0 v_0 + u_1 v_1 + u_2 v_2,$$

where $\mathbf{u} = (u_0, u_1, u_2)$ and $\mathbf{v} = (v_0, v_1, v_2)$ are vectors in $\mathbb{R}_1^3$.

A curve $\gamma(u)$ in $\mathbb{R}_1^3$ is called a null curve or lightlike curve if $\langle \gamma'(u), \gamma'(u) \rangle = 0$ and $\gamma'(u) \neq 0$ for all u. We know that $\langle \gamma''(u), \gamma''(u) \rangle \geq 0$ for a null curve $\gamma(u)$. We say that a null curve $\gamma(u)$ is parametrized by the pseudo-are length if $\langle \gamma''(u), \gamma''(u) \rangle = 1$. If a null curve $\gamma(u)$ satisfies $\langle \gamma''(u), \gamma''(u) \rangle \neq 0$, then the pseudo-arc length function of the null curve $\gamma(u)$ is defined by

$$s(u) = \int_{u_0}^{u} \langle \gamma''(t), \gamma''(t) \rangle^{\frac{1}{4}} dt.$$

Let $\gamma(s)$ be a null curve parametrized by the pseudo-arc length s in $\mathbb{R}_1^3$. Then there exists a null Frenet frame given by $\{\mathbf{e}_1 = \gamma', \mathbf{e}_2, \mathbf{e}_3\}$ of the null curve $\gamma(s)$ such that

$$\begin{aligned} &\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 0, \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 0, \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 1, \\ &\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 1, \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = 0, \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 0. \end{aligned} \tag{2.1}$$

Also, the Lorentz cross product in $\mathbb{R}_1^3$ leads to

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_1, \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_2.$$

Furthermore, the Frenet equation of the null Frenet frame $F(s) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ becomes

$$\frac{\partial F}{\partial s} = AF, \tag{2.2}$$

where A is the matrix as the form

$$A = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & 0 & \tau \\ -\tau & -\kappa & 0 \end{pmatrix}.$$

Here κ and τ are the curvature and the torsion of the null curve $\gamma(s)$ as follows:

$$\kappa = \langle \gamma'', \gamma'' \rangle, \tau = \frac{1}{2}\langle \gamma''', \gamma''' \rangle,$$

respectively.

Let $\gamma(\cdot, t)$ be a null curve parametrized by pseudo-arc length s in $\mathbb{R}^3_1$. By considering (2.1) the time evolution equation of null Frenet frames along the null curve $\gamma(\cdot, s)$ is given by

$$\frac{\partial F}{\partial t} = EF, \tag{2.3}$$

where E is the matrix as the form

$$E = \begin{pmatrix} \omega & 0 & -\nu \\ 0 & -\omega & -\mu \\ \mu & \nu & 0 \end{pmatrix}.$$

Here ω, μ and ν are smooth functions of parameters s and t. With the help of (2.2) and (2.3), the compatibility condition $\frac{\partial^2 F}{\partial s\, \partial t} = \frac{\partial^2 F}{\partial t\, \partial s}$ implies

$$\frac{\partial A}{\partial t} - \frac{\partial E}{\partial s} + [A, E] = 0_{3\times 3}, \tag{2.4}$$

where $[,]$ denotes Lie bracket operator. The last equation gives the integrable system in the intrinsic invariants κ and τ of the null curve, that is, we have

$$\begin{aligned} \frac{\partial \omega}{\partial s} &= \mu\kappa - \nu\tau, \\ \frac{\partial \kappa}{\partial t} &= -\frac{\partial \nu}{\partial s} + \kappa\omega, \\ \frac{\partial \tau}{\partial t} &= -\frac{\partial \mu}{\partial s} - \tau\omega. \end{aligned} \tag{2.5}$$

3. Acceleration flow of null curves

In this section, we study a geometric flow by an acceleration vector field of the null curve $\gamma(\cdot, t)$ in $\mathbb{R}^3_1$. First of all, we consider a geometric flow

$$\frac{\partial^2 \gamma}{\partial t^2} = KF, \tag{3.1}$$

where $K = (f_1 f_2 f_3)$ is a 1×3-matrix. Here f_1, f_2 and f_3 are the tangential, principal normal and binormal accelerations which are the functions of the intrinsic quantities κ and τ of the null curve through the time evolution. The geometric flow (3.1) is called an acceleration flow of the null curve in Minkowski 3-space.

On the other hand, differentiation of (2.3) with respect to t leads to

$$\frac{\partial^2 F}{\partial t^2} = \left(\frac{\partial E}{\partial t} + E^2\right) F \tag{3.2}$$

and differentiation of (3.1) with respect to s gives

$$\frac{\partial}{\partial s}\left(\frac{\partial^2\gamma}{\partial t^2}\right)=\left(\frac{\partial K}{\partial s}+KA\right)F. \tag{3.3}$$

Thus, we can obtain time evolution equations for the null Frenet frame of the null curve with the aid of (3.2) as follows:

Theorem 3.1: *The acceleration flow (3.1) implies the time evolution equation of the null Frenet frame of the null curve in Minkowski 3-space as follows:*

$$\frac{\partial^2 F}{\partial t^2}=\Lambda F, \tag{3.4}$$

where

$$\Lambda=\begin{pmatrix}\frac{\partial\omega}{\partial t}+\omega^2-\mu\nu & -\nu^2 & -\frac{\partial\nu}{\partial t}-\omega\nu\\ -\mu^2 & -\frac{\partial\omega}{\partial t}+\omega^2-\mu\nu & -\frac{\partial\mu}{\partial t}+\mu\\ \frac{\partial\mu}{\partial t}+\omega\mu & \frac{\partial\nu}{\partial t}-\omega\nu & -2\mu\nu\end{pmatrix}.$$

On the other hand, (3.3) is equivalent to

$$\frac{\partial^2}{\partial t^2}(\mathbf{e}_1)=\left(\frac{\partial K}{\partial s}+KA\right)F,$$

which implies the following result.

Theorem 3.2: *If the null curve γ is a solution of the acceleration flow (3.1) in Minkowski 3-space, then the functions ω, μ and ν are given by in terms of the acceleration functions f_1, f_2 and f_3 as the following system of PDEs:*

$$\begin{aligned}\frac{\partial\omega}{\partial t}+\omega^2-\mu\nu\quad &=\frac{\partial f_1}{\partial s}-f_3\tau,\\ \nu^2\quad &=f_3\kappa-\frac{\partial f_2}{\partial s},\\ \frac{\partial\nu}{\partial t}+\omega\nu\quad &=-\frac{\partial f_3}{\partial s}-f_1\kappa-f_2\tau.\end{aligned} \tag{3.5}$$

3.1 *Evolution of plane curves*

In this subsection, we investigate the time evolution equations of plane curves. In this case, we can take $\tau=0,\mu=0,\nu=0$ and $f_3=0$. It follows that from (2.5) we have

$$\frac{\partial\omega}{\partial s}=0,\frac{\partial\kappa}{\partial t}=\kappa\omega. \tag{3.6}$$

Also, (3.5) is reduced to

$$\frac{\partial\omega}{\partial t}+\omega^2=\frac{\partial f_1}{\partial s},\frac{\partial f_2}{\partial s}=0,\kappa f_1=0. \tag{3.7}$$

Since $\kappa\neq 0$, the function f_1 is identically zero, which implies that we have $\frac{\partial\omega}{\partial t}+\omega^2=0$. Then its solution is given by

$$\omega(t)=\frac{1}{t+c_1} \tag{3.8}$$

because the function ω depends only on the parameter t by the first equation of (3.6). Here c_1 is a constant. Thus we can solve the second equation of (3.6), its solution is expressed as

$$\kappa(s,t) = e^{-c_2\delta(s)}(t + c_1), \tag{3.9}$$

where $\delta(s)$ is any smooth function and c_2 is a constant.

Figure 1 is showed the curvature (3.9) with $c_1 = 0, c_2 = -1$ and $\delta(s) = \cos s$.

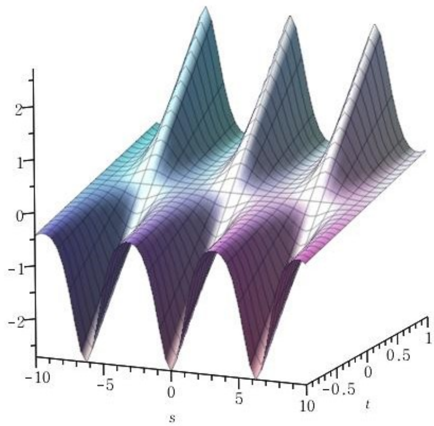


Figure 1
A soliton solution of a plane curve.

3.2 *Null binormal acceleration flow*

Now, we take the accelerations $\{f_1, f_2, f_3\} = \{0,0,\kappa\}$ in (3.1). In this case we have

$$\frac{\partial^2\gamma}{\partial t^2} = \kappa\mathbf{e}_3, \tag{3.10}$$

which is called a null binormal acceleration (NBA) flow of a null curve γ in Minkowski 3-space.

For simplicity, suppose that $\omega = 0$. Then (3.5) is reduced to

$$\mu\nu = \kappa\tau, \nu^2 = \kappa^2, \frac{\partial\nu}{\partial t} = -\frac{\partial\kappa}{\partial s}. \tag{3.11}$$

In the second equation of (3.11), we choose $\nu = -\kappa$. In this case one obtains $\mu = -\tau$. Thus, from (3.5) we have the following result.

Lemma 3.3: *If a null curve γ is a soliton solution of the NBA flow $\frac{\partial^2\gamma}{\partial t^2} = \kappa\boldsymbol{e}_3$ and $\omega = 0$, then the time evolution equation of the null Frenet frame is given by*

$$\frac{\partial^2 F}{\partial t^2} = \Lambda F, \tag{3.12}$$

where

$$\Lambda=\begin{pmatrix}\kappa\tau & -\kappa^2 & \kappa_t\\ -\tau^2 & -\kappa\tau & \tau_t\\ -\tau_t & -\kappa\tau & -2\kappa\tau\end{pmatrix}.$$

Since $\frac{\partial\nu}{\partial t}=-\frac{\partial\kappa}{\partial s}$ and $\nu=-\kappa$, one obtains $\frac{\partial\kappa}{\partial t}=\frac{\partial\kappa}{\partial s}$, its solution becomes $\kappa(s,t)=p(s+t)$ for $p\in C^1(\mathbb{R},\mathbb{R})$. Also, the third equation of (2.5) implies $\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial s}$ because of $\mu=-\tau$, which implies $\tau(s,t)=q(s+t)$ for $q\in C^1(\mathbb{R},\mathbb{R})$. Consequently, we have

Theorem 3.4: *If a null curve γ is a soliton solution of the NBA flow $\frac{\partial^2\gamma}{\partial t^2}=\kappa\boldsymbol{e}_3$ and $\omega=0$, then its curvature κ and torsion τ are given by*

$$\kappa(s,t)=p(s+t),\tau(s,t)=q(s+t)$$

for $f,g\in C^1(\mathbb{R},\mathbb{R})$.

We know that from (2.2) $\frac{\partial^2\gamma}{\partial s^2}=\kappa\mathbf{e}_3$. Therefore (3.10) implies

$$\frac{\partial^2\gamma}{\partial t^2}=\frac{\partial^2\gamma}{\partial s^2}, \tag{3.13}$$

this equation is a special case of d'Alamber's PDE.

Now, in order to solve (3.13) we take a null curve

$$\gamma(s,t)=(x(s,t),y(s,t),z(s,t)) \tag{3.14}$$

such that

$$x_{ss}=x_{tt},y_{ss}=y_{tt},z_{ss}=z_{tt},$$

where the subscript denotes the derivative with respect to parameters s and t. A solution of (3.13) depends on the initial conditions $\gamma(s,0)$ and $\gamma_t(s,0)$, that is,

$$\begin{aligned} x(s,0) &= \alpha_1(s),x_t(s,0)=\beta_1(s),\\ y(s,0) &= \alpha_2(s),y_t(s,0)=\beta_2(s),\\ z(s,0) &= \alpha_3(s),z_t(s,0)=\beta_3(s)\end{aligned}$$

at $t=0$. It consists of separate terms for the initial conditions $\gamma(s,0)$ and $\gamma_t(s,0)$. Thus, the general solution of (3.13) is expressed by

$$\begin{aligned} x(s,t) &=\tfrac{1}{2}\Big(\alpha_1(s+t)+\alpha_1(s-t)+\int_{s-t}^{s+t}\beta_1(u)du\Big),\\ y(s,t) &=\tfrac{1}{2}\Big(\alpha_2(s+t)+\alpha_2(s-t)+\int_{s-t}^{s+t}\beta_2(u)du\Big),\\ z(s,t) &=\tfrac{1}{2}\Big(\alpha_3(s+t)+\alpha_3(s-t)+\int_{s-t}^{s+t}\beta_3(u)du\Big)\end{aligned} \tag{3.15}$$

and the condition

$$-x_s^2+y_s^2+z_s^2=0$$

must be satisfied.

Example 3.5: To get a solution of (3.13), setting

$$\begin{aligned} &x(s,0)=\alpha_1(s)=\tfrac{1}{2}s, && x_t(s,0)=\beta_1(s)=s,\\ &y(s,0)=\alpha_2(s)=\cos s+s\sin s, && y_t(s,0)=\beta_2(s)=s\cos s,\\ &z(s,0)=\alpha_3(s)=\sin s-s\cos s, && z_t(s,0)=\beta_3(s)=s\sin s. \end{aligned} \tag{3.16}$$

Then the general solution of (3.13) with the initial conditions (3.16) becomes

$$\begin{aligned} &x(x,t)=\tfrac{1}{2}(s+t)^2,\\ &y(s,t)=\cos(s+t)+(s+t)\sin(s+t),\\ &z(s,t)=\sin(s+t)-(s+t)\cos(s+t). \end{aligned} \tag{3.17}$$

Thus, $\gamma(s,t)=(x(s,t),y(s,t),z(s,t))$ is a soliton solution of the NBA flow $\frac{\partial^2\gamma}{\partial t^2}=\kappa(s,t)\mathbf{e}_3$. Also, the curvature and the torsion of γ lead to

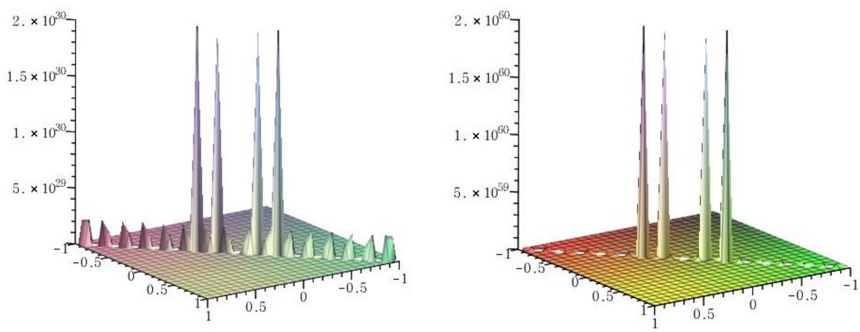
$$\begin{aligned} &\kappa(s,t)=(s+t)^2,\\ &\tau(s,t)=2+\frac{(s+t)^2}{2}. \end{aligned}$$

Example 3.6: Consider the initial conditions of (3.13) as

$$\begin{aligned} &x(s,0)=\alpha_1(s)=\sqrt{2}s, && x_t(s,0)=\beta_1(s)=\sqrt{2},\\ &y(s,0)=\alpha_2(s)=s\sin(\ln s), && y_t(s,0)=\beta_2(s)=\sin(\ln s)+\cos(\ln s),\\ &z(s,0)=\alpha_3(s)=s\cos(\ln s), && z_t(s,0)=\beta_3(s)=-\sin(\ln s)+\cos(\ln s). \end{aligned} \tag{3.18}$$

From the initial conditions, we have the general solution

$$\begin{aligned} &x(x,t)=\sqrt{2}(s+t),\\ &y(s,t)=(s+t)\sin(\ln(s+t)),\\ &z(s,t)=(s+t)\cos(\ln(s+t)). \end{aligned} \tag{3.19}$$



(a) a soliton solution of the curvature (b) a soliton solution of the torsion

Figure 2

A soliton solution of the NBA flow in Example 3.6.

Thus, $\gamma(s,t)$ determined by (3.19) is a soliton solution of the NBA flow $\frac{\partial^2\gamma}{\partial t^2}=\kappa(s,t)\mathbf{e}_3$. On the other hand, the curvature and the torsion of γ become

$$\begin{aligned}\kappa(s,t)&=2\left(\frac{1}{s+t}\right)^2,\\ \tau(s,t)&=2\left(\frac{1}{s+t}\right)^4.\end{aligned}\tag{3.20}$$

We plot the soliton solution of (3.20) in Figure 2, respectively.

Example 3.7: To solve the NBA flow (3.10) for the null curve in Minkowski 3-space, our ansatz is given as follows:

$$\gamma(s,t)=(p(s,t),\cos q(s,t),\sin q(s,t)),\tag{3.21}$$

where $p(s,t)$ and $q(s,t)$ are smooth functions. Since γ is a null curve, we have

$$p_s=\pm q_s.\tag{3.22}$$

On the other hand, the PDE (3.13) implies the following system:

$$p_{ss}=p_{tt}, q_{ss}=q_{tt}, \quad q_s=q_t.\tag{3.23}$$

In fact, we know that

$$\begin{aligned}p(s,t)&=(c_1\cosh s+c_2\sinh s)(c_3\cosh t+c_4\sinh t),\\ q(s,t)&=(c_5\cosh s+c_6\sinh s)(c_7\cosh t+c_8\sinh t)\end{aligned}\tag{3.24}$$

are partial solutions of the first and the second equations of (3.23), respectively, where c_i $(i=1,\ldots,8)$ are constant. Also, (3.22) and the third equation of (3.23) imply $c_1=c_2=c_5=c_6$ and $c_3=c_4=c_7=c_8$. Thus, for some constants a and b one finds

$$p(s,t)=q(s,t)=a^2b^2e^{s+t},\tag{3.25}$$

it follows that we have

$$\gamma(s,t)=(a^2b^2e^{s+t},\cos(a^2b^2e^{s+t}),\sin(a^2b^2e^{s+t})),\tag{3.26}$$

which is a soliton solution of the NBA flow $\frac{\partial^2\gamma}{\partial t^2}=\kappa(s,t)\mathbf{e}_3$. On the other hand, the curvature and the torsion are

$$\begin{aligned}\kappa(s,t)&=a^8b^8e^{4(s+t)},\\ \tau(s,t)&=\frac{a^8b^8}{2}e^{4(s+t)}(a^4b^4e^{2(s+t)}+7).\end{aligned}\tag{3.274}$$

4. Bäcklund transformation of the NBA flow

In this section, we construct Bäcklund transformation of the NBA flow of the null curve in $\mathbb{R}_1^3$.

Consider another null curve in $\mathbb{R}_1^3$ related the null curve γ by

$$\tilde{\gamma}(s,t)=\gamma(s,t)+p(s,t)\mathbf{e}_1+q(s,t)\mathbf{e}_2+r(s,t)\mathbf{e}_3,\tag{4.1}$$

where p,q and r are smooth functions of s and t. Using (2.2), a direct computation leads to

$$\frac{\partial\tilde{\gamma}}{\partial s}=(1+p_2-r\tau)\mathbf{e}_1+(q_s-r\kappa)\mathbf{e}_2+(r_s+\kappa p+\tau q)\mathbf{e}_3,$$

$$\frac{\partial^2\tilde{\gamma}}{\partial s^2}=((1+p_s-r\tau)_s-\tau(r_s+\kappa p+\tau q))\mathbf{e}_1+((q_s-r\kappa)_s$$
$$-\kappa(r_s+\kappa p+\tau q))\mathbf{e}_2+((r_s+\kappa p+\tau q)_s+\kappa(1+p_s-r\tau)+\tau(q_s-r\kappa))\mathbf{e}_3. \tag{4.2}$$

Let $\tilde{s}$ be the pseudo-arc length function of the null curve $\tilde{\gamma}$. Then

$$\begin{aligned} d\tilde{s} &= \langle\frac{\partial^2\tilde{\gamma}}{\partial s^2},\frac{\partial^2\tilde{\gamma}}{\partial s^2}\rangle^{1/4}ds \\ &= [2((1+p_s-r\tau)_s-\tau(r_s+\kappa p+\tau q))((q_s-r\kappa)_s-\kappa(r_s+\kappa p+\tau q)) \\ &\quad +(((r_s+\kappa p+\tau q)_s+\kappa(1+p_s-r\tau)+\tau(q_s-r\kappa)))^2]^{1/4}ds:=\Theta ds, \end{aligned}$$

where Θ is a non zero smooth function. It follows that the curvature $\tilde{\kappa}$ of the null curve $\tilde{\gamma}$ is given by

$$\tilde{\kappa}=\Theta^4.$$

On the other hand, the unit tangent vector of the null curve $\tilde{\gamma}$ is determined by

$$\tilde{\mathbf{e}}_1=\Phi_1\mathbf{e}_1+\Phi_2\mathbf{e}_2+\Phi_3\mathbf{e}_3, \tag{4.3}$$

where $\Phi_1=\Theta^{-1}(1+p_s-r\tau), \Phi_2=\Theta^{-1}(q_s-r\kappa)$ and $\Phi_3=\Theta^{-1}(r_s+\kappa p+\tau q)$. (4.3) implies

$$\tilde{\mathbf{e}}_{1\tilde{s}}=\frac{\Phi_{1s}-\Phi_3\tau}{\Theta}\mathbf{e}_1+\frac{\Phi_{2s}-\Phi_3\tau}{\Theta}\mathbf{e}_2+\frac{\Phi_{3s}\Phi_1+\kappa+\Phi_2\tau}{\Theta}\mathbf{e}_3.$$

Since $\tilde{\mathbf{e}}_{1\tilde{s}}=\tilde{\kappa}\tilde{\mathbf{e}}_3$, the binormal vector $\tilde{e}_3$ is given by

$$\tilde{\mathbf{e}}_3=\frac{\Phi_{1s}-\Phi_3\tau}{\Theta^5}\mathbf{e}_1+\frac{\Phi_{2s}-\Phi_3\tau}{\Theta^5}\mathbf{e}_2+\frac{\Phi_{3s}+\Phi_1\kappa+\Phi_2\tau}{\Theta^5}\mathbf{e}_3. \tag{4.4}$$

Suppose that $\tilde{\gamma}$ is a solution of the NBA flow, that is,

$$\frac{\partial^2\tilde{\gamma}}{\partial t^2}=\tilde{\kappa}\tilde{\mathbf{e}}_3. \tag{4.5}$$

By using (2.3) and differentiating with respect to t, one obtains

$$\frac{\partial^2\tilde{\gamma}}{\partial t^2}=(A_{1t}+\omega A_1+\mu A_3)\mathbf{e}_1+(A_{2t}-\omega A_2+\nu A_3)\mathbf{e}_2+(\kappa+A_{3t}-\nu A_1-\mu A_2)\mathbf{e}_3. \tag{4.6}$$

Consequently, the Bäcklund transformation of the NBA flow with the help of (4.4), (4.5) and (4.6) turns out to be the following result:

Theorem 4.1: *The NBA flow (4.5) is admits the Bäcklund transformation (4.1) in Minkowski 3-space if the system*

$$\begin{aligned} A_{1t}+\omega A_1+\mu A_3 &= (\Phi_{1s}-\Phi_3\tau)\Omega^{-1}, \\ A_{2t}-\omega A_2+\nu A_3 &= (\Phi_{2s}-\Phi_3\tau)\Omega^{-1}, \\ \kappa+A_{3t}-\nu A_1-\mu A_2 &= (\Phi_{3s}+\Phi_1\kappa+\Phi_2\tau)\Omega^{-1} \end{aligned}$$

is satisfied.

5. Applications

In this section, we discuss soliton solutions of integrable system (2.5) in the curvature κ and the torsion τ by using hyperbolic functions as ansatz.

Case 1: Consider the velocities $\{\nu, \omega, \mu\} = \left\{0, \tau, \frac{\tau_s}{\kappa}\right\}$ in (2.5), then the evolution equation (2.5) becomes

$$\begin{aligned} \kappa_t &= \kappa\tau, \\ \tau_t &= -\left(\frac{\tau_s}{\kappa}\right)_s - \tau^2. \end{aligned} \tag{5.1}$$

To obtain a solitary wave solution of (5.1), we assume the ansatz of the form:

$$\begin{aligned} \kappa(s,t) &= \lambda_1 \coth^{p_1}\xi, \\ \tau(s,t) &= \lambda_2 \coth^{p_2}\xi, \end{aligned} \tag{5.2}$$

where $\xi = \eta(s - \rho t)$. Here λ_1, λ_2 and η are constants and ρ is the velocity of the solitary wave (cf. [9]). If we substitute (5.2) into (5.1), then we have

$$-\lambda_1 p_1 \rho\eta \coth^{-1+p_1}\xi + \lambda_1 p_1 \rho\eta \coth^{1+p_1}\xi - \lambda_1\lambda_2 \coth^{p_1+p_2}\xi = 0, \tag{5.3}$$

$$\begin{aligned} &-\lambda_2 p_2 \rho\eta \coth^{-1+p_2}\xi + \lambda_2^2 \coth^{2p_2}\xi + \lambda_2 p_2 \rho\eta \coth^{1+p_2}\xi \\ &+\frac{1}{\lambda_1}\{(-\lambda_2 p_2 \eta^2 - \lambda_2 p_1 p_2 \eta^2 + \lambda_2 p_2^2 \eta^2) \coth^{-2+p_1+p_2}\xi \\ &+(2\lambda_2 p_1 p_2 \eta^2 - 2\lambda_2 p_2^2 \eta^2 + \lambda_2 p_2^2 \eta^2) \coth^{-p_1+p_2}\xi \\ &+(\lambda_2 p_2 \eta^2 - \lambda_2 p_1 p_2 \eta^2 + \lambda_2 p_2^2 \eta^2) \coth^{2-p_1+p_2}\xi\} = 0. \end{aligned} \tag{5.4}$$

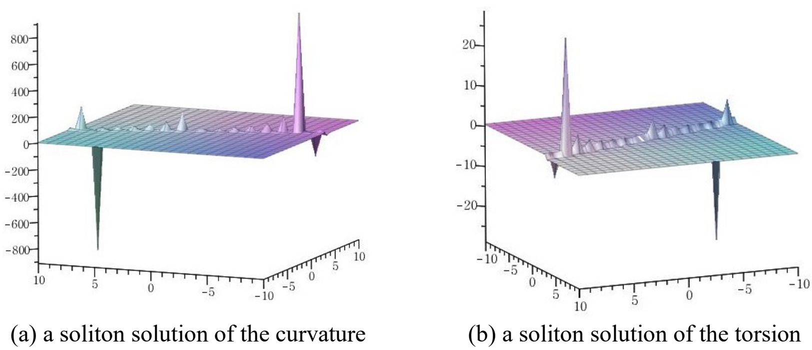
Now, equating the exponent $(1 + p_1, p_1 + p_2)$ in (5.3) and equating the exponent $(2 - p_1 + p_2, 1 + p_2)$ in (5.4) lead to

$$p_1 = p_2 = 1.$$

Since the coefficients of $\coth^{p_1+p_2+j}\xi$ in (5.3) and (5.4) must be zero, one finds $\rho = -\frac{\eta}{\lambda_1}$ or $\rho = \frac{\lambda_2}{\eta}$, where $\eta^2 = -\lambda_1\lambda_2$. Thus, we obtain the following singular soliton solutions

$$\begin{aligned} \kappa(s,t) &= \lambda_1 \coth(\eta(s - \rho t)), \\ \tau(s,t) &= \lambda_2 \coth(\eta(s - \rho t)). \end{aligned} \tag{5.5}$$

In Figure 3, we illustrate the singular soliton solutions of the curvature and the torsion in (5.5) with $\lambda_1 = -5$ and $\lambda_2 = 0.5$, respectively.



(a) a soliton solution of the curvature (b) a soliton solution of the torsion

Figure 3

A soliton solution of integrable system with the velocities $\{\nu, \omega, \mu\} = \{0, \tau, \frac{\tau_s}{\kappa}\}$

Case 2: Take the velocities $\{\nu,\omega,\mu\}=\left\{\kappa,\kappa_s,\frac{\kappa_{ss}+\kappa\tau}{\kappa}\right\}$ in (2.5), the evolution equations of the null curve lead to

$$\begin{aligned}\kappa_t &= -\kappa_s+\kappa\kappa_s\\ \tau_t &= -\left(\frac{\kappa_{ss}+\kappa\tau}{\kappa}\right)_s-\tau\kappa_s.\end{aligned}\tag{5.6}$$

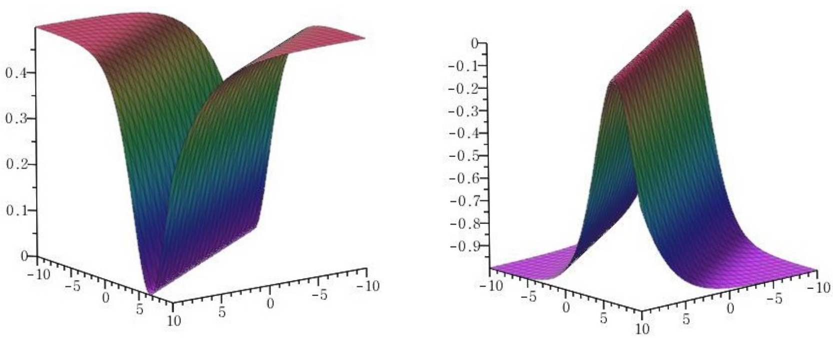
Then, the solutions of (5.6) become

$$\begin{aligned}\kappa(s,t) &= \lambda_3\tanh^{p_3}\xi,\\ \tau(s,t) &= \lambda_4\tanh^{p_4}\xi,\end{aligned}\tag{5.7}$$

where $\xi=\eta(s-\rho t)$. Here λ_3,λ_4 and η are constants and ρ is the velocity of the solitary wave (cf. [9]). Substituting (5.7) into (5.6) yields

$$\begin{aligned}&\lambda_3p_3\eta(1-\rho)\tanh^{p_3-1}\xi+\lambda_3p_3\eta(-1+\rho)\tanh^{p_3+1}\xi\\ &-\lambda_3^2p_3\eta\tanh^{-1+2p_3}\xi+\lambda_3^2p_3\eta\tanh^{1+2p_3}\xi=0,\end{aligned}\tag{5.8}$$

$$\begin{aligned}&\lambda_3\lambda_4p_3\eta(-\rho+1)\tanh^{-1+p_4}\xi+\lambda_3\lambda_4p_4\eta(\rho+1)\tanh^{1+p_4}\xi\\ &+(\lambda_3^2\lambda_4p_3\eta)\tanh^{-1+p_3+p_4}\xi+(-\lambda_3^2\lambda_4p_3\eta)\tanh^{1+p_3+p_4}\xi\\ &+\lambda_3p_3^2(-1+p_3)\eta^3(-3+p_3)\tanh^{-3}\xi+\lambda_3(-1+p_3)p_3\eta^3(-3\lambda_3(-2+p_3)\\ &-4\lambda_3+3p_3)\tanh^{-1}\xi-2\lambda_3p_3\eta^3\tanh\xi\\ &+\lambda_3p_3\eta^3(1-4p_3-p_3^2)\tanh^3\xi=0.\end{aligned}\tag{5.9}$$



(a) a soliton solution of the curvature (b) a soliton solution of the torsion

Figure 4

A shock wave solution of integrable system with the velocities $\{\boldsymbol{\nu},\boldsymbol{\omega},\boldsymbol{\mu}\}=\left\{\boldsymbol{\kappa},\boldsymbol{\kappa_s},\frac{\kappa_{ss}+\kappa\tau}{\kappa}\right\}$

Thus, equating the exponent $(1+p_1,-1+2p_1)$ in (5.8) and equating the exponent $(1+p_2,3)$ in (5.9) lead to

$$p_3=p_4=2.$$

Since the coefficients of $\tanh^{p_1+p_2+j}\xi$ in (5.8) and (5.9) must be zero, we can obtain $\rho = \lambda_3 + 1$ and $\eta = \frac{\lambda_4(\lambda_3+1)}{-8}$. Thus, we have the following shock soliton solutions

$$\begin{aligned} \kappa(s,t) &= \lambda_3 \tanh^2(\eta(s-\rho t)), \\ \tau(s,t) &= \lambda_4 \tanh^2(\eta(s-\rho t)). \end{aligned} \tag{5.10}$$

In Figure 4, we illustrate the shock wave solutions of (5.10) with $\lambda_3 = 0.5$ and $\lambda_4 = -1$.

6. Conclusion

The evolution of curve-based geometric flows offers one of the most intriguing links between such flows and soliton solutions. In this paper, we study time evolution equations of geometric flow $\frac{\partial^2\gamma}{\partial t^2} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$ in terms of an acceleration vector field of a null curve γ in Minkowski 3-space. In particular, we consider the null binormal acceleration (NBA) flow $\frac{\partial^2\gamma}{\partial t^2} = \kappa\mathbf{e}_3$ along a null curve γ and investigate the NBA flow along a planar null curve. As a result of time evolution equations of the NBA flow, we give some examples of soliton solutions of the NBA flow, and construct Bäcklund transformations of the NBA flows. Finally, we discuss integrable system in the curvature κ and the torsion τ of the null curve and we know that hyperbolic functions are special solutions of the integrable system along the null curve in Minkowski 3-space.

Declaration of Competing Interest

The authors report no declarations of interest.

Acknowledgements

D.W. Yoon was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C101043211).

References

[1] N.H. Abdel-All and M.T. Al-Dossary, "Motion of curves specified by acceleration field in $\mathbb{R}^n$," *Applied Mathematical Sciences*, vol. 7, no. 69, pp. 3403-3418 (2013).

[2] S.C. Anco and E. Asadi, "Hasimoto variables, generalized vortex filament equations, Heisenberg models and Schrödinger maps arising from group-invariant NLS systems," *Journal of Geometry and Physics*, vol. 144, pp. 324-357 (2019).

[3] M.E. Aydin, A. Mihai, A.O. Ogrenmis and M. Ergut, "Geometry of the solutions of localized induction equation in the pseudo-Galilean space," *Advances in Mathematical Physics*, Article ID 905978, p. 7 (2015).

[4] Q. Ding and J. Inoguchi, "Schrodinger flows, binormal motion for curves and the second AKNS-hierarchies," *Chaos Solitons Fractals,* vol. 21, no. 3, pp. 669--677 (2004).

[5] Q. Gamayun and Q. Lisovyy, "On self-similar solutions of the vortex filament equation," *Journal of Mathematical Physics,* vol. 60, no. 4, 083510 (2019).

[6] M. Grbovic' and E. Nešovic', "On Bäcklund transformation and vortex filament equation for null Cartan curve in Minkowski 3-space," *Mathematical Physics, Analysis and Geometry,* vol. 19, Article number: 23 (2016).

[7] N. E. Gürbüz and D. W. Yoon, "Visco-Da Rios equation in 3-dimensional Riemannian manifold," *Frontiers in Physics*, vol. 9, Article 810920 (2022).

[8] N.E. Gürbüz, Z.K. Yuzbasi and D.W. Yoon, "Hasimoto Maps for Nonlinear Schrödinger Equations in Minkowski Space," *Journal of Nonlinear Mathematical Physics,* vol. 29, pp. 761-775 (2022).

[9] M. Inc, E. Ulutas, E.C. Aslan and A. Biswas, "Singular 1-Soliton Solution of the K(m, n) Equation with Generalized Evolutions and its Subsidiaries," *Acta Physica Polonica B*, vol. 44, no. 9, pp. 1825-1836 (2013).

[10] M. Kass, A. Witkin and D. Terzopoulos, "Snakes: active contour models," in: Proc. 1st Int. Conference on Computer Vision, pp. 259-268 (1987).

[11] T. Körpinar and E. Turhan, "Inextensible flows of biharmonic S-curves according to Sabban frame in Heisenberg group Heis_3," *Journal of Interdisciplinary Mathematics*, vol. 21, no. 1, pp. 17-27 (2018).

[12] G. Muiraja and G. Lakshmanan, "Motion of space curves in 3-dimensional Minkowski space $\mathbb{R}_1^3$, $SO(2,1)$ spin equation and defocusing nonlinear Schrödinger equation," *International Journal of Geometric Methods in Modern Physics*, vol. 7, no. 6, pp. 1043-1049 (2010).

[13] T. Tsurumi, H. Segur, K. Nakayama and M. Wadati, "Motion of curves specified by accelerations," *Physics Letters A*, vol. 224, no.4-5, pp. 253-263 (1997).

[14] A. Yavuz and M. Erdoğdu, "Congruence of generate surface along pseudo null curve and Landau-Lifshitz equation," *Journal of Geometry and Physics,* vol. 178, p. 104553 (2022).

[15] D.W. Yoon, Z.K. Yüzbas and E.C. Aslan, "Evolution of space-likes and special time-like ruled surfaces in the Minkowski space," *Indian Journal of Physics,* vol. 96, pp. 995-999 (2022).

[16] Z.K. Yüzbas and S.C. Anco, "Elastic null curve flows, nonlinear C-integrable systems, and geometric realization of Cole-Hopf transformations," *Journal of Nonlinear Mathematical Physics,* vol. 27, pp. 357-392 (2020).

[17] Z.K. Yüzbas, "A new approach to the connections between moving curves and a family of complex KdV type systems," *Mathematical Methods in the Applied Sciences*, vol. 46, no. 4, pp. 4279-4288 (2023).

Received December, 2023