

Generalized B-W-F-contraction in cone-b metric space with application

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Abstract

This manuscript investigates a new generalized F-contraction of Boyd-Wong type with respect to cone-b metric space. First, we prove a main theoretical result using this new contraction, then to support the theoretical findings, an illustration is also presented. In addition, an application is also provided for the solution of dynamic programming problems.

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1. Introduction

Fixed Point Theory (FPT) is an area of keen interest for many researchers. The field has evolved a lot since 1922, especially following the development of Banach Contraction Principle (BCP). FPT finds its broad relevance in the field of computer science, physics, and applied mathematics [14]. Czerwik [1] first proposed the idea of a b-metric space $\mathfrak{b}$ -MS, extending the notion of a metric space (MS). In their research work, Long-Guang and Xian [6] established the cone-metric space (ρ_c -MS) by generalizing the MS. They introduced the concept of real Banach space to replace the set of real numbers. Subsequently, Hussain and Shah [5], developed cone-b metric spaces ($\rho_{c,b}$ -MS) as a further generalization of $\mathfrak{b}$ -MS and ρ_c -MS. Moreover, Krishnakumar and Dhamodharan [7] presented common fixed-point theorems for $\rho_{c,b}$ -MS by means of cone C-class function. Recently, [11] proved (FP) results in cone m-hemi metric space

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via various mappings such as Meir-Keeler and Boyd-Wong. Following this, many significant generalizations were introduced, one of which is presented in [13].

On the other hand, In 2012, Wardowski [4] pioneered the field by formulating F-contraction and established fixed-point result as a mere generalization of BCP, which was further modified by many authors (see [2], [3], [8]). The objective of this manuscript is to study the results of generalized F-contraction regarding the cone-b metric space. To validate the versatility of the result, an illustration is presented in the section below along with an application. we often use $\mathfrak{R}, \mathfrak{N}, \mathfrak{R}^+$ to represent standard notations in number system.

Definition 1.1: [6] Consider Ω be real Banach space also $\dot{\mathcal{B}} \subseteq \Omega$. Their exist zero element of Ω which is denoted by ϑ and int $\dot{\mathcal{B}}$ is denoted as interior of $\dot{\mathcal{B}}$. Then $\dot{\mathcal{B}}$ is referred as cone if,

1. $\dot{\mathcal{B}} \neq \{\emptyset\}, \{\dot{\epsilon}, \vartheta\} \subseteq \Omega$;
2. $\dot{\mathcal{B}} \cap (-\dot{\mathcal{B}}) = \vartheta$.
3. $b, c \in \mathfrak{R}; b, c \geq 0; \lambda, \wp \in \dot{\mathcal{B}} \implies b * \lambda + c * \wp \in \dot{\mathcal{B}}$;

Partial ordering is defined by $\preceq$ w.r.t λ by $\lambda \preceq \wp$ iff $\lambda - \wp \in \dot{\mathcal{B}}$. The notation $\lambda = \wp$ is used to indicate that $\wp - \lambda \in \text{int} \dot{\mathcal{B}}$.

Definition 1.2: [9] Suppose $\mathbb{A} \neq \{\emptyset\}$, and $\exists$ a constant $s > 1$ where Ω be a Banach space. A function $\rho_{c,b}: \mathbb{A} \times \mathbb{A} \rightarrow \Omega$ is cone-b metric if $\forall \pi, \mathfrak{f}, \sigma \in \mathbb{A}$,

1. $\rho_{c,b}(\pi, \mathfrak{f}) = 0$ if and only if $\pi = \mathfrak{f}$;
2. $\rho_{c,b}(\pi, \mathfrak{f}) = \rho_{c,b}(\mathfrak{f}, \pi)$;
3. $\rho_{c,b}(\pi, \sigma) \preceq s[\rho_{c,b}(\pi, \mathfrak{f}) + \rho_{c,b}(\mathfrak{f}, \sigma)]$;

A combination $(\mathbb{A}, \rho_{c,b})$ is referred to as $\rho_{c,b}$ -MS.

Lemma 1.3: [9] For a given $\rho_{c,b}$ -MS, $(\mathbb{A}, \rho_{c,b})$ with a sequence $\{v_n\}$. Therefore:

1. $\{v_n\}$ is defined as Cauchy sequence, if for $\rho \in \Omega$ with $\vartheta \ll \rho$, $\exists \mathfrak{N}$ where $n, m > \mathfrak{N}$, $\rho_{c,b}(v_n, v_m) \ll \rho$;
2. $\{v_n\}$ is said to be convergent sequence, if for every ρ in Ω with $\vartheta = \rho$, there $\exists$ some fixed $\mathfrak{N} \in \mathbb{A}$ so that for $n > \mathfrak{N}$, $\rho_{c,b}(v_n, \varsigma) \ll \rho$.
3. The $\rho_{c,b}$ -MS is complete provided that each cauchy sequence converges to some point of $\mathbb{A}$.

Lemma 1.4: [12] Consider a MS $(\mathbb{A}, \rho_{c,b})$. Consider a sequence $\{u_n\}$ in $\mathbb{A}$ so that $\rho_{c,b}(u_n, u_{n+1}) \rightarrow 0$ whenever $n \rightarrow \infty$. If possible suppose, $\{u_n\}$ is not cauchy then for $\varepsilon > 0$ along with two increasing sequences m_t and n_t taking $m_t > n_t > \mathfrak{k}$ so that $\rho_{c,b}(u_{m_t}, u_{n_t}) \geq \delta$, $\rho_{c,b}(u_{m_{t-1}}, u_{n_t}) < \delta$ and

$$\lim_{t \rightarrow \infty} \rho_{c,b}(u_{m_{t-1}}, u_{n_{t+1}}) = \delta \lim_{t \rightarrow \infty} \rho_{c,b}(u_{m_t}, u_{n_t}) = \delta \lim_{t \rightarrow \infty} \rho_{c,b}(u_{m_{t-1}}, u_{n_k}) = \delta.$$

Definition 1.5: [4] Consider $F: \mathfrak{R}^+ \rightarrow \mathfrak{R}$ be a function that satisfies:

1. or some $t, \omega \in \mathfrak{R}^+$, given $t < \omega \implies F(t) < F(\omega)$ which implies that F is monotonically increasing;
2. also $\exists$ a sequence $\{\mathfrak{I}_n\}_{n=1}^\infty$ of positive reals, so that $\lim_{n \rightarrow \infty} F(\mathfrak{I}_n) = -\infty$;
3. $\exists \mathfrak{p} \in (0,1)$, such that $\lim_{n \rightarrow \infty} \mathfrak{I}^{\mathfrak{p}} F(\hat{\mathfrak{u}}) = 0$.

The set of all functions satisfying 1-3 is represented by $\mathcal{F}$. Further, Huang and Xu [9] improvised the condition 3 by,

1. F is continuous on $(0, \infty)$.
we represent the condition 1, 2 and 3, together by $\Delta_{\mathcal{F}}$

Definition 1.6: [3] Let $\Phi = \{\phi: (0, \infty) \rightarrow (0, \infty); t_1 \leq t_2 \implies \phi(t_1) \leq \phi(t_2)\}$
 $\phi(t_1) < t_1$ for any $t_1 > 0$, ϕ is given continuous.

Definition 1.7: [3] A function Ψ is expressed as $\{\psi: (0, \infty) \rightarrow (0, \infty)\}$.

2. Fixed Point results

Theorem 2.1: Assume $(\mathbb{A}, \rho_{c,b})$ as a cone-b MS and $G: \mathbb{A} \rightarrow \mathbb{A}$ is satisfying generalized B-W-F-contraction if $\exists \mathcal{F} \in \Delta_{\mathcal{F}}$, and functions $\psi \in \Psi$ as well as $\phi \in \Phi$ for every pair of points $q, \tau \in \mathbb{A}$.

$$\psi(\rho_{c,b}(q, \tau) + \mu(q, \tau)\mathcal{F}(s \cdot \rho_{c,b}(Gq, G\tau))) \preceq \mathcal{F}(\phi(\mathcal{M}(q, \tau))) \quad (1)$$

in which,

$$\mathcal{M}(q, \tau) = \max\{\rho_{c,b}(q, \tau), \rho_{c,b}(q, Gq), \rho_{c,b}(\tau, G\tau), \frac{1}{2}[\rho_{c,b}(q, G\tau) + \rho_{c,b}(\tau, Gq)]\} \quad (2)$$

Consequently, G possesses a single fixed-point $\mathfrak{ij} \in \mathbb{A}$.

Proof: Assume q_0 be any element of $\mathbb{A}$ and suppose there is a sequence $\{q_i\}$ such that

$$q = q_0 \text{ and } q_{\xi+1} = Gq_{\xi} \text{ for every } \xi \in \mathfrak{N}$$

If we have, $\xi_0 \in \mathfrak{N}$ provided that $q_{\xi_0} = q_{\xi_0+1}$ then $Gq_{\xi_0} = q_{\xi_0}$. Thus, q_{ξ_0} is indeed the fixed point of G .

consequently, assume $q_{\xi} \neq q_{\xi+1}$ or $\rho_{c,b}(Gq_{\xi-1}, Gq_{\xi}) > 0$ for every $\xi \in \{1, 2, 3 \dots\}$

Further utilising (1)

$$\begin{aligned} \psi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) + \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) &= \psi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) \\ &\quad + \mathcal{F}(s \cdot \rho_{c,b}(Gq_{\xi-1}, Gq_{\xi})) \\ &\preceq \mathcal{F}(\phi(\mathcal{M}(q_{\xi-1}, q_{\xi}))) \end{aligned} \quad (3)$$

where,

$$\begin{aligned}
\mathcal{M}(q_{\xi-1}, q_{\xi}) &= \max\{\rho_{c,b}(q_{\xi-1}, q_{\xi}), \rho_{c,b}(q_{\xi-1}, Gq_{\xi-1}), \rho_{c,b}(q_{\xi}, Gq_{\xi}), \\
&\quad \frac{1}{2}[\rho_{c,b}(q_{\xi-1}, Gq_{\xi}) + \rho_{c,b}(q_{\xi}, Gq_{\xi-1})]\} \\
&= \max\{\rho_{c,b}(q_{\xi-1}, q_{\xi}), \rho_{c,b}(q_{\xi-1}, q_{\xi}), \frac{1}{2}[\rho_{c,b}(q_{\xi-1}, q_{\xi+1}) + \rho_{c,b}(q_{\xi}, q_{\xi+1})]\} \quad (4) \\
&= \max\{\rho_{c,b}(q_{\xi-1}, q_{\xi}), \rho_{c,b}(q_{\xi}, q_{\xi+1})\} \quad (5)
\end{aligned}$$

For some $\xi \in \mathfrak{N}$, $\mathcal{M}(q_{\xi-1}, q_{\xi}) = \rho_{c,b}(q_{\xi}, q_{\xi+1}) \forall \xi \in \mathfrak{N}$ it follows from (3) this shows that,

$$\psi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) + \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) \preceq \mathcal{F}(\phi(\rho_{c,b}(q_{\xi}, q_{\xi+1})))$$

Based on property of ϕ, ψ , and (F1)

$$\psi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) + \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) < \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1}))$$

As $\psi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) > 0$, we arrive at a contradiction. Therefore,

$\mathcal{M}(q_{\xi-1}, q_{\xi}) = \rho_{c,b}(q_{\xi-1}, q_{\xi}) \forall \xi \in \mathfrak{N}$ From (3), making use of property (F1), ψ , we get

$$\begin{aligned}
\mathcal{F}(\rho_{c,b}(q_{\xi}, q_{\xi+1})) &\preceq \mathcal{F}(\phi(\rho_{c,b}(q_{\xi-1}, q_{\xi})) - \psi(\rho_{c,b}(q_{\xi-1}, q_{\xi}))) \\
&\preceq \mathcal{F}(\phi(\rho_{c,b}(q_{\xi-1}, q_{\xi}))) \\
\mathcal{F}(\rho_{c,b}(q_{\xi}, q_{\xi+1})) &< \mathcal{F}(\rho_{c,b}(q_{\xi-1}, q_{\xi})) \quad (6)
\end{aligned}$$

which yields,

$$\rho_{c,b}(q_{\xi}, q_{\xi+1}) < \rho_{c,b}(q_{\xi-1}, q_{\xi}) \forall \xi \in \mathfrak{N}$$

Consequently, $\rho_{c,b}(q_{\xi}, q_{\xi+1})$ is a decreasing sequence in $\mathfrak{N}^+$. As a result, using the property of ϕ and (6) gives,

$$\begin{aligned}
\mathcal{F}(s \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) &< \mathcal{F}(\rho_{c,b}(q_{\xi-1}, q_{\xi}) - \psi(\rho_{c,b}(q_{\xi-1}, q_{\xi}))) \\
\mathcal{F}(s^2 \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) &< \mathcal{F}(\rho_{c,b}(q_{\xi-2}, q_{\xi-1})) - 2\psi(\rho_{c,b}(q_{\xi-2}, q_{\xi-1}))
\end{aligned}$$

Also,

$$\mathcal{F}(s^n \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) < \mathcal{F}(\rho_{c,b}(q_0, q_0)) - 2\xi\psi(\rho_{c,b}(q_0, q_1)) \quad (7)$$

Given that, $\mathcal{F} \in \Delta\mathcal{F}$, when $\xi \rightarrow \infty$ in (7) we obtain,

$$\lim_{\xi \rightarrow \infty} \mathcal{F}(s^{\xi} \cdot \rho_{c,b}(q_{\xi}, q_{\xi+1})) = -\infty \implies \lim_{\xi \rightarrow \infty} s^{\xi} \rho_{c,b}(q_{\xi}, q_{\xi+1}) = 0 \quad (8)$$

To demonstrate $\{q_{\xi}\}$ is cauchy sequence in $\mathbb{A}$, assuming the contradiction condition, $\{q_{\xi}\}$ is not cauchy in $(\mathbb{A}, \rho_{c,b})$ then one can find $\varepsilon > 0$ for a given integer $\mathfrak{k}$, $q_{\xi_{\mathfrak{k}}} > q_{\zeta_{\mathfrak{k}}} \geq \mathfrak{k}$ such that, $\rho_{c,b}(q_{\zeta_{\mathfrak{k}}}, q_{\xi_{\mathfrak{k}}}) \geq \varepsilon$, which yields,

$$\rho_{c,b}(q_{\zeta_{\mathfrak{k}-1}}, q_{\xi_{\mathfrak{k}}}) < \varepsilon \quad (9)$$

also by lemma, (1.4),

$$\lim_{\mathfrak{k} \rightarrow \infty} \rho_{c,b}(q_{\zeta_{\mathfrak{k}}}, q_{\xi_{\mathfrak{k}}}) = \varepsilon, \lim_{\mathfrak{k} \rightarrow \infty} \rho_{c,b}(q_{\xi_{\mathfrak{k}-1}}, q_{\xi_{\mathfrak{k}+1}}) = \varepsilon, \lim_{\mathfrak{k} \rightarrow \infty} \rho_{c,b}(q_{\zeta_{\mathfrak{k}-1}}, q_{\xi_{\mathfrak{k}}}) = \varepsilon \quad (10)$$

For some $\xi_0 \in \mathfrak{N}$,

$$\rho_{c,b}(q_{\xi_{\mathfrak{k}}}, q_{\xi_{\mathfrak{k}+1}}) = \frac{\varepsilon}{4}, \rho_{c,b}(q_{\zeta_{\mathfrak{k}}}, q_{\zeta_{\mathfrak{k}+1}}) = \frac{\varepsilon}{4}$$

Now we assert that

$$\rho_{c,b}(Gq_{\xi_t}, Gq_{\zeta_t}) = \rho_{c,b}(q_{\xi_{t+1}}, q_{\zeta_{t+1}}) > 0 \quad (11)$$

On contrary, assume that $\rho_{c,b}(q_{\xi_{t+1}}, q_{\zeta_{t+1}}) = 0$ Then,

Which is not true. Therefore $\rho_{c,b}(q_{\xi_{t+1}}, q_{\zeta_{t+1}}) > 0$

Then, we observe by the property of ψ that

$$\begin{aligned} \psi(\rho_{c,b}(q_{\xi_t}, q_{\zeta_t})) + \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi_{t+1}}, q_{\zeta_{t+1}})) &\preceq \psi(\rho_{c,b}(q_{\xi_t}, q_{\zeta_t})) \\ &\preceq \mathcal{F}(\phi(\mathcal{M}(q_{\xi_t}, q_{\zeta_t}))) \\ \mathcal{F}(s \cdot \rho_{c,b}(q_{\xi_{t+1}}, q_{\zeta_{t+1}})) &\preceq \mathcal{F}(\phi(\mathcal{M}(q_{\xi_t}, q_{\zeta_t}))) \\ \mathcal{M}(q_{\xi_t}, q_{\zeta_t}) &= \max\{\rho_{c,b}(q_{\xi_t}, q_{\zeta_t}), \rho_{c,b}(q_{\xi_t}, Gq_{\xi_t}), \rho_{c,b}(q_{\zeta_t}, Gq_{\zeta_t}), \\ &\quad \frac{1}{2}[\rho_{c,b}(q_{\xi_t}, Gq_{\zeta_t}) + \rho_{c,b}(q_{\zeta_t}, Gq_{\xi_t})]\} \end{aligned} \quad (12)$$

By definition of $\mathcal{M}(q, \tau)$, (11), (12)

$$\lim_{t \rightarrow \infty} \mathcal{M}(q_{\xi_t}, q_{\zeta_t}) = \varepsilon \quad (13)$$

Letting $t \rightarrow \infty$ in (13) and through (8), (10), (11), (12)

We arrive at, $\mathcal{F}(\varepsilon) \preceq \mathcal{F}(\phi(\varepsilon)) < \mathcal{F}(\varepsilon)$, that leads to inconsistency. Hence, the sequence is cauchy. Since $\mathbb{A}$ is complete, it guarantees the existence of a point $\bar{y}$.

$$\lim_{\xi, \zeta \rightarrow \infty} \rho_{c,b}(q_{\xi}, q_{\zeta}) = \lim_{\xi \rightarrow \infty} \rho_{c,b}(q_{\xi}, \bar{y}) = 0 \quad (14)$$

Infact, $\bar{y}$ is the F. P of G . Since $\mathcal{F} \in \Delta\mathcal{F}$, it is clear from (14), $(\mathcal{F}_3)$, $\mathcal{F}$ is continuous. This leads to two possible cases:

Case (I): For every k where $k \in \mathfrak{N}$, one can find $\xi \in \mathfrak{N}$, so that $\rho_{c,b}(q_{\xi+1}, G\bar{y}) = 0$ i.e $q_{\xi+1} = G\bar{y}$ where, $q_{\xi} > q_{\xi-1}$, we get $\bar{y} = \lim_{\xi \rightarrow \infty} q_{\xi+1} = \lim_{\xi \rightarrow \infty} G\bar{y} = G\bar{y}$

Hence, the fixed point of G is $\bar{y}$

Case (II): $\exists k_2 \in \mathfrak{N}$, $\rho_{c,b}(Gq_{\xi}, G\bar{y}) \neq 0 \forall \xi \geq k_2$

So,

$$\begin{aligned} \mathcal{F}(\rho_{c,b}(G\bar{y}, q_{\xi+2})) &\preceq \mu(\bar{y}, q_{\xi+1})\mathcal{F}(s \cdot \rho_{c,b}(G\bar{y}, Gq_{\xi+1})) \\ &\preceq \mathcal{F}(\mathcal{M}(\bar{y}, q_{\xi+1})) - \psi(\rho_{c,b}(\bar{y}, q_{\xi+1})) \end{aligned} \quad (15)$$

where

$$\begin{aligned} \mathcal{M}(\bar{y}, q_{\xi+1}) &= \max\{\rho_{c,b}(\bar{y}, q_{\xi+1}), \rho_{c,b}(\bar{y}, G\bar{y}), \rho_{c,b}(q_{\xi+1}, q_{\xi+2}), \\ &\quad \frac{1}{2}[\rho_{c,b}(\bar{y}, q_{\xi+2}) + \rho_{c,b}(q_{\xi+1}, G\bar{y})]\} \end{aligned} \quad (16)$$

If $\rho_{c,b}(\bar{y}, G\bar{y}) > 0$,

$$\lim_{\xi \rightarrow \infty} \rho_{c,b}(\bar{y}, q_{\xi+1}) = \lim_{\xi \rightarrow \infty} \rho_{c,b}(\bar{y}, q_{\xi+2}) = \lim_{\xi \rightarrow \infty} \rho_{c,b}(q_{\xi+1}, q_{\xi+2}) = 0$$

Since $\exists, \xi_3 \in \mathfrak{N}$ for all $\xi \geq \xi_3$, so that

$$\max\{\rho_{c,b}(\mathfrak{j}, q_{\xi+1}), \rho_{c,b}(\mathfrak{j}, G\mathfrak{j}), \rho_{c,b}(q_{\xi+1}, q_{\xi+2}), \frac{1}{2}[\rho_{c,b}(\mathfrak{j}, q_{\xi+2}) + \rho_{c,b}(q_{\xi+1}, G\mathfrak{j})]\} = \rho_{c,b}(\mathfrak{j}, G\mathfrak{j}) \quad (17)$$

Employing (15) and (17),

$\mathcal{F}(\rho_{c,b}(G\mathfrak{j}, q_{\xi+2})) \preceq \mathcal{F}(\phi(\rho_{c,b}(\mathfrak{j}, G\mathfrak{j}))) \quad \forall \xi \geq \{\xi_2, \xi_3\}$ as $\xi \rightarrow \infty$ in above equation, which implies

$\mathcal{F}(\rho_{c,b}(G\mathfrak{j}, \mathfrak{j})) < \mathcal{F}(\rho_{c,b}(G\mathfrak{j}, \mathfrak{j}))$, this gives rise to a contradiction. Therefore, G admits a fixed point $\mathfrak{j}$.

Uniqueness: The uniqueness can be established trivially by considering two distinct points. we arrive at the contradiction in the end which proves our result.

Example 2.2: Consider $\Omega = \mathfrak{R}^2$ and $\alpha > 1$ is a constant. Assume $\mathcal{C} = \{(\dot{q}, \dot{t}) \in \mathfrak{R}^2; \dot{q}, \dot{t} \geq 0, \dot{q} \neq \dot{t}\} \subset \Omega$, $\mathbb{A} \in [0, 1]$ and $\rho_{c,b}: \mathbb{A} \times \mathbb{A} \rightarrow \Omega$ then

$$\rho_{c,b}(\dot{q}, \dot{t}) = \{|\dot{q} - \dot{t}|^\alpha, |\dot{q} - \dot{t}|\}^\alpha \quad \forall \dot{q}, \dot{t} \in \mathbb{A}$$

Apparently, the space $(\mathbb{A}, \rho_{c,b})$ is complete under cone-b metric. Define $G: \mathbb{A} \rightarrow \mathbb{A}$ as $G\dot{q} = \frac{1}{2}\dot{q} - \frac{1}{4}\dot{q}^2 \quad \forall \dot{q} \in \mathbb{A}$, then G is continuous. Let $F(g) = \log(g)$, $\forall g \in \mathfrak{R}^+$. Consider a decreasing function ψ , where $\psi(k) = \frac{1}{100^{*(k+1)}}$ and let ϕ be given by $\phi(k) = \frac{120k+1}{123}$.

Proof: We can easily claim that the mapping, G defines a generalized Boyd-Wong type F -contraction on $\mathbb{A}$.

$$L.H.S = \frac{1}{100^{*(|\dot{q}-\dot{t}|^2+1)}}, R.H.S = \log\left(\frac{120\mathcal{M}(\dot{q}, \dot{t})+1}{123}\right).$$

In Figure 1, $L.H.S$ is shown by blue line however, the $R.H.S$ is represented by red line. As observed from the plot, contraction (1) holds for all $q, \tau \in \mathbb{A}$.

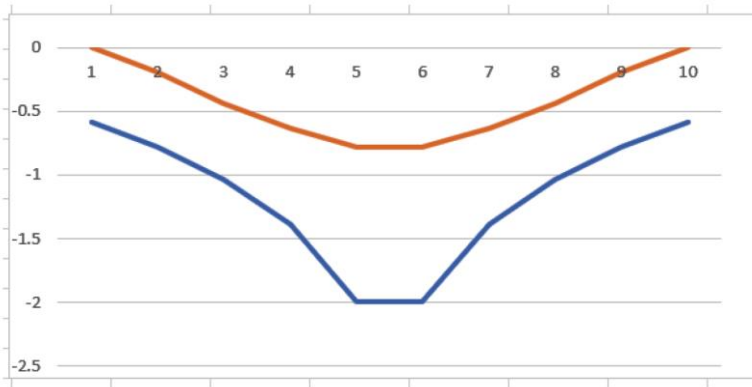


Figure 1
Validation of contraction

3. Application

This section investigated the existence and uniqueness of solution associated with functional equations that is commonly encountered in dynamic programming. Bellman and Lee [10] were the first to introduce the elementary version of functional equation that arises in dynamic programming. It is a multi-stage process including 'n' variables which is further sub-divided into m stages, each being a decision-making problem in one variable. Consider $\mathcal{B}(Q) = \{f: Q \rightarrow \mathfrak{R}\}$ represent all real-valued mappings over Q .

$$\|I\| = \sup\{I(r): r \in Q\}$$

The functional equation is given by,

$$q(r) = \sup_{w \in W} \{g(r, w) + \mathfrak{G}(r, w, q(d(r, w)))\} \quad (18)$$

where

r = state vector and w = decision vectors.

In this context, $q(r)$ refers to optimal return function;

d signifies the transformation of the process.

Also, $g: \mathcal{E} \times W \rightarrow \mathfrak{R}$, $\mathfrak{G}: Q \times W \times \mathfrak{R} \rightarrow \mathfrak{R}$ and $d: Q \times W \rightarrow Q$; $Q \subseteq \mathcal{L}$ and $W \subseteq \mathcal{E}$ are the state space and decision space respectively. $\mathcal{L}$ and $\mathcal{E}$ are the respective Banach spaces. Also, $\Gamma: \mathcal{B}(Q) \rightarrow \mathcal{B}(Q)$ For an arbitrary $l \in \mathcal{B}(Q)$,

$$\Gamma(r) = \sup_{w \in W} \{g(r, w) + \mathfrak{G}(r, w, \Gamma(d(r, w)))\} \quad (19)$$

Theorem 3.1: Consider Γ be an operator mentioned in (19), $g: Q \times W \rightarrow \mathfrak{R}$ and let $\mathfrak{G}: Q \times W \times \mathfrak{R} \rightarrow \mathfrak{R}$ then $\exists$ two continuous functions which are bounded.

Suppose, ψ is taken as a non-decreasing function, also ϕ is considered as upper-semi-continuous which satisfies,

1. for every $t_1 > 0$ one can find the function value $\phi(t_1) < t_1$;
2. $|\mathfrak{G}(r, w, I(r)) - \mathfrak{G}(r, w, \mathfrak{f}(r))| \preceq e^{-\psi(\|I - \mathfrak{f}\| + b)} \phi(\mathcal{M}(I, \mathfrak{f}))$, with

$$\mathcal{M}(I, \mathfrak{f}) = \max\{\rho_{c,b}(I, \mathfrak{f}), \rho_{c,b}(I, \Gamma I), \rho_{c,b}(\mathfrak{f}, \Gamma \mathfrak{f}), \frac{1}{2}[\rho_{c,b}(I, \Gamma \mathfrak{f}) + \rho_{c,b}(\mathfrak{f}, \Gamma I)]\}$$

$\forall I, \mathfrak{f} \in \mathcal{B}(Q), r \in Q$ and $w \in W$. Then the system has a unique solution.

4. Conclusion

Our findings have established fixed point results obtained by employing the extended version of Boyd-Wong F contraction in cone-b metric setting over Banach space. To validate the result, we have formulated an example that fulfills the hypothesis of our main results. As a consequence, we have investigated the existence and uniqueness in dynamic programming using the functional equation as an application.

References

- [1] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Math. Inform. Univ. Ostraviensis*, vol. 1, pp. 5–11 (1993).

- [2] D. Singh, V. Chauhan, P. Kumam, and V. Joshi, "Some applications of fixed point results for generalized two classes of Boyd-Wong's F-contraction in partial b-metric spaces," *Mathematical Sciences*, vol. 12, no. 2, pp. 111–127 (2018).
- [3] D. Singh, V. Chaouhan, R. Wangkeeree, P. Kumam, V. Joshi, and P. Thounthong, "Applications of fixed point results for cyclic Boyd-Wong type generalized F- ψ -contractions to dynamic programming," *J. Math. Comput. Sci.*, vol. 17, pp. 200–215 (2017).
- [4] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, pp. 1–6 (2012).
- [5] N. Hussain and M. H. Shah, "KKM mappings in cone b-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684 (2011).
- [6] H. Long-Guang and Z. Xian, "Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 332, no. 2, pp. 1468–1476 (2007).
- [7] R. Krishnakumar and D. Dhamodharan, "Cone C-class Function with Common Fixed Point Theorems for Cone b-metric Space," *Journal of Mathematics and Informatics*, vol. 8, pp. 83–94 (2017).
- [8] H. Piri and P. Kumam, "Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, p. 210 (2014).
- [9] H. Huang and S. Xu, "Fixed point theorems of contractive mappings in cone b-metric spaces and applications," *Fixed Point Theory and Applications*, vol. 2013, no. 1, p. 112 (2013).
- [10] R. Bellman and E. Lee, "Functional equations in dynamic programming," *Aequationes Mathematicae*, vol. 17, no. 1, pp. 1–18 (1978).
- [11] L. Guo, R. Bibi, A. Alshejari, E. Savas, T. Kamran, and U. Ishtiaq, "Fixed-Point Results for Krasnoselskii, Meir-Keeler, and Boyd-Wong-Type Mappings with Applications to Dynamic Market Equilibrium," *Axioms*, vol. 13, no. 12, p. 867 (2024).

- [12] GVR. Babu and PD. Sailaja, "A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces," *Thai Journal of Mathematics*, vol. 9, no. 1, pp. 1–10 (2011).
- [13] A. Raturi and S. S. Chauhan, "New results in bipolar controlled b dislocated metric space and its applications for solving fractional differential equations," *Journal of Interdisciplinary Mathematics*, vol. 27, no. 8, pp. 1797–1830 (2024).
- [14] M. Ghods and S. Ghobadi, "Fixed point results for single and multi-valued mappings with application in graph theory," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 8, pp. 2445–2455 (2022).

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