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Generalized average box dimensions of representative compact metric spaces

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Abstract

The significance of this study lies in its extension of Olsen's foundational result concerning the classical average box dimension [24], thus broadening its reach and relevance. Specifically, this paper explores the general average box dimensions of typical compact metric spaces ($\mathcal{TCM}$s) within the Gromov-Hausdorff space ($\mathcal{GH}$s), utilizing the Gromov-Hausdorff metric ($\delta^{\mathcal{GH}}$). We show that the general box dimensions demonstrate significant divergence, even after a "smoothing" effect is applied using sophisticated averaging methods. Finally, our main result is used to reveal that the general box dimensions related to higher-order Hölder and Cesàro averages (h-$\mathcal{HCA}$) exhibit considerable variation.

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1. Introduction

Classical geometry proves inadequate when it comes to describing irregular and fragmented shapes, leading to the development of fractal geometry, a mathematical framework designed to capture the complexity of structures that defy traditional concepts of smoothness and dimension. Unlike Euclidean objects, which are defined by integer dimensions and regular shapes, fractals are characterized by self-similarity and scale invariance, properties that persist across multiple levels of magnification. These features make conventional geometric measures insufficient for fully representing the nature of such objects [14].

The mid-20th century marks the beginning of fractal geometry with key contributions from mathematicians like Benoît B. Mandelbrot [20, 21].

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Mandelbrot played a pivotal role in formalizing the concept, which had been informally observed in nature and other fields for centuries. In his seminal work The Fractal Geometry of Nature, he introduced the idea that fractals are objects that exhibit intricate structure at every scale, challenging traditional geometric views. His research bridged the gap between mathematics and natural phenomena, showing how fractals could describe patterns found in coastlines, mountains, clouds, and biological structures. Since then, fractal analysis has become a crucial tool in various disciplines, including physics, computer science, biology, and even economics [13, 22, 38, 40]. The ability to quantify the complexity of irregular objects has provided new insights into systems that cannot be understood using traditional geometric approaches. As a result, fractal geometry has not only broadened our understanding of complex systems but has also transformed the way we model and analyze real-world phenomena.

Within this framework, the box-counting dimension has emerged as a key concept for quantifying the complexity of self-similar structures. Unlike Euclidean shapes, fractals exhibit intricate patterns that persist across scales, making traditional dimensional methods inadequate for their characterization. This concept has become fundamental in the analysis of fractals and in exploring the scaling and self-similarity properties inherent in many natural and artificial systems.

The box dimension is determined using the box-counting method, where a fractal is covered with boxes of side length r, and the number of required boxes N_r is analyzed as r decreases. It is defined, for some non-empty bounded set E, as $\dim_B(E) = \lim_{r\to 0} \frac{\log N_r(E)}{\log(1/r)}$, capturing how the number of covering elements scales with resolution. This measure often results in non-integer values, reflecting the fractal's degree of self-similarity and space-filling properties. The definition can be traced back to at least the 1930s. The box dimension is widely applied in physics, biology, and data science, from analyzing turbulence and chaotic systems to quantifying the complexity of natural patterns such as coastline irregularities, tumor growth, and network structures. By providing a rigorous way to describe fractal structures, the box dimension serves as a powerful tool in understanding complex systems. For a more comprehensive discussion on the box dimension, we direct the reader to [12].

Previous studies have focused on box dimensions, particularly in the context of $\mathcal{TCM}$s. Specifically, Rouyer [32] established that the lower box dimension of a $\mathcal{TCM}$s is the least attainable, while the upper box dimension is as large as possible. This phenomenon of duality is also observed in other research on typical compact sets. For instance, Gruber [15] demonstrated that the lower box dimension of a $\mathcal{TCM}$s attains its minimum, whereas the upper box dimension frequently reaches its highest value. You can also refer to Myjak and Rudnicki [23]. Furthermore, Olsen [24] noted that for a $\mathcal{TCM}$s in the $\mathcal{GH}$s, the box-counting function exhibits such irregularity that it continues to diverge, regardless of the averaging scheme used. A variety of scholars have explored the challenge of averaging certain dimensions, specific distances, and distinct measures, aiming to investigate the different methods by which these factors can be integrated or assessed in an averaged form. Their work contributes to a larger framework of understanding these variables, while also striving to establish robust methodologies that ensure their consistent application in both theoretical and practical domains. For readers eager to delve deeper into these studies, those with a strong interest in the topic may refer to seminal works, including [8, 9, 25, 26, 27, 28, 29, 37].

In a series of works [1, 2, 3, 4, 5, 6, 7, 10, 11, 18, 19, 33, 34, 35, 36, 41, 42], the authors have proposed and developed an original generalization of fractal dimensions, which serve as essential tools in the study of complex geometrical structures and scaling phenomena. Specifically, in [4], they derive estimates for the lower and upper box dimensions ($\mathcal{LUB}$) of standard measures, employing the framework of Baire category theory.

This paper offers a comprehensive exploration of the general box dimensions of a $\mathcal{TCM}$s within the $\mathcal{GH}$s, a framework that encompasses the entirety of compact metric spaces ($\mathcal{CMs}$) endowed with the $\delta^{\mathcal{GH}}$. The significance of this study lies in its extension of Olsen's seminal result, broadening its scope and applicability. By generalizing Olsen's findings, this work provides deeper insights into the underlying geometric structure of $\mathcal{CM}$s, offering a more robust understanding of their behavior in the Gromov-Hausdorff space. This extension not only enriches the theoretical landscape but also opens avenues for further research in the study of metric spaces and their dimensional properties, highlighting the profound implications for various areas of mathematical analysis and geometry.

Section 2 is dedicated to providing the necessary background. In particular, Subsection 2.1 presents a rigorous and nuanced formulation of the general $\mathcal{LUB}$ dimensions, elucidating their fundamental significance in encapsulating the scaling dynamics and intricate geometric structure of $\mathcal{CM}$s. By providing clear and rigorous formulations, this subsection lays the groundwork for a deeper understanding of the dimensional characteristics of spaces equipped with the $\delta^{\mathcal{GH}}$, setting the stage for the subsequent developments in the paper.

Subsection 2.2 serves as an introduction to the $\mathcal{GH}$s and the associated metric that defines the structure of this space. It meticulously outlines the foundational concepts necessary to grasp the intricate relationships between compact metric spaces within the Gromov-Hausdorff framework. This subsection establishes the theoretical background required to comprehend the complexities involved in the study of box dimensions within this context.

Section 3 is devoted to the statement of the main result, Theorem 1, which shows that for any $\mathcal{TCM}$s $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, the general box dimensions exhibit such a high degree of irregularity that no averaging procedure, however refined, can ensure convergence.

Section 4 shifts focus to the exploration of two distinct types of averages, $h - \mathcal{HCA}$. This section provides an in-depth examination of these averaging methods, elucidating their significance in the study of box dimensions. It highlights how these averages serve as tools for refining our understanding of the behavior of $\mathcal{CM}$s, especially when viewed through the lens of their dimensional properties. The discussion is carefully crafted to underscore the relevance of these higher-order averages within the broader framework of geometric and analytical studies.

Finally, Section 5 presents the rigorous proof of the main result of the paper, which represents an extension of Olsen's original findings. In this section, we meticulously demonstrate how our approach generalizes Olsen's result, elaborating on its implications for the theory of box dimensions in $\mathcal{CM}$s. The proof not only solidifies the theoretical advancements proposed in the paper but also provides a crucial contribution to the ongoing development of the mathematical understanding of $\mathcal{GH}$s and their associated dimensional properties. Through this final section, the paper concludes by reinforcing the importance of the extended result, opening new pathways for future research in the field.

2. Preliminaries

2.1 *General Box Dimensions*

Definition 1: Let $c \in (0, +\infty]$ and consider two functions $h: (0, c) \to [0, +\infty]$ and $g: (0, c) \to [0, +\infty]$ such that these functions meet the three conditions:

(1) continuous and strictly decreasing,

(2) $\lim_{x \to c} h(x) = 0$,

(3) $\lim_{x \to 0} h(x) = \lim_{x \to 0} g(x) = +\infty$.

We will now introduce the definitions of both the general $\mathcal{LUB}$ dimensions, with greater precision and depth.

Definition 2: Let $\Phi = (h, g)$, where h and g are two functions of Definition 1. The generalized lower and upper $\mathcal{LUB}$ dimensions of a non-empty, bounded metric space $\mathbb{M}$, denoted by $\underline{\dim}_B^{\Phi}(\mathbb{M})$ and $\overline{\dim}_B^{\Phi}(\mathbb{M})$, respectively, are defined as follows:

$$\underline{\dim}_B^{\Phi}(\mathbb{M}) = \underline{\lim}_{r \to 0} \frac{h\left(\frac{1}{N_r(\mathbb{M})}\right)}{g(r)}, \overline{\dim}_B^{\Phi}(\mathbb{M}) = \overline{\lim}_{r \to 0} \frac{h\left(\frac{1}{N_r(\mathbb{M})}\right)}{g(r)},$$

where $N_r(\mathbb{M})$ is defined as

$N_r(\mathbb{M}) = \inf\{{}^{\#}\{B_i(x, r)\}_i$ such that the collection $\{B_i(x, r)\}_i$ consists of closed balls in $\mathbb{M}$ whose union covers $\mathbb{M}\}$.

If $\underline{\dim}_B^{\Phi}(\mathbb{M}) = \overline{\dim}_B^{\Phi}(\mathbb{M})$, then the general box dimension of $\mathbb{M}$ is said to exist, and we denote this common value by $\dim_B^{\Phi}(\mathbb{M})$, i.e.,

$$\dim_B^{\Phi}(\mathbb{M}) = \underline{\dim}_B^{\Phi}(\mathbb{M}) = \overline{\dim}_B^{\Phi}(\mathbb{M}).$$

Remark 1: It is important to emphasize that the box dimensions represent a specific instance of the general box dimension. This generalization makes it possible to retrieve the classical box dimension by choosing, for example, $h(r) = g(r) = -\log(r)$. For a thorough and structured discussion on box dimensions, refer to [12].

2.2 *The* $\mathcal{GH}$*s and the* $\delta^{\mathcal{GH}}$

Definition 3: Let $\mathcal{K}^{\mathcal{GH}}$ denote the collection of all pre-$\mathcal{GH}$s structures of compact, non-empty metric spaces.

(1) For $\mathbb{M}, \mathbb{M}' \in \mathcal{K}^{\mathcal{GH}}$, we introduce the equivalence relation $\mathcal{R}$ such that

$$\mathbb{M}\ \mathcal{R}\ \mathbb{M}' \Leftrightarrow \exists\ \mathcal{I}: \mathbb{M} \to \mathbb{M}' \text{a bijective isometry.}$$

We define the space of $\mathcal{GH}$s, denoted by $\mathbb{K}^{\mathcal{GH}}$, as the quotient space $\mathbb{K}^{\mathcal{GH}} = \mathcal{K}^{\mathcal{GH}}{}_{/\mathcal{R}}$. Though technically defined as equivalence classes of $\mathcal{CM}$s, the elements of $\mathbb{K}^{\mathcal{GH}}$ are often treated as individual representatives. As a result, we consider an element of $\mathbb{K}^{\mathcal{GH}}$ to be $\mathcal{CM}$s in his own right, rather than abstract equivalence class.

(2) Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two compact subsets of a metric space $\mathbb{M}$. The Hausdorff distance between $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined as

$$\delta^{\mathcal{H}}(\mathcal{E}_1, \mathcal{E}_2) = \max\left(\sup_{x\in\mathcal{E}_1} d(x, \mathcal{E}_2), \sup_{y\in\mathcal{E}_2} d(y, \mathcal{E}_1)\right),$$

where for a point $z \in \mathbb{M}$ and a set $E \subset \mathbb{M}$, the distance from z to E is given by

$$d(z, E) = \inf_{x\in E} \text{dist}(z, x).$$

Now, for $\mathbb{M}$ and $\mathbb{M}'$ in $\mathbb{K}^{\mathcal{GH}}$, we define the Gromov-Hausdorff distance $\delta^{\mathcal{GH}}$ as

$$\delta^{\mathcal{GH}}(\mathbb{M}, \mathbb{M}') = \inf\big\{\delta^{\mathcal{H}}\big(\mathcal{I}_1(\mathbb{M}), \mathcal{I}_2(\mathbb{M}')\big); \mathcal{I}_1: \mathbb{M} \to \text{Z and } \mathcal{I}_2: \mathbb{M}' \to \text{Z are isometries and Z is a metric space}\big\}.$$

For a more detailed discussion of the $\mathcal{GH}$s and their metric, we refer the reader to Chapter 10 of [31].

3. Overview of the Main Results

We initiate by reviewing the concept of an averaging (or summability) system. For an in-depth and systematic analysis of such systems, one is directed to Hardy's authoritative text [16].

Definition 4:

(1) We call a family $\Gamma = (\Gamma_t)_{t\geq t_0}$ with $t_0 > 0$, an averaging system if:

(a) Γ_t represents a finite Borel measure on $[t_0, \infty)$,

(b) the support of Γ_t is compact,

(c) for $L: [t_0, \infty) \to [0, \infty)$ a positive, measurable function such that

$$\lim_{t\to\infty} L(t) = l \in \mathbb{R},$$

the following holds:

$$\lim_{t\to\infty} \int L\, d\Gamma_t = l.$$

This property is known as the consistency condition.

(2) For a positive, measurable function $L: [t_0, \infty) \to [0, \infty)$, then we define lower and upper Γ-averages of L, respectively by

$$\underline{\mathcal{A}}_\Gamma L = \underline{\lim}_{t\to\infty} \int L d\Gamma_t,$$

and

$$\overline{\mathcal{A}}_\Gamma L = \overline{\lim}_{t\to\infty} \int L d\Gamma_t.$$

We introduce, for a metric space $\mathbb{M}$, the general box-counting function as

$$\mathcal{B}_{\mathbb{M}}(s) = \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})}$$

where h and g are two functions that satisfy the conditions in Definition 1. This formulation leads us directly to our central definition.

Definition 5: Let a metric space $\mathbb{M}$ be given, and let Γ denote an averaging system $(\Gamma_t)_{t\geq t_0}$. The general Γ-average $\mathcal{LUB}$ dimensions of $\mathbb{M}$ are then defined as follows:

$$\underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \underline{\mathcal{A}}_\Gamma \mathcal{B}_{\mathbb{M}} = \underline{\lim}_{t\to\infty} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s)$$

and

$$\overline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \overline{\mathcal{A}}_\Gamma \mathcal{B}_{\mathbb{M}} = \overline{\lim}_{t\to\infty} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s).$$

Remark 2: The classical box dimension emerges as a particular case of the general average box dimension. This formulation provides a broader perspective on box dimensions. Consider a metric space $\mathbb{M}$, δ_a the Dirac measure concentrated at a, and define the averaging system $\Gamma = (\delta_a)_{a\geq 1}$. By choosing $g(r) = h(r) = -\log(r)$, we arrive at the following result:

$$\overline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \overline{\dim}_{\Gamma,\mathrm{B}}(\mathbb{M}) \text{ and } \underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \underline{\dim}_{\Gamma,\mathrm{B}}(\mathbb{M}). \tag{3.1}$$

We now introduce the central result of this article, specifically Theorem 1 below. This theorem asserts that for a $\mathcal{TCM}$s $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, the behavior of the general box-counting function $\mathcal{B}_{\mathbb{M}}$ is characterized by such pronounced irregularity that it remains divergent, even when subjected to arbitrary averaging processes.

Theorem 1: *Let $\Phi = (h, g)$, where h and g are two functions satisfying the conditions specified in Definition 1, $\mathbb{M}$ be a representative $\mathcal{CM}$s in the $\mathcal{GH}$s $\mathbb{K}^{\mathcal{GH}}$ and $\Gamma = (\Gamma_t)_{t\geq t_0}$ denote an averaging system. Then*

$$\underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = 0. \tag{3.2}$$

In addition, if $h(xy) \geq h(x) + h(y)$ and $\forall\, N \in \mathbb{N}, \forall r > 0, \exists\, \mathcal{C}$ a compact subset of $\mathcal{V} = \{x \in \mathbb{R}^N / |x| \leq \frac{r}{8}\}$ such that $\overline{\dim}^{\Phi}_{\Gamma,B}(\mathcal{C}) = N$ then

$$\overline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \infty. \tag{3.3}$$

4. $h-\mathcal{HCA}$ of the general box dimension of a $\mathcal{TCM}$s

In this section, we focus on two specific types of averages: the $h-\mathcal{HCA}$. Both $h-\mathcal{HCA}$ represent examples of averages derived from averaging systems. Consequently, it is natural and instructive to examine how the general results established in Section 3 apply to these two types of averages.

We now proceed to define the $h-\mathcal{HCA}$ and apply them to the general box-counting function of a metric space $\mathbb{M}$,

$$\mathcal{B}_{\mathbb{M}}(t) = \frac{h\left(\frac{1}{N_{e^{-t}}(\mathbb{M})}\right)}{g(e^{-t})}.$$

Definition 6: Let $c > 0$ and $g:(c,\infty) \to [0,\infty)$ be a positive measurable function. We introduce two operators, $\mathcal{M}_g:(c,\infty) \to [0,\infty)$ and $\mathcal{I}_g:(c,\infty) \to [0,\infty)$, defined by

$$\mathcal{M}_g(t) = \frac{1}{t}\int_c^t g(s) \text{ and } \mathcal{I}_g(t) = \int_c^t g(s).$$

(1) For $n \in \mathbb{N}$, we define the lower and upper n-th order Hölder averages of g as:

$$\underline{\mathcal{H}_n} g = \underline{\lim}_{t\to\infty} \mathcal{M}_g^n(t) \text{ and } \overline{\mathcal{H}}_n g = \overline{\lim}_{t\to\infty} \mathcal{M}_g^n(t),$$

where $\mathcal{M}_g^n$ represents the n-fold application of the operator $\mathcal{M}$.

(2) The lower and upper n-th order Cesàro averages of g are defined as:

$$\underline{\mathcal{C}_n} g = \underline{\lim}_{t\to\infty} \frac{n!}{t^n}\mathcal{I}_g^n(t) \text{ and } \overline{\mathcal{C}}_n g = \overline{\lim}_{t\to\infty} \frac{n!}{t^n}\mathcal{I}_g^n(t),$$

for $n \in \mathbb{N}$ and where $\mathcal{I}_g^n$ represents the n-fold integral of g.

A well-known fact is that the $h-\mathcal{HCA}$ fulfill the following inequalities:

$$\underline{\lim}_{t\to\infty} g(t) = \underline{\mathcal{H}_0} g \le \underline{\mathcal{H}_1} g \le \underline{\mathcal{H}_2} g \le \cdots \le \overline{\mathcal{H}}_2 g \le \overline{\mathcal{H}}_1 g \le \overline{\mathcal{H}}_0 g = \overline{\lim}_{t\to\infty} g(t), \tag{4.1}$$

and

$$\underline{\lim}_{t\to\infty} g(t) = \underline{\mathcal{C}_0} g \le \underline{\mathcal{C}_1} g \le \underline{\mathcal{C}_2} g \le \cdots \le \overline{\mathcal{C}}_2 g \le \overline{\mathcal{C}}_1 g \le \overline{\mathcal{C}}_0 g = \overline{\lim}_{t\to\infty} g(t). \tag{4.2}$$

Additionally, it is well-established that both the $h-\mathcal{HCA}$ are examples of averaging systems as defined in Section 3. More specifically, we define $\Gamma_n^{\mathcal{H}} = (\Gamma_{n,t}^{\mathcal{H}})_{t\ge t_0}$, the averaging system, for a given $n \in \mathbb{N}$ as follows:

$$\Gamma_{n,t}^{\mathcal{H}}(F) = \int \frac{1_{[t_0,t]\cap F}(\log t - \log s)^{n-1}}{(n-1)!\,t}\,ds,$$

where F is a Borel subset of $[t_0,\infty)$. Consequently, we have the following relations:

$$\underline{\mathcal{H}_n}g = \underline{\lim}_{t\to\infty} \int g \, d\Gamma_{n,t}^{\mathcal{H}} \text{ and } \overline{\mathcal{H}}_n g = \overline{\lim}_{t\to\infty} \int g \, d\Gamma_{n,t}^{\mathcal{H}}.$$

Consult page 675 in [17] for more details. In a similar manner, we define for $n \in \mathbb{N}$ the averaging system $\Gamma_n^{\mathcal{C}} = (\Gamma_{n,t}^{\mathcal{C}})_{t\geq t_0}$ as

$$\Gamma_{n,t}^{\mathcal{C}}(F) = \int \frac{1_{[t_0,t]\cap F} n(t-s)^{n-1}}{t^n} ds,$$

then we can express the Cesàro averages as:

$$\underline{\mathcal{C}_n}g = \underline{\lim}_{t\to\infty} \int g \, d\Gamma_{n,t}^{\mathcal{C}} \text{ and } \overline{\mathcal{C}}_n g = \overline{\lim}_{t\to\infty} \int g \, d\Gamma_{n,t}^{\mathcal{C}},$$

as seen, for instance, on pages $110 - 111$ in [16]. These results showcase the utility of $h - \mathcal{HCA}$ in analyzing and summarizing the behavior of functions over large intervals by leveraging the structure of averaging systems.

By utilizing both $h - \mathcal{HCA}$, we are now able to define the $h - \mathcal{HCA}$ general box dimensions. This is accomplished by applying these averaging procedures to the function $\mathcal{B}_{\mathbb{M}}(t) = \frac{h\left(\frac{1}{N_{e^{-t}}(\mathbb{M})}\right)}{g(e^{-t})}$. Consequently, we obtain the following definition.

Definition 7: Let $\mathbb{M}$ be a metric space and $\Phi = (h, g)$ where h and g are two functions of Definition 1.

(1) The Hölder nth-order general average $\mathcal{LUB}$ dimensions of $\mathbb{M}$, denoted by $\underline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M})$ and $\overline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M})$, are defined as the Hölder nth-order averages $\mathcal{LUB}$ of the function $\mathcal{B}_{\mathbb{M}}(t)$ for all $t \geq 1$ as follows:

$$\underline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M}) = \underline{\mathcal{H}_n}\mathcal{B}_{\mathbb{M}}$$

and

$$\overline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M}) = \overline{\mathcal{H}}_n\mathcal{B}_{\mathbb{M}}.$$

(2) The Cesàro nth-order general average $\mathcal{LUB}$ dimensions of $\mathbb{M}$, denoted by $\underline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M})$ and $\overline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M})$, are given by:

$$\underline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M}) = \underline{\mathcal{C}_n}\mathcal{B}_{\mathbb{M}}$$

and

$$\overline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M}) = \overline{\mathcal{C}}_n\mathcal{B}_{\mathbb{M}}.$$

Remark 3: These higher-order $h - \mathcal{HCA}$ general box dimensions form an infinite double hierarchy which contains at least countably many levels. More specifically, by utilizing equations (4.1) and (4.2), we obtain the following relationships:

$$\begin{aligned}\underline{\dim}_{B}^{\Phi}(\mathbb{M}) = \underline{\dim}_{B,0}^{\mathcal{H},\Phi}(\mathbb{M}) \leq \underline{\dim}_{B,1}^{\mathcal{H},\Phi}(\mathbb{M}) \leq \cdots \leq \overline{\dim}_{B,1}^{\mathcal{H},\Phi}(\mathbb{M}) \\ \leq \overline{\dim}_{B,0}^{\mathcal{H},\Phi}(\mathbb{M}) = \overline{\dim}_{B}^{\Phi}(\mathbb{M}) \qquad (4.3)\end{aligned}$$

and

$$\underline{\dim}_B^{\Phi}(\mathbb{M}) = \underline{\dim}_{B,0}^{\mathcal{C},\Phi}(\mathbb{M}) \le \underline{\dim}_{B,1}^{\mathcal{C},\Phi}(\mathbb{M}) \le \cdots \le \overline{\dim}_{B,1}^{\mathcal{C},\Phi}(\mathbb{M}) \le \overline{\dim}_{B,0}^{\mathcal{C},\Phi}(\mathbb{M}) = \overline{\dim}_B^{\Phi}(\mathbb{M}). \tag{4.4}$$

These inequalities capture the hierarchical structure of the dimensions, illustrating the relationships between the lower and upper bounds of the $h-\mathcal{HCA}$ at various levels. Each level within this hierarchy represents a more refined approximation of the box dimension, providing a detailed understanding of the dimensional properties of the space $\mathbb{M}$ across different scales and averaging methods.

As a corollary of Theorem 1, we now demonstrate that the dynamics of a $\mathcal{TCM}$ s, $\mathbb{M} \in K^{\mathcal{GH}}$, exhibit such profound irregularity that the hierarchies delineated in equations (4.3) and (4.4), constructed through the $h-\mathcal{HCA}$ across all orders, are insufficient to adequately "regularize" the behavior of the box-counting function $\mathcal{B}_{\mathbb{M}}(t) = \frac{h\left(\frac{1}{N_{e^{-t}}(\mathbb{M})}\right)}{g(e^{-t})}$ as $t \to \infty$. This result emphasizes that no matter how refined the hierarchical structure of the averages may be, the intrinsic irregularity of the $\mathcal{CM}$s is such that these averages cannot eliminate the oscillations or fluctuations in the box counting function. As $t \to \infty$, the function $\mathcal{B}_{\mathbb{M}}(t)$ retains its erratic behavior, resisting any attempt at simplification through these averaging methods. This highlights the extreme complexity and unpredictability of $\mathcal{TCM}$s in this framework, showing that their dimensional characteristics cannot be fully captured through any finite or countably infinite family of averaging techniques.

Corollary 1: *Let* $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$ *be a* $\mathcal{TCM}$ *s. Then for every* $n \in \mathbb{N} \cup \{0\}$ *the following hold:*

$$\underline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M}) = \underline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M}) = 0$$

and

$$\overline{\dim}_{B,n}^{\mathcal{H},\Phi}(\mathbb{M}) = \overline{\dim}_{B,n}^{\mathcal{C},\Phi}(\mathbb{M}) = \infty.$$

Proof: This result stems directly from Theorem 1, which establishes that, for $\mathcal{TCM}$s in $\mathbb{K}^{\mathcal{GH}}$, the behavior of the box-counting function is so irregular that the $h-\mathcal{HCA}$ of all orders fail to provide a meaningful measure of the box dimension. The $\mathcal{LUB}$ dimensions, as defined by both Hölder and Cesàro methods, collapse to the extreme values 0 and ∞, respectively. These results reflect the highly erratic and unpredictable nature of $\mathcal{LUB}$ dimensions.

5. Proof of Theorem 1

To begin with, we define the notions of an r-covering and an r-packing sets, together with the associated r-covering number $N_r(\mathbb{M})$ and r-packing number $M_r(\mathbb{M})$ for a metric space $\mathbb{M}$.

Definition 8: Let $(\mathbb{M}, \delta)$ be a metric space, and fix $r > 0$.

(1) A subset $\Upsilon \subset \mathbb{M}$ is said to r-cover $\mathbb{M}$ if, for every $m_1 \in \mathbb{M}$, one can find $m_2 \in \Upsilon$ satisfying

$$d(m_1, m_2) \leq r.$$

(2) The r-covering number $N_r(\mathbb{M})$ of $\mathbb{M}$ is defined as the smallest possible cardinality of an r-covering subset of $\mathbb{M}$, given by

$$N_r(\mathbb{M}) = \inf\{|\Upsilon| \,|\, \Upsilon \text{ is an } r - \text{covering subset of } \mathbb{M}\}.$$

(3) A subset $\Upsilon \subset \mathbb{M}$ is said to be an r-separated subset of $\mathbb{M}$ if every pair of distinct points $m_1, m_2 \in \Upsilon$ satisfies $\delta(m_1, m_2) \geq r$.

(4) The r-packing number $M_r(\mathbb{M})$ of $\mathbb{M}$ is defined as the largest cardinality of any r-separated subset of $\mathbb{M}$, given by

$$M_r(\mathbb{M}) = \sup\{|\Upsilon| \,|\, \Upsilon \text{ is an } r - \text{separated subset of } \mathbb{M}\}.$$

As a preliminary, we recall several notable properties of the covering number $N_r(\mathbb{M})$ and the packing number $M_r(\mathbb{M})$, which play a key role in the forthcoming analysis and will be frequently referenced in the subsequent sections.

Lemma 1: *Let $r > 0$. The following statements are valid:*

(1) *the mapping $N_r: \mathbb{K}^{\mathcal{GH}} \to \mathbb{N}$ is lower semi-continuous,*

(2) *the mapping $M_r: \mathbb{K}^{\mathcal{GH}} \to \mathbb{N}$ is upper semi-continuous,*

(3) *for every $\mathcal{CM}s$ $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, we have the inequality $N_r(\mathbb{M}) \leq M_r(\mathbb{M})$.*

Proof: This statement follows directly from [32, Lemma 9]. See also [15] for complementary references.

5.1 *Proof of Theorem 1.(3.2):*

Let's begin with a quick reminder of this Lemma:

Lemma 2: [39] *Consider the measure space $(\Omega, \mathcal{A}, \lambda)$ and the sequence $(f_k)_k$ of positive measurable functions defined on Ω. If $\int sup_k f_k d\lambda < \infty$, then*

$$\overline{\lim_k} \int f_k d\lambda \leq \int \overline{\lim_k} f_k d\lambda.$$

For the next step, it is necessary to prove the following lemma:

Lemma 3: *For an averaging system $\Gamma = (\Gamma_t)_{t \geq t_0}$, $c \in \mathbb{R}$ and $t \geq t_0$, the set below is an open set in $\mathbb{K}^{\mathcal{GH}}$*

$$\left\{\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,\middle|\, \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) < c\right\}.$$

Proof: Let the set:

$$\begin{aligned}\mathcal{F} &= \mathbb{K}^{\mathcal{GH}} \setminus \left\{ \mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,\middle|\, \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) < c \right\} \\ &= \left\{ \mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,\middle|\, \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) \geq c \right\}.\end{aligned}$$

To demonstrate this, we consider a sequence $(\mathbb{M}_n)_n$ in $\mathcal{F}$ and an element $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$ such that $\mathbb{M}_n \underset{n\to\infty}{\to} \mathbb{M}$. Our goal is to show that $\mathbb{M} \in \mathcal{F}$, meaning we must prove that

$$\int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) \geq c.$$

For conciseness, define the functions $\psi, \psi_n\colon [t_0, \infty) \to [0, \infty)$ as follows:

$$\psi(s) = \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} \text{ and } \psi_n(s) = \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}_n)}\right)}{g(e^{-s})}.$$

We now turn our attention to proving:

Claim 5.1: We have $\int \sup_n \psi_n d\Gamma_t < \infty$.

Proof: Since $\operatorname{supp}\Gamma_t$, the support of the measure Γ_t, is compact so we can take $t_0 \leq T_0$ such that the support of Γ_t is contained within the interval $[t_0, T_0]$. Furthermore, observe that for all $n, s \in [t_0, T_0]$, it follows that

$$\psi_n(s) = \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}_n)}\right)}{g(e^{-s})} \leq \frac{h\left(\frac{1}{M_{e^{-T_0}}(\mathbb{M}_n)}\right)}{g(e^{-t_0})}.$$

Thus, the inclusion $\operatorname{supp}\Gamma_t \subseteq [t_0, T_0]$ immediately leads to the conclusion that

$$\int \sup_n \psi_n \, d\Gamma_t = \int_{t_0}^{T_0} \sup_n \psi_n \, d\Gamma_t \leq \frac{h\left(\frac{1}{M_{e^{-T_0}}(\mathbb{M}_n)}\right)}{g(e^{-t_0})} \Gamma_t([t_0, T_0]) < \infty.$$

The claim 5.1 is thus proved.

Claim 5.2: One can deduce that $c \leq \int \overline{\lim_n} \psi_n d\Gamma_t$.

Proof: Since $\mathbb{M}_n \in \mathcal{F}$, we conclude that

$$c \leq \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}_n)}\right)}{g(e^{-s})} d\Gamma_t(s) = \int \psi_n \, d\Gamma_t$$

for all n, which implies

$$c \leq \overline{\lim_n} \int \psi_n \, d\Gamma_t. \tag{5.1}$$

Claim 5.1 and Lemma 2 yield that

$$\overline{\lim_n} \int \psi_n \, d\Gamma_t \le \int \overline{\lim_n} \psi_n \, d\Gamma_t. \tag{5.2}$$

The expected result arises directly from equations (5.1) and (5.2), thus completing the demonstration of Claim 5.2.

Claim 5.3: For all $s \ge t_0$, it follows that $\overline{\lim_n} \psi_n(s) \le \psi(s)$.

Proof: Fix $s \ge t_0$. Since for all $r > 0$, M_r is an upper semicontinuous function (see Lemma 1) and $\mathbb{M}_n \underset{n\to+\infty}{\rightarrow} \mathbb{M}$ in $\mathbb{K}^{\mathcal{GH}}$, we conclude that

$$\overline{\lim_n} M_{e^{-s}}(\mathbb{M}_n) \le M_{e^{-s}}(\mathbb{M}),$$

which implies

$$\begin{aligned} \overline{\lim_n} \psi_n(s) &= \overline{\lim_n} \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}_n)}\right)}{g(e^{-s})} \\ &= \frac{h\left(\overline{\lim_n} \frac{1}{M_{e^{-s}}(\mathbb{M}_n)}\right)}{g(e^{-s})} \\ &\le \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} \\ &= \psi(s). \end{aligned}$$

The claim 5.3 is thus proven.

In conclusion, we infer from Claim 5.2 and Claim 5.3 that

$$\begin{aligned} c &\le \int \overline{\lim_n} \psi_n d\Gamma_t \\ &\le \int \psi d\Gamma_t \\ &= \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s). \end{aligned}$$

This proves Lemma 3.

We shall now embark on the detailed proof of Theorem 1 (3.2), carefully examining each step of the argument.

We are required to demonstrate that for a $\mathcal{TCM}$s $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, we have

$$\underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = 0.$$

Since it is clear that $\underline{\dim}_{\Gamma,\mathrm{B}}(\mathbb{M}) \ge 0$, it is enough to show that $E = \left\{\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} | \underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) > 0\right\}$ is a meagre set. Recall that a set is meager if its complement is co-meager. For more details see [30].

For $e > 0$, define

$$E_e = \left\{\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} | \underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) > e\right\}.$$

Since

$$E = \bigcup_{e\in\mathbb{Q},e>0} E_e,$$

to prove that E is meagre it suffices to show that for all positive e in $\mathbb{Q}$, E_e is meagre. Let fix $e \in \mathbb{Q}$ with $e > 0$. We need to demonstrate that there is a countable family of open and dense subsets $(\mathcal{O}_k)_k$ of $\mathbb{K}^{\mathcal{GH}}$ such that

$$\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \setminus E_e.$$

For $t \geq t_0$, define

$$O_t = \left\{ \mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \mid \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) < e \right\},$$

and for each $k \in \mathbb{N}$, let

$$\mathcal{O}_k = \bigcup_{t\geq k} O_t.$$

In what follows, we establish that the collection $(\mathcal{O}_k)_k$ forms a family of open and dense subsets in $\mathbb{K}^{\mathcal{GH}}$ satisfying

$$\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \setminus E_e.$$

This will be addressed through the next claims.

Claim 5.4: In the topology of $\mathbb{K}^{\mathcal{GH}}$, the set $\mathcal{O}_k$ is open.

Proof: By Lemma 3, for all $t \geq t_0$, O_t is open set, and thus $\mathcal{O}_k = \bigcup_{t\geq k} O_t$ is an open set too. This finishes the proof of Claim 5.4.

Claim 5.5: The set $\mathcal{O}_k$ is dense in $\mathbb{K}^{\mathcal{GH}}$.

Proof: Consider $r > 0$ and $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$. We aim to construct $\mathbb{M}' \in \mathbb{K}^{\mathcal{GH}}$ satisfying

$$\delta^{\mathcal{GH}}(\mathbb{M}, \mathbb{M}') < r \text{ and } \mathbb{M}' \in \mathcal{O}_k.$$

By the compactness of $\mathbb{M}$, there exists a finite (and hence compact) subset $\mathbb{M}' \subset \mathbb{M}$ such that

$$\delta^{\mathcal{H}}(\mathbb{M}, \mathbb{M}') < r.$$

In particular, we have $\delta^{\mathcal{GH}}(\mathbb{M}, \mathbb{M}') \leq \delta^{\mathcal{H}}(\mathbb{M}, \mathbb{M}') < r$. Now, we show that $\mathbb{M}' \in \mathcal{O}_k$.

Indeed, since $\mathbb{M}'$ is finite, we conclude that

$$\frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} \to 0 \text{ as } t \to \infty,$$

additionally, the consistency condition leads to the conclusion that

$$\int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) \to 0 \text{ as } t \to \infty.$$

From this, we readily conclude the existence of a real number $t \geq k$ such that

$$\int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) < e,$$

and therefore $\mathbb{M}' \in O_t \subseteq \mathcal{O}_k$. This completes the proof of Claim 5.5.

Claim 5.6: We have $\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \backslash E_e$.

Proof: Let $\mathbb{M} \in \bigcap_k \mathcal{O}_k$. For each $k \in \mathbb{N}$, we can find $k \leq t_k$ such that $\mathbb{M} \in O_{t_k}$, which implies

$$\int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_{t_k}(s) < e$$

for all $k \in \mathbb{N}$. Therefore,

$$\varliminf_{t \to \infty} \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) \leq \varliminf_{k \to \infty} \int \frac{h\left(\frac{1}{M_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_{t_k}(s) \leq e.$$

Combining this with Lemma 1, we directly obtain that

$$\underline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \varliminf_{t \to \infty} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) \leq e,$$

and hence $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \backslash E_e$. This concludes the proof of Claim 5.6.

The set E_e can be concluded to be meagre by combining Claim 5.6, Claim 5.5, and Claim 5.4.

5.2 *Proof of Theorem 1(3.3):*

$$\overline{\dim}^{\Phi}_{\mathrm{B},\Gamma}(\mathbb{M}) = \infty \text{ for a } \mathcal{TCM}\text{s } \mathbb{M} \in \mathbb{K}^{\mathcal{GH}}.$$

Lemma 4: *For an averaging system $\Gamma = (\Gamma_t)_{t \geq t_0}$, $c \in \mathbb{R}$ and $t \geq t_0$ it follows that*

$$\left\{ \mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,\middle|\, \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) > c \right\} \textit{is an open set in } \mathbb{K}^{\mathcal{GH}}.$$

Proof: The proof of Lemma 4 is very similar to that of Lemma 3, utilizing Fatou's Lemma and the lower semi-continuity of the function $N_r\colon \mathbb{K}^{\mathcal{GH}} \to \mathbb{N}$ instead of the Reverse Fatou's Lemma and the upper semi-continuity of the function $M_r\colon \mathbb{K}^{\mathcal{GH}} \to \mathbb{N}$. Consequently, we have chosen to omit the proof.

To prove Theorem 1 (3.3), we need to show that for a $\mathcal{TCM}$s $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, we have $\overline{\dim}^{\Phi}_{\Gamma,\mathrm{B}}(\mathbb{M}) = \infty$. In other words, we need to show that the set

$$\mathcal{U} = \left\{ \mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \middle| \overline{\dim}_{\Gamma,\mathrm{B}}(\mathbb{M}) < \infty \right\}$$

is meagre. Let $u > 0$ and define

$$\mathcal{U}_u = \left\{\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,|\, \overline{\dim}_{\Gamma,\mathrm{B}}(\mathbb{M}) < u\right\}.$$

Since

$$\mathcal{U} = \bigcup_{\substack{u \in \mathbb{Q} \\ u>0}} \mathcal{U}_u,$$

it is enough to show that for all positive u in $\mathbb{Q}$, the set $\mathcal{U}_u$ is meagre. Therefore, we fix u a positive element of $\mathbb{Q}$. It remains to establish a countable family $(\mathcal{O}_k)_k$ of open dense subsets of $\mathbb{K}^{\mathcal{GH}}$ such that

$$\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \backslash \mathcal{U}_u.$$

For $t \geq t_0$, let

$$O_t = \left\{\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \,|\, \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) > u\right\},$$

and for every positive integer k, define

$$\mathcal{O}_k = \bigcup_{t \geq k} O_t.$$

In what follows, we will demonstrate that for all k, $\mathcal{O}_k$ is an open dense subsets of $\mathbb{K}^{\mathcal{GH}}$ and that

$$\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \backslash \mathcal{U}_u.$$

The following three claims encapsulate the content to be discussed.

Claim 5.7: In the topology of $\mathbb{K}^{\mathcal{GH}}$, the set $\mathcal{O}_k$ is open.

Proof: Indeed, since Lemma 4 ensures that for all $t \geq t_0$ O_t is open, consequently, $\mathcal{O}_k$ is open as a union of open sets, thereby completing the proof of Claim 5.7.

Claim 5.8: $\mathcal{O}_k$ is a dense set in $\mathbb{K}^{\mathcal{GH}}$.

Proof: Let $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}}$, $r > 0$ and δ a metric on $\mathbb{M}$. We seek $\mathbb{M}' \in \mathbb{K}^{\mathcal{GH}}$ that satisfie $\delta^{\mathcal{GH}}(\mathbb{M}, \mathbb{M}') < r$ and $\mathbb{M}' \in \mathcal{O}_k$. Since $\mathbb{M}$ is compact, there exists a finite subset $\mathcal{J} \subset \mathbb{M}$ such that $\delta^{\mathcal{H}}(\mathbb{M}, \mathcal{J}) < \frac{r}{2}$. We then choose $N > u$ a positive integer. There exists a compact subset $\mathcal{C}$ of $\mathcal{V} = \{x \in \mathbb{R}^N : |x| \leq \frac{r}{8}\}$ such that

$$\overline{\dim}^{\Phi}_{\mathrm{B},\Gamma}(\mathcal{C}) = N.$$

Finally, define

$$\mathbb{M}' = \mathcal{J} \times \mathcal{C},$$

and endow $\mathbb{M}'$ with the metric $\mathcal{D}$, defined by

$$\mathcal{D}((p_1, p_2), (p_1', p_2')) = \max_{(p_1, p_1') \in \mathcal{J}^2; (p_2, p_2') \in \mathcal{C}^2} (\delta(p_1, p_1'), |p_2 - p_2'|).$$

It is readily apparent that $\mathbb{M}'$ is compact, so $\mathbb{M}' \in \mathbb{K}^{\mathcal{GH}}$.

Next, we will show that $\delta^{\mathcal{GH}}(\mathbb{M}, \mathbb{M}') < r$ and $\mathbb{M}' \in \mathcal{O}_k$.

As a first step, we establish that

$$\delta^{\mathcal{GH}}(\mathbb{M},\mathbb{M}') < r.$$

To get this, we begin by showing that

$$\delta^{\mathcal{GH}}(\mathcal{I},\mathbb{M}') < \frac{r}{2}. \tag{5.3}$$

Indeed, fix $y_0 \in \mathcal{C}$ and define $m_1\colon \mathcal{I} \to \mathbb{M}'$ and $m_2\colon \mathbb{M}' \to \mathbb{M}'$ by $m_1(x) = (x, y_0)$ and $m_2(x,y) = (x,y)$. Both m_1 and m_2 are isometries. Therefore, we conclude that

$$\begin{aligned} \delta^{\mathcal{GH}}(\mathcal{I},\mathbb{M}') \quad &\leq \delta^{\mathcal{H}}(m_1(\mathcal{I}), m_2(\mathbb{M}')) \\ &= \delta^{\mathcal{H}}(\mathcal{I}\times\{y_0\}, \mathcal{I}\times\mathcal{C}) \\ &\leq diam(\mathcal{C}) \\ &< \frac{r}{2}. \end{aligned}$$

This proves that $\delta^{\mathcal{GH}}(\mathcal{I},\mathbb{M}') < \frac{r}{2}$. From this, we immediately deduce that

$$\begin{aligned} \delta^{\mathcal{GH}}(\mathbb{M},\mathbb{M}') \quad &\leq \delta^{\mathcal{GH}}(\mathbb{M},\mathcal{I}) + \delta^{\mathcal{GH}}(\mathcal{I},\mathbb{M}') \\ &< \delta^{\mathcal{GH}}(\mathbb{M},\mathcal{I}) + \frac{r}{2} \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

This proves $\delta^{\mathcal{GH}}(\mathbb{M},\mathbb{M}') < r$.

Next, we prove that

$$\mathbb{M}' \in \mathcal{O}_k.$$

Let $r_{\min} = \frac{1}{2}\min_{a,b\in\mathcal{I},a\neq b}\delta(a,b)$. Moreover, since $\mathbb{M}' = \mathcal{I}\times\mathcal{C}$, we obtain, for all $r < r_{\min}$,

$$N_r(\mathbb{M}') = |\mathcal{I}|\, N_r(\mathcal{C}).$$

We conclude that if we define $t_{\min} = \max(-\log r_{\min}, t_0)$, then

$$N_{e^{-t}}(\mathbb{M}') = |\mathcal{I}| N_{e^{-t}}(\mathcal{C}) \text{ for all } t > t_{\min}. \tag{5.4}$$

Next, consider the function $\mathcal{L}\colon [t_0,\infty) \to \mathbb{R}$ defined by

$$\mathcal{L}(t) = \begin{cases} \dfrac{h\left(\frac{N_{e^{-t}}(\mathcal{C})}{N_{e^{-t}}(\mathbb{M}')}\right)}{g(e^{-t})} & \text{for } t < t_{\min}, \\ \dfrac{h\left(\frac{1}{|\mathcal{I}|}\right)}{g(e^{-t})} & \text{for } t \geq t_{\min}. \end{cases}$$

It follows from (5.4) that

$$\int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})}\, d\Gamma_t(s)$$

$$= \int_{[t_0,t_{\min})} \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})}\, d\Gamma_t(s) + \int_{[t_{\min},\infty)} \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})}\, d\Gamma_t(s)$$

$$= \int_{[t_0,t_{\min})} \frac{h\left(\frac{N_{e^{-s}}(\mathcal{C})}{N_{e^{-s}}(\mathcal{C})N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) + \int_{[t_{\min},\infty)} \frac{h\left(\frac{|\mathcal{I}|}{|\mathcal{I}|N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s)$$

$$\geq \int_{[t_0,t_{\min})} \frac{h\left(\frac{N_{e^{-s}}(\mathcal{C})}{N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) + \int_{[t_0,t_{\min})} \frac{h\left(\frac{1}{N_{e^{-s}}(\mathcal{C})}\right)}{g(e^{-s})} d\Gamma_t(s)$$

$$+ \int_{[t_{\min},\infty)} \frac{h\left(\frac{1}{|\mathcal{I}|}\right)}{g(e^{-s})} d\Gamma_t(s) + \int_{[t_{\min},\infty)} \frac{h\left(\frac{1}{N_{e^{-s}}(\mathcal{C})}\right)}{g(e^{-s})} d\Gamma_t(s)$$

$$= \int \mathcal{L}(s) d\Gamma_t(s) + \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathcal{C})}\right)}{g(e^{-s})} d\Gamma_t(s). \tag{5.5}$$

By observing that $|\mathcal{L}(t)| \to 0$, we conclude that

$$\left|\int \mathcal{L}\, d\Gamma_t\right| \leq \int |\mathcal{L}|\, d\Gamma_t \underset{t\to\infty}{\to} 0,$$

and therefore, from (5.5), we have

$$\begin{aligned} \overline{\lim_{t}} \int \frac{h\left(\frac{1}{N_{e^{-t}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) &\geq \overline{\lim_{t}} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathcal{C})}\right)}{g(e^{-s})} d\Gamma_t(s) \\ &= \overline{\dim}_{\Gamma,\mathrm{B}}^{\Phi}(\mathcal{C}) \\ &= N \\ &> u. \end{aligned} \tag{5.6}$$

Equation (5.6) implies that there exists $t \geq k$ such that

$$\int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M}')}\right)}{g(e^{-s})} d\Gamma_t(s) > u,$$

which implies that $\mathbb{M}' \in O_t \subseteq \mathcal{O}_k$. This proves that $\mathbb{M}' \in \mathcal{O}_k$.

Finally, Claim 5.8 follows from the equation (5.3) and the equation (5.4). This marks the end of the claim demonstration.

Claim 5.9: It holds that $\bigcap_k \mathcal{O}_k \subseteq \mathbb{K}^{\mathcal{GH}} \backslash \mathcal{U}_u$.

Proof: Let $\mathbb{M} \in \bigcap_k \mathcal{O}_k$. Then, for all $k \in \mathbb{N}$, there is an $k \leq t_k$ for which $\mathbb{M} \in O_{t_k}$. Consequently, for all $k \in \mathbb{N}$, we have:

$$\int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})}\, d\Gamma_{t_k}(s) > u.$$

From this, we conclude that

$$\begin{aligned} \overline{\dim}_{\Gamma,\mathrm{B}}^{\Phi}(\mathbb{M}) &= \overline{\lim_{t\to\infty}} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_t(s) \\ &\geq \overline{\lim_{k\to\infty}} \int \frac{h\left(\frac{1}{N_{e^{-s}}(\mathbb{M})}\right)}{g(e^{-s})} d\Gamma_{t_k}(s) \geq u, \end{aligned}$$

and therefore $\mathbb{M} \in \mathbb{K}^{\mathcal{GH}} \backslash \mathcal{U}_u$. This finalizes the demonstration of Claim 5.9. Building on the prior three claims, it becomes that $\mathcal{U}_u$ is meager.

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