Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
DOI : 10.47974/JIM-2430

Fractional beam dynamics with nonlinear foundation : Theory and singular perturbation analysis

Iqbal Jebril †
Department of Mathematics
Al-Zaytoonah University of Jordan
Amman 11733
Jordan

Zoubir Dahmani *
Laboratory of Discrete and Applied Mathematics and Operations Research
LAMDA-RO
University of Blida 1
09000 Blida
Algeria

Abstract

This paper investigates the dynamics of a nonlinear elastic beam governed by a space-fractional differential equation of Caputo type, where the fractional order lies in the range $z \in (3,4)$. A small parameter ϵ is incorporated to capture singular perturbation effects, and a composite approximation is constructed through matched asymptotic expansions. The leading-order solution is derived using the Tanh method for the reduced integer-order problem. The fractional perturbation introduces algebraically decaying corrections, which are analyzed and matched with the outer-region behavior. Numerical plots illustrate how varying ϵ and the fractional order z affects the beam profile, especially the far-field tails. The analysis shows that higher fractional orders amplify long-range perturbative effects in the solution profile.

Subject Classification (2020): *35C07, 35Q53, 34E10.*

Keywords: *Beam equation, Caputo derivative, Singular perturbation, Tanh method, Matched asymptotic expansions.*

† *ORCID-ID:* **0000-0003-4348-6197**
* *ORCID-ID:* **0000-0003-4659-0723**

† *E-mail:* **iqbal501@hotmail.com**
* *E-mail:* **zzdahmani@yahoo.fr (Corresponding Author)**

© Taru Publications

1. Introduction

The study of beam structures has long been an important topic in mechanics. The Euler-Bernoulli beam theory presents a powerful work to understand the bending behavior of beams [8, 25, 26]. This classical model has been enriched with various generalizations, including linear and nonlinear elastic responses and higher-order corrections to capture more behavior.

Recently, the incorporation of fractional differential equations into structural models has opened up new ways for modeling memory effects, spatial nonlocality, and anomalous diffusion. This theory has given effective in viscoelasticity and anomalous wave propagation [3, 11, 15, 19, 20]. In particular, space-fractional beam problems have been developed to account for long-range interactions in micro-structured and heterogeneous materials [5, 10, 14, 22, 23, 24, 27].

Among the definitions of fractional derivatives, the Caputo derivative is very applied in physical applications due to its compatibility with classical initial and boundary conditions [4, 6]. It has been used in both time-fractional and space-fractional mechanical models.

In parallel, singular perturbation methods have been extensively used in differential equations to study problems with multiscale behavior and boundary layers [12, 13, 16]. The extension of these techniques to fractional-order problems has gained considerable attention. Recent works by Alikhanov and Huang [1], Baleanu and Diethelm [2], and Li and Zeng [18] have explored analytical and numerical approaches for fractional perturbed problem over long time. Also, in 2024, Langthjem and Jensen [17] have applied matched asymptotic expansion method of a buckle wave propagation problem with perturbation theory.

Few studies have combined space-fractional derivatives and singular perturbation techniques in the context of nonlinear beam equations [7, 9, 21].

The present work completes this gap. We study a nonlinear beam equation involving a Caputo space-fractional derivative of order $z \in (3,4)$, with a small parameter ε. Through a traveling wave method and matched asymptotic expansions, we analyze the leading-order nonlinear solution and derive composite approximations capturing both the core and far-field behavior of the beam.

From a theoretical perspective, this paper enriches the field of applied mechanics by proposing and investigating an $\alpha-$order differential problem involving nonlinear foundations and dynamic effects. Analytical approaches to fractional differential equations with physical boundary conditions are still relatively underdeveloped. Our study contributes to the development of asymptotic techniques in fractional differential equations.

This paper is organized as follows: In Section 2, we present the mathematical model and derive the reduced and full perturbed equations. In the same section, we apply the Tanh method to solve the leading-order nonlinear beam equation. In addition, this section presents the singular perturbation analysis, including inner/outer expansions and composite matching. We provide numerical results and illustrations in the same section. Section 3 concludes the paper and suggests directions for further study.

2. Beam Problem with Fractional Derivative

We consider a nonlinear beam equation that involves a space-fractional derivative of Caputo type:

$$\alpha\varepsilon^2\, {}^C D_{\bar{x}}^z \bar{w} - \tau \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + (1 - a\bar{w} + b\bar{w}^2)\bar{w} + \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = \bar{p}, \tag{2.1}$$

where, $\bar{w}(\bar{x}, \bar{t})$ is the beam deflection, $a^2 > 3b$, $\alpha = H^{4-z}$ ensures physical consistency of units for the fractional derivative, $\varepsilon^2 = \frac{EI}{k_0 H^4}$ arises from bending stiffness normalization, $\tau = \frac{T}{k_0 H^2}$ reflects dimensionless tension, and $z \in (3,4)$ is the order of the space-fractional Caputo derivative.

It should be noted that in the case where $\alpha = 4$, we obtain the problem studied in [17].

Caputo Derivative Justification.

The Caputo derivative ${}^C D_x^z f(x)$ is defined for $n - 1 < z < n = 4$ by:

$${}^C D_x^z f(x) = \frac{1}{\Gamma(n-z)} \int_0^x (x - s)^{n-z-1} f^{(n)}(s)\, ds, n = \lceil z \rceil. \tag{2.2}$$

This choice guarantees that physical boundary conditions are respected.

2.1 *Traveling Wave Method*

We suppose the wave variable:

$$\bar{w}(\bar{x}, \bar{t}) = W(\xi), \xi = \bar{x} - c\bar{t}, \tag{2.3}$$

where, c is the wave speed to be determined.

We have:

$$\frac{\partial \bar{w}}{\partial \bar{x}} = W'(\xi), \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = W''(\xi), \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = c^2 W''(\xi), D_{\bar{x}}^z \bar{w} = D_\xi^z W.$$

Substituting into Eq. (2.1), we obtain:

$$\alpha\varepsilon^2\, {}^C D_\xi^z W + (c^2 - \tau)W'' + (1 - aW + bW^2)W = \bar{p}. \tag{2.4}$$

Let $\kappa = c^2 - \tau$ and rewrite:

$$\alpha\varepsilon^2\, {}^C D_\xi^z W + \kappa W'' + (1 - aW + bW^2)W = \bar{p}. \tag{2.5}$$

2.2 *Reduced Problem of Singular Beam*

We begin by studying the reduced problem given by $\varepsilon = 0$:

$$\kappa W'' + (1 - aW + bW^2)W = \bar{p}. \tag{2.2}$$

We seek traveling wave solutions of the form:

$$W(\xi) = a_0 + a_1 \tanh(\xi). \tag{2.3}$$

Using the identity:

$$\tanh'(\xi) = 1 - \tanh^2(\xi), \tanh''(\xi) = -2\tanh(\xi)(1 - \tanh^2(\xi)),$$

we can write:

$$W' = a_1(1 - T^2),$$
$$W'' = -2a_1 T + 2a_1 T^3,$$

where, $T = \tanh(\xi)$.

Next, we compute:

$$W^2 = a_0^2 + 2a_0a_1T + a_1^2T^2,$$
$$(1 - aW + bW^2)W = C_0 + C_1T + C_2T^2 + C_3T^3,$$

with,

$$C_0 = (1 - aa_0 + ba_0^2)a_0,$$
$$C_1 = a_1(1 - 2aa_0 + 3ba_0^2),$$
$$C_2 = a_1^2(-a + 3ba_0),$$
$$C_3 = ba_1^3.$$

Therefore, we have

$$-2\kappa a_1T + 2\kappa a_1T^3 + C_0 + C_1T + C_2T^2 + C_3T^3 - \bar{p} = 0,$$

that is

$$(C_0 - \bar{p}) + (C_1 - 2\kappa a_1)T + C_2T^2 + (C_3 + 2\kappa a_1)T^3 = 0,$$

which implies that

$$(1 - aa_0 + ba_0^2)a_0 = \bar{p}, \tag{2.4a}$$
$$a_1(1 - 2aa_0 + 3ba_0^2) = 2\kappa a_1, \tag{2.4b}$$
$$a_1^2(-a + 3ba_0) = 0, \tag{2.4c}$$
$$ba_1^3 + 2\kappa a_1 = 0. \tag{2.4d}$$

From (2.4c), and thanks to $a_1 \neq 0$, we get

$$a_0 = \frac{a}{3b}. \tag{2.5}$$

Substituting into (2.4b), we get

$$\kappa = \frac{1}{2}\left(1 - \frac{a^2}{3b}\right). \tag{2.6}$$

From (2.4d), we have

$$a_1^2 = \frac{1}{b}\left(\frac{a^2}{3b} - 1\right). \tag{2.7}$$

Thus, the leading-order solution is the following:

$$W_0(\xi) = \frac{a}{3b} + a_1\tanh(\xi), \text{where}, a_1^2 = \frac{1}{b}\left(\frac{a^2}{3b} - 1\right). \tag{2.8}$$

2.3 *Perturbed Problem*

We now return to the full perturbed beam equation:

$$\alpha\varepsilon^2\, {}^{C}D_{\xi}^{z}W + \kappa W'' + (1 - aW + bW^2)W = \bar{p}. \tag{2.9}$$

2.3.1 Inner Region Expansion

We use a regular expansion in the inner region:

$$W(\xi) = W_0(\xi) + \varepsilon^2W_1(\xi) + O(\varepsilon^4). \tag{2.10}$$

Leading Order: The term W_0 solves our reduced problem:

$$\kappa W_0'' + (1 - aW_0 + bW_0^2)W_0 = \bar{p}, \tag{2.11}$$

First-Order Correction: The $\mathcal{O}(\varepsilon^2)$ equation is:

$$\kappa W_1'' + (1 - 2aW_0 + 3bW_0^2)W_1 = -\alpha\, {}^C D_\xi^z W_0, \tag{2.12}$$

which is a linear ODE with variable coefficients; its source term involves Caputo derivative:

$${}^C D_\xi^z W_0 = a_1\, {}^C D_\xi^z \tanh(\xi). \tag{2.13}$$

The term W_1 will be important for matching with the outer expansion.

Evaluation of the Source Term: The Caputo derivative of $\tanh(\xi)$ for $z \in (3,4)$ exhibits the asymptotic behavior:

$${}^C D_\xi^z \tanh(\xi): \begin{cases} \xi^{4-z}, & \xi \to 0^+, \\ \xi^{-z+3}, & \xi \to +\infty. \end{cases} \tag{2.14}$$

This algebraic decay will induce algebraic tails in the first-order correction $W_1(\xi)$.

2.3.2 Outer Region Expansion

In the region $|\xi| \gg 1$, we seek the asymptotic behavior of the solution $W(\xi)$ that matches the inner expansion derived earlier.

From the above analysis, the first-order correction term $W_1(\xi)$ satisfies:

$$W_1(\xi): \xi^{-z+3}, \text{ as } \xi \to +\infty.$$

Hence, we can anticipate the full solution to be as follows:

$$W(\xi): \varepsilon^2 \xi^{-z+3}, \text{ as } \xi \to +\infty, \tag{2.6}$$

where, $z \in (3,4)$.

So, let us examine whether (2.6) is asymptotically valid with the above perturbed equation.

Substituting $W(\xi) = \varepsilon^2 \xi^\lambda$, with $\lambda = -z + 3 \in (-1,0)$, we have:

- The Caputo derivative:

$${}^C D_\xi^z W(\xi): \varepsilon^2\, \xi^{\lambda - z} = \varepsilon^2\, \xi^{-2z+3},$$

this is because fractional calculus theory ensures that this decay holds even when the standard identity for ${}^C D_\xi^z \xi^\lambda$ is not applicable (because $\lambda < z - 1$).

- The second derivative:

$$W''(\xi): \varepsilon^2 \xi^{\lambda - 2} = \varepsilon^2 \xi^{-z+1}.$$

- The nonlinear term:

$$(1 - aW + bW^2)W: W: \varepsilon^2 \xi^{-z+3}.$$

Since $-2z + 3 < -3$ for $z \in (3,4)$, the fractional derivative term decays faster than all others. Therefore, we can wrtie

$${}^C D_\xi^z W(\xi) \to 0 \text{ rapidly as } \xi \to \infty.$$

Now, neglecting the fractional term at leading order, our equation reduces to:

$$\kappa W'' + (1 - aW + bW^2)W = \bar{p}.$$

But, since $W(\xi) \to 0$ in the far field, the nonlinear term also decays, and we would require $\kappa W'' : \bar{p}$, leading to a parabolic behavior $W(\xi) : \bar{p}\xi^2$. This contradicts the decay assumption (2.6), unless $\bar{p} = 0$ in the outer region.

The only outer solution that matches the inner tail is:

$$W(\xi) : \varepsilon^2 \xi^{-z+3}.$$

This confirms our anticipation.

2.3.3 Composite Approximation and Matching

We now pass to construct a composite approximation that captures the behavior of $W(\xi)$ across the full domain by matching the inner and outer expansions.

From the inner expansion, we have:

$$W(\xi) = W_0(\xi) + \varepsilon^2 W_1(\xi) + O(\varepsilon^4),$$

with,

$$W_0(\xi) = \frac{a}{3b} + a_1 \tanh(\xi), W_1(\xi) : \xi^{-z+3} \text{ as } \xi \to +\infty.$$

Thus, the inner behavior for large ξ is:

$$W_{\text{inner}}(\xi) : W_0(\infty) + \varepsilon^2 \xi^{-z+3}, \text{ where } W_0(\infty) = \frac{a}{3b} + a_1.$$

On the other hand, the outer expansion predicts:

$$W_{\text{outer}}(\xi) : \varepsilon^2 \xi^{-z+3}.$$

To match both behaviors, we subtract the constant $W_0(\infty)$ from the inner approximation. This yields the composite approximation:

$$W_{\text{comp}}(\xi) = W_0(\xi) + \varepsilon^2 W_1(\xi) - W_0(\infty).$$

This composite solution satisfies the following two asymptotics:

i: When $\xi \to 0$: $W_{\text{comp}}(\xi) : W_0(\xi) - W_0(\infty) + O(\varepsilon^2)$; it preserves the inner core behavior up to a constant shift.

ii: If $\xi \to \infty$: $W_0(\xi) \to W_0(\infty)$, so $W_{\text{comp}}(\xi) : \varepsilon^2 \xi^{-z+3}$, consistently matching the outer expansion.

Hence, this approximation connects the nonlinear center region with the algebraically decaying far field, ensuring asymptotic consistency over the entire domain.

2.3.4 Effect of varying ε, z

Case 2: We fix $a = 2.6$, $b = 1$, and $z = 3.5$, so that $a_1 \approx 1.04$.

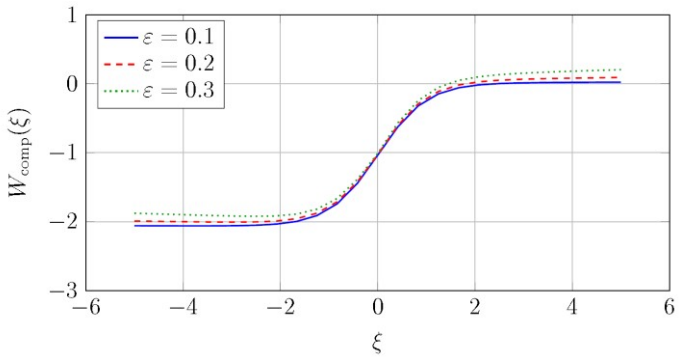

Figure 1
Composite solutions for some values of ε; with $z = 3.5$.

Case 2: We take $a = 2.6$, $b = 1$, and $z = 3.95$. Then $a_1 \approx 1.04$.

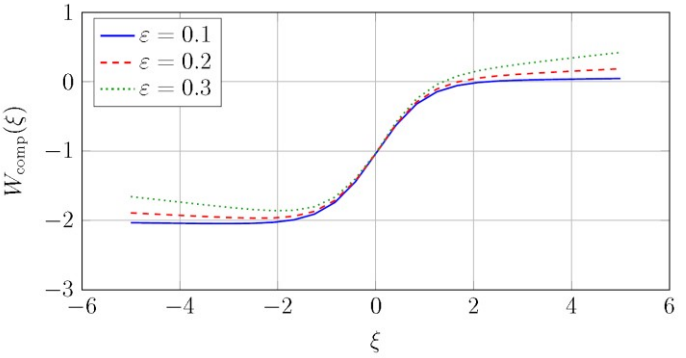

Figure 2
Composite solutions for some values of ε, with $z = 3.95$: Higher fractional order amplifies the perturbation at large $|\xi|$.

Discussions: The comparison between the two cases in Figure 1 and Figure 2 shows the influence of the parameter z on the behavior of the composite solution. As z increases (from 3.5 to 3.95), the algebraic correction term decays slowly, leading to a big contribution from the perturbation in the outer region. Consequently, the composite profile becomes sensitive to variations in ε, especially for large values of ξ. Near $\xi = 0$, the center core remains governed by the hyperbolic tangent structure, the tails are amplified for larger z, resulting in separation between solution curves. This shows that higher fractional orders enhance the long-range effects of the perturbation.

Conclusion

In this work, we have analyzed a nonlinear beam equation involving a space-fractional Caputo derivative of order $z \in (3,4)$, involving a perturbation

parameter ε. Using a traveling wave variable, we reduced the fractional problem to a nonlinear fractional ODE. For the reduced problem, with $\varepsilon = 0$, we applied the Tanh method to obtain explicit solutions. We then used a matched asymptotic analysis to account for the fractional perturbation effects. The inner region expansion revealed algebraically decaying corrections driven by the Caputo derivative, and the outer region behavior was consistent with power-law decay. By matching these two regions, we obtained a composite approximation that captures the full-domain behavior of the solution. Numerical simulations for some different ε and fractional orders z showed the influence of fractional effects on the beam deflection profile. In particular, higher values of z led to slower decay and long-range effects, underscoring the nonlocal influence of the fractional operator.

This study shows the usefulness of combining classical singular perturbation techniques with fractional calculus in modeling complex mechanical systems.

Future work could include extending this paper to time-fractional beam dynamics incorporating damping effects, or exploring multidimensional geometries.

References

[1] A. A. Alikhanov and C. Huang, "A class of time-fractional diffusion equations with generalized fractional derivatives," *Journal of Computational and Applied Mathematics*, vol. 414, Art. no. 114424 (Nov. 2022).

[2] D. Baleanu and K. Diethelm, Fractional calculus: models and numerical approaches, *Mathematics*, vol. 8, no. 3, Art. no. 369 (2020).

[3] I. M. Batiha, S. A. O. Albteiha, and S. Momani, "On Q-fractional calculus: Foundations and theoretical insights," in Recent Developments in Fractional Calculus: Theory, Applications, and Numerical Simulations, L. Kaur and P. Kumar, Eds., Studies in Systems, Decision and Control, vol. 235. Cham, Switzerland: Springer (2025), doi: 10.1007/978-3-031-84955-8_6.

[4] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, *Geophys. J. Int.*, vol. 13, no. 5, pp. 529-539 (1967).

[5] A. Carpinteri, P. Cornetti, and A. Sapora, Nonlocal elasticity: an appraoch based on fractional calculus, Meccanica, vol. 49 (2014).

[6] K. Diethelm, The Analysis of Fractional Differential Equations, Springer (2010).

[7] D. Dönmez Demir, N. Bildik, and B. G. Sinir, Application of fractional calculus in the dynamics of beams, Boundary Value Problems, vol. 2012, Art. no. 135 (2012).

[8] L. Euler, A method for finding curved lines enjoying properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense, Lausanne and Geneva (1744).

[9] T. Feng and M. Ni, "Asymptotic solution for a system of singularly perturbed delay differential equations," *Journal of Applied Analysis & Computation,* vol. 16, no. 1, pp. 458-478 (2026). doi:10.11948/20250097.

[10] R. E. George and B. O. Oyelami, "Analytical solution of the fractional-order Euler–Bernoulli beam equation," *Electronic Journal of Mathematics*, vol. 9, pp. 102-112 (2025).

[11] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific (2000).

[12] R.F. Kadam and K.M.M Al-Abrahemee, Neuro-fuzzy system for solving fuzzy singular perturbation problems, *Journal of Interdisciplinary Mathematics*, 25(5), 1509-1524 (2022).

[13] R. Kareem and K.M.M Al-Abraheme, Modification artificial neural networks for solving singular perturbation problems, *Journal of Interdisciplinary Mathematics*, 25(5), 1535-1549, 2022.

[14] A. N. Khan, U. Younis, M. Q. Mehmood, and M. Zubair, Atmospheric propagation of space-fractional Gaussian-beam waves in a FSO communication system, *Opt. Express,* vol. 30, no. 2, pp. 1570-1586, Jan. 2022.

[15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier (2006).

[16] J. Kevorkian and J. D. Cole, Multiple Scale and Singular Perturbation Methods, Springer (1996).

[17] M. A. Langthjem and H.M. Jensen, On matched asymptotic expansion solutions of a buckle wave propagation problem, ZAMM" *J. Appl. Math. Mech.* (2024).

[18] C. Li and F. Zeng, "Finite difference methods for fractional differential equations," *International Journal of Bifurcation and Chaos*, vol. 22, no. 04, p. 1230014 (2012).

[19] J. A. T. Machado, Some applications of fractional calculus in engineering, *Math. Probl. Eng.,* Art. ID 639801 (2010).

[20] I. Podlubny, Fractional Differential Equations, Academic Press (1999).

[21] P. D. Spanos and G. Malara, Nonlinear vibrations of beams and plates with fractional derivative elements subject to combined harmonic and random excitations, *Probabilist. Eng. Mech.,* vol. 59, Art. no. 103043 (Jan. 2020).

[22] P. Stempin and W. Sumelka, "Space-fractional Euler-Bernoulli beam model--theory and identification for silver nanobeam bending," *Int. J. Mech. Sci.,* vol. 186, Art. no. 105902 (Nov. 2020).

[23] P. Stempin and W. Sumelka, "Dynamics of space-fractional Euler-Bernoulli and Timoshenko beams," *Materials*, vol. 14, no. 8, Art. no. 1817 (Apr. 2021).

[24] W. Sumelka, "Fractional calculus for continuum mechanics—anisotropic non-locality," *Bull. Polish Acad. Sci. Tech. Sci.*, vol. 64, no. 2, pp. 1-10 (2016).

[25] S. Timoshenko, History of Strength of Materials, McGraw-Hill, New York (1953).

[26] C. Truesdell, The Rational Mechanics of Flexible or Elastic Bodies 1638-1788, Venditioni Exponunt Orell Fussli Turici (1960).

[27] L. Wang, G. Wang, X. Dong, X. Gao, and S. Zhuang, Creation of a stable vector vortex beam with dual fractional orbital angular momentum, *Sci. Rep.,* vol. 15, Art. no. 576 (2025).

Received July, 2025