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Fixed point theorem for three pairs of mappings in parametric metric space

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Abstract

Fixed point results are used in program verification, semantics of programming languages and recursive function theory, where multiple mappings may represent different processes or transformations. We established a common fixed-point theorem in Parametric metric space, utilizing weakly compatible mappings and implicit relations. By moving to Parametric Metric spaces, it covers cases where the distance function is Dynamic, which is not addressed in classical settings. These findings generalizes and enhances the existing common fixed point theorems based on the implicit relation. The use of implicit relations generalizes traditional contractive conditions, encompassing a wider range of possible inequalities and making the theorem applicable to diverse settings.

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1. Introduction

After the introduction of Banach's contraction principle [1] in 1922, the fixed point theory got attention from many researchers. Since it is one of the most important tools of analysis and has a vast application due to which it has become one of the most potential research area at present. In 1905, metric space was introduced by Frechet [2]. Since then, that research in Mathematics and applied sciences rely heavily on metric spaces. Metric space has been explored by researchers time to time and as a result we can find number of variants of metric space in the literature [3, 4, 5, 6, 7, 8, 9, 10, 11]. Hussain et al. [12] and Hussain, Salimi, and Parvaneh [13] introduced Parametric metric space in 2014 and Parametric b-Metric Space in 2015 as an extension of metric space and proved many results based on these spaces [14, 15, 16, 17, 18].

Definition 1: [12] Assume Let Γ be a set containing at least one element, and $\delta: \mathbb{X} \times \mathbb{X} \times (0,\infty) \to (0,\infty)$ be a function which is Parametric Metric on $\mathbb{X}$ if,

1. $\delta(\varkappa,\curlyvee,t) = 0 \ \ \forall t > 0$ iff $\varkappa = \curlyvee$
2. $\delta(\varkappa,\curlyvee,t) = \delta(\curlyvee,\varkappa,t) \ \ \forall \varkappa,\curlyvee \in \Gamma, t > 0$
3. $\delta(\varkappa,\curlyvee,t) \leq [\delta(\varkappa,\varpi,t) + \delta(\varpi,\curlyvee,t)] \ \ \forall \varkappa,\curlyvee,\varpi \in \mathbb{X}, t > 0$

Then, the pair (Γ,δ) is a PMS.

The present study extends and generalises some common FP theorems [19] and proved the development of new FP theorems for multiple pairs of mappings in generalized metric spaces, where PMS, is a significant area of research. Fixed point theorems are crucial tools in mathematics for establishing the existence of solutions to various equations such as differential equations, integral equations, operator equations, particularly in systems where dynamics or parameters play a role [20, 21, 22, 23, 24, 25].

2. Main Results

Theorem 1: *Let* F, G, H, f, g, h *be six self maps of PMS* (Γ,δ) *with* $\overline{H(\Gamma)} \subseteq f(\Gamma), \overline{G(\Gamma)} \subseteq h(\Gamma), \overline{F(\Gamma)} \subseteq g(\Gamma)$ *and for all* $\varkappa,\curlyvee,\varpi \in \Gamma, t > 0$ *and* $\psi \in \Psi$

Then;

$$\psi(\delta(F\varkappa, G\curlyvee, t), \delta(G\curlyvee, H\varpi, t), \delta(f\varkappa, g\curlyvee, t), \delta(g\curlyvee, h\varpi, t), \delta(f\varkappa, F\varkappa, t),$$
$$\delta(g\curlyvee, G\curlyvee, t), \ \delta(h\varpi, H\varpi, t), \delta(f\varkappa, H\varpi, t), \delta(g\curlyvee, F\varkappa, t), \delta(h\varpi, G\curlyvee, t)) \leq 0 \tag{2.1}$$

There is one point of coincidence in Γ *for a complete subspace where* Γ *may be represented as* $\overline{H(\Gamma)}$,*G(*Γ*),* $\overline{F(\Gamma)}$. *Moreover,* Γ *have a unique common FP if* F, G, H, f, g, h *are weakly compatible.*

Proof: Assume $\varkappa_0$ be an arbitrary point as $H(\Gamma) \subseteq \overline{H(\Gamma)}, G(\Gamma) \subseteq \overline{G(\Gamma)}, F(\Gamma) \subseteq \overline{F(\Gamma)}$. We have, $F(\Gamma) \subseteq f(\Gamma), G(\Gamma) \subseteq g(\Gamma), H(\Gamma) \subseteq h(\Gamma)$. The sequence $\{\varkappa_n\}, \{\curlyvee_n\}, \{\varpi_n\} \in \Gamma$ can be inductively defined as follows :

$$\begin{aligned}\varpi_{2n-2} &= F\varkappa_{2n-2} = g\varkappa_{2n-1},\\ \varpi_{2n-1} &= H\varkappa_{2n-1} = f\varkappa_{2n},\\ \varpi_{2n} &= G\varkappa_{2n} = h\varkappa_{2n+1} \forall n \in N\end{aligned} \tag{2.2}$$

From (2.1) with $\varkappa = \varkappa_{2n+1}, \curlyvee = \varkappa_{2n+2}, \varpi = \varkappa_{2n+3}$, We get for all $t > 0$ and all $n \in N$;

$$\begin{aligned}&\psi(\delta(F\varkappa_{2n+1}, G\varkappa_{2n+2}, t), \delta(G\varkappa_{2n+2}, H\varkappa_{2n+3}, t), \delta(f\varkappa_{2n+1}, g\varkappa_{2n+2}, t),\\ &\delta(g\varkappa_{2n+2}, h\varkappa_{2n+3}, t), \delta(f\varkappa_{2n+1}, F\varkappa_{2n+1}, t), \delta(g\varkappa_{2n+2}, G\varkappa_{2n+2}, t),\\ &\delta(h\varkappa_{2n+3}, H\varkappa_{2n+3}, t), \delta(f\varkappa_{2n+1}, H\varkappa_{2n+3}, t), \delta(g\varkappa_{2n+2}, F\varkappa_{2n+1}, t),\\ &\delta(h\varkappa_{2n+3}, G\varkappa_{2n+2}, t)) \le 0\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\psi(\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n+2}, \varpi_{2n+3}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t), \delta(\varpi_{2n+1}, \varpi_{2n+2}, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+3}, t), \delta(\varpi_{2n}, \varpi_{2n+3}, t), \delta(\varpi_{2n+1}, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+2}, t)) \le 0\end{aligned}$$

That is

$$\begin{aligned}&\psi(\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n+2}, \varpi_{2n+3}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t), \delta(\varpi_{2n+1}, \varpi_{2n+2}, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+3}, t), \delta(\varpi_{2n}, \varpi_{2n+3}, t), 0, 0) \le 0\end{aligned} \tag{2.4}$$

As ψ is a non-increasing in variable u_8, u_9, u_{10}, we obtain;

$$\delta(\varpi_{2n}, \varpi_{2n+3}, t) \le (\delta(\varpi_{2n}, \varpi_{2n+1}, t) + \delta(\varpi_{2n+1}, \varpi_{2n+2}, t) + \delta(\varpi_{2n+2}, \varpi_{2n+3}, t) \tag{2.5}$$

From (2.4), we derive that,

$$\begin{aligned}&\psi(\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n+2}, \varpi_{2n+3}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(\varpi_{2n}, \varpi_{2n+1}, t), \delta(\varpi_{2n+1}, \varpi_{2n+2}, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+3}, t), (\delta(\varpi_{2n}, \varpi_{2n+1}, t) + \delta(\varpi_{2n+1}, \varpi_{2n+2}, t) +\\ &\delta(\varpi_{2n+2}, \varpi_{2n+3}, t), 0, 0) \le 0\end{aligned} \tag{2.6}$$

Now using property Ψ_{2a} we have;

$$\delta(\varpi_{2n+2}, \varpi_{2n+3}, t) \le k\delta(\varpi_{2n+1}, \varpi_{2n+2}, t) \tag{2.7}$$

Again using (2.1) with,

$\varkappa = \varkappa_{2n}, \curlyvee = \varkappa_{2n+1}, \varpi = \varkappa_{2n+2}$, we get $\forall\ t > 0, n \in N$;

$$\begin{aligned}&\psi(\delta(F\varkappa_{2n}, G\varkappa_{2n+1}, t), \delta(G\varkappa_{2n+1}, H\varkappa_{2n+2}, t), \delta(f\varkappa_{2n}, g\varkappa_{2n+1}, t),\\ &\delta(g\varkappa_{2n+1}, h\varkappa_{2n+2}, t), \delta(f\varkappa_{2n}, F\varkappa_{2n}, t), \delta(g\varkappa_{2n+1}, G\varkappa_{2n+1}, t),\\ &\delta(h\varkappa_{2n+2}, H\varkappa_{2n+2}, t), \delta(f\varkappa_{2n}, H\varkappa_{2n+2}, t), \delta(g\varkappa_{2n+1}, F\varkappa_{2n}, t),\\ &\delta(h\varkappa_{2n+2}, G\varkappa_{2n+1}, t)) \le 0\end{aligned} \tag{2.8}$$

$$\psi(\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n+1},\varpi_{2n+2},t),\delta(\varpi_{2n-1},\varpi_{2n},t),$$
$$\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n-1},\varpi_{2n},t),\delta(\varpi_{2n},\varpi_{2n+1},t),$$
$$\delta(\varpi_{2n+1},\varpi_{2n+2},t),\delta(\varpi_{2n-1},\varpi_{2n+2},t),\delta(\varpi_{2n},\varpi_{2n},t),$$
$$\delta(\varpi_{2n+1},\varpi_{2n+1},t))\leq 0$$

$$\psi(\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n+1},\varpi_{2n+2},t),\delta(\varpi_{2n-1},\varpi_{2n},t),$$
$$\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n-1},\varpi_{2n},t),\delta(\varpi_{2n},\varpi_{2n+1},t),$$
$$\delta(\varpi_{2n+1},\varpi_{2n+2},t),\delta(\varpi_{2n-1},\varpi_{2n+2},t),0,0)\leq 0 \tag{2.9}$$

As ψ is a non-increasing in variable u_8,u_9,u_{10}, we have;

$$\delta(\varpi_{2n-1},\varpi_{2n+2},t)\leq(\delta(\varpi_{2n-1},\varpi_{2n},t)+\delta(\varpi_{2n},\varpi_{2n+1},t)+\delta(\varpi_{2n+1},\varpi_{2n+2},t) \tag{2.10}$$

From (2.9) we obtain,

$$\psi(\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n+1},\varpi_{2n+2},t),\delta(\varpi_{2n-1},\varpi_{2n},t),$$
$$\delta(\varpi_{2n},\varpi_{2n+1},t),\delta(\varpi_{2n-1},\varpi_{2n},t),\delta(\varpi_{2n},\varpi_{2n+1},t),$$
$$\delta(\varpi_{2n+1},\varpi_{2n+2},t),(\delta(\varpi_{2n-1},\varpi_{2n},t)+\delta(\varpi_{2n},\varpi_{2n+1},t)+ \tag{2.11}$$
$$\delta(\varpi_{2n+1},\varpi_{2n+2},t))\leq 0$$

Yielding thereby (due to (Ψ_{2a});

$$\delta(\varpi_{2n+1},\varpi_{2n+2},t)\leq k\delta(\varpi_{2n},\varpi_{2n+1},t) \tag{2.12}$$

Combining (2.7) and (2.12), we have;

$$\delta(\varpi_{2n+2},\varpi_{2n+3},t)\leq k^2\delta(\varpi_{2n},\varpi_{2n+1},t) \tag{2.13}$$

Now by induction , we obtainfor each $n=0,1,2,\ldots$.

$$\delta(\varpi_{2n+2},\varpi_{2n+3},t)\leq k\delta(\varpi_{2n+1},\varpi_{2n+2},t)\leq\ldots\ldots\leq k^{2n+2}\delta(\varpi_0,\varpi_1,t) \tag{2.14}$$

and by a routine calculation, we have;

$$\delta(\varpi_{2n+2},\varpi_{2n+3},t)\leq k\delta(\varpi_{2n+1},\varpi_{2n+2},t)\leq\ldots\ldots\leq k^{n+2}\delta(\varpi_0,\varpi_1,t) \tag{2.15}$$

Hence for each $n>m$, we obtain;

$$\delta(\varpi_n,\varpi_m,t)\leq\delta(\varpi_n,\varpi_{n-1},t)+\delta(\varpi_{n-1},\varpi_{n-2},t)+\cdots\ldots+\delta(\varpi_{m+1},\varpi_m,t)$$
$$\leq(k^{n-1}+k^{n-2}+\cdots.+k^m)\delta(\varpi_0,\varpi_1,t)\leq\frac{k^m}{1-k}\delta(\Upsilon_0,\Upsilon_1,t) \tag{2.16}$$

So, the sequence $\{\varpi_n\}$ is Cauchy. Let $\overline{H(\Gamma)}$ is complete. The sequence $\{\varpi_{2n}\}$ is a Cauchy sequence which is contained in $\overline{H(\Gamma)}$ must a limit Ω^* in $f(x)$, that is;

$$\lim_{n\to\infty}\varpi_{2n+1}=\lim_{n\to\infty}Hx_{2n+1}=\lim_{n\to\infty}fx_{2n+2}\in\overline{H(\Gamma)}\in\Gamma\subseteq f(\Gamma)\in\Gamma,$$
$$\lim_{n\to\infty}\varpi_{2n+1}=\lim_{n\to\infty}Hx_{2n+1}=\lim_{n\to\infty}fx_{2n+2}=\Omega^*\in f(\Gamma) \tag{2.17}$$

It is easy to see

$$\Omega^*=\lim_{n\to\infty}\varpi_{n+1}=\lim_{n\to\infty}Hx_{2n+1}=\lim_{n\to\infty}fx_{2n+2}=\lim_{n\to\infty}Gx_{2n+1}$$
$$=\lim_{n\to\infty}fx_{2n+2}=\lim_{n\to\infty}Fx_{2n-2}=\lim_{n\to\infty}gx_{2n+1} \tag{2.18}$$

We can find $\Omega \in \Gamma$ where $f\Omega = \Omega^*$. So, $F\Omega = f\Omega = \Omega^*$.

Otherwise;

$\delta(F\Omega, \Omega^*, t) > 0$. Using (2.1), with $\varkappa = \Omega$,$\Upsilon = \varkappa_{2n+1}, \varpi = \varkappa_{2n+2}$, We have;

$$\begin{aligned}&\psi(\delta(F\Omega, G\varkappa_{2n+1}, t), \delta(G\varkappa_{2n+1}, H\varkappa_{2n+2}, t), \delta(f\Omega, g\varkappa_{2n+1}, t),\\ &\delta(g\varkappa_{2n+1}, h\varkappa_{2n+2}, t), \delta(f\Omega, F\Omega, t), \delta(g\varkappa_{2n+1}, G\varkappa_{2n+1}, t),\\ &\delta(h\varkappa_{2n+2}, H\varkappa_{2n+2}, t), \delta(f\Omega, H\varkappa_{2n+2}, t), \delta(g\varkappa_{2n+1}, F\Omega, t),\\ &\delta(h\varkappa_{2n+2}, G\varkappa_{2n+1}, t)) \le 0\end{aligned}$$

$$\begin{aligned}&\psi(\delta(F\Omega, \varpi_{2n+1}, t), \delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(f\Omega, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+1}, \varpi_{2n+2}, t), \delta(f\Omega, F\Omega, t), \delta(\varpi_{2n+1}, \varpi_{2n+1}, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+2}, t), \delta(f\Omega, \varpi_{2n+2}, t), \delta(\varpi_{2n+1}, F\Omega, t),\\ &\delta(\varpi_{2n+2}, \varpi_{2n+1}, t)) \le 0\end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in the above inequality, using (2.18) and continuity of ψ, we have;

$$\psi(\delta(F\Omega, \Omega^*, t),0,0,0, \delta(\Omega^*, F\Omega, t),0,0, \delta(\Omega^*, F\Omega, t),0) \le 0 \tag{2.20}$$

Yielding thereby (due to (ψ_{2b}), $\delta(F\Omega, \Omega^*, t) \le 0$, where $\delta(F\Omega, \Omega^*, t) = 0$, which is a contradiction. Then we have $F\Omega = f\Omega = \Omega^*$, which shows that Ω is a coincidence point of F and f, that is Ω^* is a point of coincidence of F and f. Since, $\Omega^* = F\Omega \in F(\Gamma) \subseteq \overline{F(\Gamma)} = g(\Gamma)$, there exist $\Omega', \Omega'' \in \Gamma$ as $g\Omega' = h\Omega'' = \Omega^*$, So, $G\Omega' = H\Omega'' = \Omega^*$. Otherwise, $P(G\Omega', \Omega^*, t) \le 0, \delta(H\Omega'', \Omega^*, t) \le 0$. Using (2.1); with $\varkappa = \Omega$,$\Upsilon = \Omega', \varpi = \Omega''$, we have;

$$\begin{aligned}&\psi(\delta(F\Omega, G\Omega', t), \delta(G\Omega', H\Omega'', t), \delta(f\Omega, g\Omega', t),\\ &\delta(g\Omega', h\Omega'', t), \delta(f\Omega, F\Omega, t), \delta(g\Omega', G\Omega', t),\\ &\delta(h\Omega'', H\Omega'', t), \delta(f\Omega, H\Omega'', t), \delta(g\Omega', F\Omega, t),\\ &\delta(h\Omega'', G\Omega', t)) \le 0\end{aligned}$$

$$\psi(\delta(\Omega^*, G\Omega', t),0,0,0,0, \delta(\Omega^*, H\Omega'', t), \delta(\Omega^*, H\Omega'', t),0, \delta(\Omega^*, G\Omega', t)) \le 0 \tag{2.21}$$

Yielding thereby (due to (Ψ_{2b}), $\delta(\Omega^*, G\Omega', t) \le 0$, then $\delta(\Omega^*, G\Omega', t) = 0$, and $\delta(\Omega^*, H\Omega'', t) \le 0$ then, $\delta(\Omega^*, H\Omega'', t) = 0$. Thus, our supposition that $\delta(G\Omega', \Omega^*, t) > 0, \delta(H\Omega'', \Omega^*, t) > 0$ was wrong. Therefore,$G\Omega' = g\Omega' = \Omega^*$ and $H\Omega'' = h\Omega'' = \Omega^*$, which shows that Ω', Ω'' are the point of coincidence of G, g and H, h that is Ω^* is a point of coincidence of G, g, H and h. Let, Ω_* is another coincidence point of F and f that is

$\Omega_* = F\overline{\Omega} = f\overline{\Omega}$ for some $\overline{\Omega} \in \Gamma$. Using (2.1); we have,

$$\begin{aligned}&\psi(\delta(F\overline{\Omega}, G\Omega', t), \delta(G\Omega', H\Omega'', t), \delta(f\overline{\Omega}, g\Omega', t),\\ &\delta(g\Omega', h\Omega'', t), \delta(f\overline{\Omega}, F\overline{\Omega}, t), \delta(g\Omega', G\Omega', t),\\ &\delta(h\Omega'', H\Omega'', t), \delta(f\overline{\Omega}, H\Omega'', t), \delta(g\Omega', F\overline{\Omega}, t),\\ &\delta(h\Omega'', G\Omega', t)) \le 0\end{aligned} \tag{2.22}$$

This implies that;

$$\begin{aligned}&\psi(\delta(\Omega_*,\Omega^*,t),\delta(\Omega^*,\Omega^*,t),\delta(\Omega_*,\Omega^*,t),\\&\delta(\Omega^*,\Omega^*,t),\delta(\Omega_*,\Omega_*,t),\delta(\Omega^*,\Omega^*,t),\\&\delta(\Omega^*,\Omega^*,t),\delta(\Omega_*,\Omega^*,t),\delta(\Omega^*,\Omega_*,t),\\&\delta(\Omega^*,\Omega^*,t))\le 0\\&\psi(\delta(\Omega_*,\Omega^*,t),0,\delta(\Omega_*,\Omega^*,t),0,0,0,0,\delta(\Omega_*,\Omega^*,t),\\&\delta(\Omega_*,\Omega^*,t),\delta(\Omega^*,\Omega_*,t),0)\le 0\end{aligned}\tag{2.23}$$

(Ψ_3) contradicts, if $\Omega_* \neq \Omega^*$. Hence, Coinciding point of F and f is unique. Now, suppose that ${\Omega_1}^*$ and ${\Omega_2}^*$ are another Coinciding point of G, g and H , h respectively, that is ${\Omega_1}^* = G\Omega_1 = g\Omega_1$ for some $\Omega_1 \in \Gamma$ and ${\Omega_2}^* = H\Omega_2 = h\Omega_2$ for some $\Omega_2 \in \Gamma$. Using (2.1); we have,

$$\begin{aligned}&\psi(\delta(F\Omega,G\Omega_1,t),\delta(G\Omega_1,H\Omega_2,t),\delta(f\Omega,g\Omega_1,t),\\&\delta(g\Omega_1,h\Omega_2,t),\delta(f\Omega,F\Omega,t),\delta(g\Omega_1,G\Omega_1,t),\\&\delta(h\Omega_2,H\Omega_2,t),\delta(f\Omega,H\Omega_2,t),\delta(g\Omega_1,F\Omega,t),\\&\delta(h\Omega_2,G\Omega_1,t))\le 0\\&\psi(\delta(\Omega^*,\Omega_1^*,t),\delta(\Omega_1^*,\Omega_2^*,t),\delta(\Omega^*,\Omega_1^*,t),\\&\delta(\Omega_1^*,\Omega_2^*,t),\delta(\Omega^*,\Omega^*,t),\delta(\Omega_1^*,\Omega_1^*,t),\\&\delta(\Omega_2^*,\Omega_2^*,t),\delta(\Omega^*,\Omega_2^*,t),\delta(\Omega_1^*,\Omega^*,t),\\&\delta(\Omega_2^*,\Omega_1^*,t))\le 0\end{aligned}$$

That is ;

$$\begin{aligned}&\psi(\delta(\Omega^*,\Omega_1^*,t),\delta(\Omega_1^*,\Omega_2^*,t),\delta(\Omega^*,\Omega_1^*,t),\delta(\Omega_1^*,\Omega_2^*,t),\\&0,0,0,\delta(\Omega^*,\Omega_2^*,t),\delta(\Omega_1^*,\Omega^*,t),\delta(\Omega_2^*,\Omega_1^*,t))\le 0\end{aligned}\tag{2.24}$$

which contradicts (Ψ_3), if $\Omega_* \neq \Omega^*$ .So, the Coinciding point of G, g and H, h is unique.Then, we proved that Ω^* is the unique Coinciding point of (F, f), (G, g) and (H, h). Also, if $(F, f), (G, g)$ and (H, h) are weakly compatible, from $F\Omega = f\Omega = \Omega^*$,$G\Omega' = g\Omega' = \Omega^*$ and $H\Omega'' = h\Omega'' = \Omega^*$, we have, $F\Omega^* = F(f\Omega) = f(F\Omega) = f\Omega^*, G\Omega^* = G(g\Omega') = g(G\Omega') = g\Omega^*$ and $H\Omega^* = H(h\Omega'') = h(H\Omega'') = h\Omega^*$. Otherwise; $F\Omega^* \neq G\Omega^* \neq H\Omega^*$ and from (2.1); we have,

$$\begin{aligned}&\psi(\delta(F\Omega^*,G\Omega^*,t),\delta(G\Omega^*,H\Omega^*,t),\delta(f\Omega^*,g\Omega^*,t),\\&\delta(g\Omega^*,h\Omega^*,t),\delta(f\Omega^*,F\Omega^*,t),\delta(g\Omega^*,G\Omega^*,t),\\&\delta(h\Omega^*,H\Omega^*,t),\delta(f\Omega^*,H\Omega^*,t),\delta(g\Omega^*,F\Omega^*,t),\\&\delta(h\Omega^*,G\Omega^*,t))\le 0\\&\psi(\delta(F\Omega^*,G\Omega^*,t),\delta(G\Omega^*,H\Omega^*,t),\delta(F\Omega^*,G\Omega^*,t),\delta(G\Omega^*,H\Omega^*,t),\\&0,0,0,\delta(F\Omega^*,H\Omega^*,t),\delta(G\Omega^*,F\Omega^*,t),\delta(H\Omega^*,G\Omega^*,t))\le 0\end{aligned}\tag{2.25}$$

By property (Ψ_3), we deduce that $\delta(F\Omega^*,G\Omega^*,t)\le 0$ that is $\delta(F\Omega^*,G\Omega^*,t) = 0$ and $\delta(G\Omega^*,H\Omega^*,t)\le 0$ that is $\delta(G\Omega^*,H\Omega^*,t) = 0$ and then our assumption that $F\Omega^* \neq G\Omega^* \neq H\Omega^*$ was wrong. Hence, $F\Omega^* = f\Omega^* = G\Omega^* = g\Omega^* = H\Omega^* = h\Omega^*$. Finally, we show that $F\Omega^* = f\Omega^* = G\Omega^* = g\Omega^* = H\Omega^* = h\Omega^*$. Again, from (2.1); we have,

$$\begin{aligned}&\psi(\delta(F\Omega,G\Omega^*,t),\delta(G\Omega^*,H\Omega^*,t),\delta(f\Omega,g\Omega^*,t),\delta(g\Omega^*,h\Omega^*,t),\\&\delta(f\Omega,F\Omega,t),\delta(g\Omega^*,G\Omega^*,t),\delta(h\Omega^*,H\Omega^*,t),\\&\delta(f\Omega,H\Omega^*,t),\delta(g\Omega^*,F\Omega,t),\delta(h\Omega^*,G\Omega^*,t))\le 0\end{aligned}$$

$$\psi(\delta(\Omega^*, G\Omega^*, t), \delta(G\Omega^*, H\Omega^*, t), \delta(\Omega^*, G\Omega^*, t), \delta(G\Omega^*, H\Omega^*, t), 0,0,0, \delta(\Omega^*, H\Omega^*, t), \delta(G\Omega^*, \Omega^*, t), \delta(H\Omega^*, G\Omega^*, t)) \leq 0 \tag{2.26}$$

Yielding thereby due to (Ψ_3), $\delta(\Omega^*, G\Omega^*, t) \leq 0$ and so $\delta(\Omega^*, G\Omega^*, t) = 0$ and $\delta(G\Omega^*, H\Omega^*, t) \leq 0$. So; $\delta(G\Omega^*, H\Omega^*, t) = 0$, a contradiction; if $\Omega^* \neq G\Omega^* \neq H\Omega^*$. Hence, $F\Omega^* = f\Omega^* = G\Omega^* = g\Omega^* = H\Omega^* = h\Omega^* = \Omega^*$. Then, Ω^* is the unique common FP of F, f, G, g, H, h. This concludes the proof.

3. Conclusion

The present work describes a common FP theorem for weakly compatible mappings which satisfies implicit function under three pairs of mappings. The implicit function is used to generalize and extend the existing fixed point theorems.

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