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Exponential convergence and boundedness criteria for differential equations using integral inequalities

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Abstract

The aim purpose of the present work is to investigate a nonlinear Gronwall-Bellman-Bihari-Gamidov's type inequality. This investigation allows us to study the asymptotic behavior of some differential equations in presence of perturbation in the sense that the solutions are bounded and exponential convergence to a small neighborhood of the origin.

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The obtained findings can be used to study some properties of boundedness of solutions for perturbed differential equations.

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1. Introduction

Numerous studies have been conducted on the qualitative behavior of time-varying differential equations, and stability issues involving perturbations in the state space have been thoroughly examined. The primary purpose of this approach is to efficiently investigate qualitative features of solutions for systems of differential, integro-differential, and functional-differential equations with impulsive perturbations. These features include attraction, boundedness, stability by Lyapunov, stability by Chetaev, and stability by Lyapunov ([22], [28], [31]-[34]). Additionally, estimates of solutions for systems with various types of (continuous, discrete, and parametric) perturbations can be written analytically ([3]-[9], [12]-[16], [18], [20]-[21], [23]-[28], [29], [35]-[41]).

The science of linear and nonlinear differential equations, as well as integral equations, has developed largely due to integral inequalities, which give explicit limitations on unknown functions ([1], [11], [32]). The Gronwall-Bellman integral inequality is one of the most well-known and often applied inequality types ([2], [19]). A number of novel extensions of the Gronwall-Bellman inequality have been constructed by many authors ([10], [14], [17], [20], [24], [25], [31]) in recent years, in light of the significant applications of this inequality. These generalizations can be effective tools in the study of certain new classes of differential and integral equations. In the meanwhile, other writers have investigated numerous extensions of the Gronwall-Bellman disparity.

Investigating a nonlinear Gronwall-Bellman-Bihari-Gamidov's type inequality is the main goal of the current paper. Some sufficient conditions ensuring the global exponential convergence and boundedness of the solutions of the system are obtained. We prove that, we are able to examine the asymptotic behavior of some differential equations in the presence of perturbations thanks to this integral inequalities. The results acquired can be applied to the study of the boundedness and exponential convergence of solutions to a small neighborhood of the origin for certain perturbed differential systems.

1.1 *Exponential convergence and boundedness analysis*

Let consider the following differential equation: $\dot{x} = \Psi(t, x)$. Unless otherwise stated, we assume throughout the paper that the function $\Psi(.,.)$ encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we will denote by $x(t, t_0, x_0)$, or simply by $x(t)$, the unique solution at time t_0 starting from the point x_0. Unless otherwise stated, we assume throughout the paper that the function encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\|.\|$ stands for the Euclidean norm vectors.

In the sequel, for dynamic systems with parameter, we will remember what uniformly ultimate boundedness and exponential convergence in the sense of [39]. Let consider the following system described by:

$$\dot{x} = \Psi^{\varepsilon}(t, x) \tag{1}$$

with Ψ^{ε} are smooth functions with $\varepsilon \in]0, \varepsilon^*[, \varepsilon^* > 0$, $t \in \mathbb{R}_+$ is the time and $x \in \mathbb{R}^n$ is the state, $\varepsilon > 0$ represents a small parameter in the system. The existence and uniqueness of the solutions are ensured by the fact that $\Psi^{\varepsilon}(t, x) := \Psi(t, x, \varepsilon)$ is continuous in (t, x, ε) and locally Lipschitz in (x, ε), uniformly in t. The solution of (1) will depend on the parameter ε, where the solution can be written as $x(t, \varepsilon)$.

Let $x(t, \varepsilon)$ the solution of (1.1) with respect the initial condition $(t_0, x(0))$. System (1.1) is said to be globally exponentially ultimately bounded if:

$$\|x(t, \varepsilon)\| \leq k\|x(0)\|e^{-\gamma(t-t_0)} + r(\varepsilon), \text{ for all } t \geq t_0 \geq 0, \tag{2}$$

with $k > 0$, $\gamma > 0, r(\varepsilon) > 0, \varepsilon > 0$.

Using the definitions introduced by [3] and [5], the last inequality is none other than practical stability in the sense that the trajectories approach a small neighborhood of the origin which is equivalent to:

$$\|x(t)\| - r(\varepsilon) \leq k\|x(0)\|e^{-\gamma(t-t_0)}, \text{ for all } t \geq t_0 \geq 0.$$

So, the solutions are globally uniformly bounded. Moreover, by taking initial conditions outside the ball B_ε centered at the origin and of radius $r(\varepsilon)$, the solutions converge exponentially to B_ε. Therefore, in the case when $r = r(\varepsilon)$ goes to zero as ε tends to zero, then for initial conditions taken outside the ball B_ε, the states approache the origin exponentially as t tends to infinity.

The class of system under consideration has the following form:

$$\dot{x}(t) = A(t)x(t) + \psi(t, x(t), \varepsilon),$$

where $A(t) \in \mathbb{R}^{n \times n}$, is matrice whose elements are continuous and bounded and the function $\psi(t, x, \varepsilon)$ is a smooth function.

In the following, we need a version of Gronwall's Lemma of the Gamidov type to be able to study the boundedness of the solutions and their asymptotic behavior.

The famous Gronwall inequality [19], can be expressed by taking into account a small parameter $\varepsilon \in]0, \varepsilon^*[, \varepsilon^* > 0$, as follows: *Let* $u^{\varepsilon}(t)$ *be a continuous function defined on the interval* $[t_0, t_1]$ *with* $\varepsilon > 0$ *and*

$$u^{\varepsilon}(t) \leq a + b\int_{t_0}^{t} u^{\varepsilon}(s)ds,$$

where a and b are nonnegative constants. Then, for all $t \in [t_0, t_1]$, *we have*

$$u^{\varepsilon}(t) \leq ae^{b(t-t_0)}.$$

The above inequality can be extended as follows:

Let a be a positive constant, $u^{\varepsilon}(t)$ and $b(t)$, $t \in [t_0, t_1]$ be real-valued continuous functions, $b(t) \geq 0$, satisfying

$$u^{\varepsilon}(t) \leq a + \int_{t_0}^{t} b(s)u^{\varepsilon}(s)ds, t \in [t_0, t_1].$$

Then, for all $t \in [t_0, t_1]$, we have

$$u^{\varepsilon}(t) \leq aexp\{\int_{t_0}^{t} b(s)ds\}.$$

The result shows that $u(t, \varepsilon)$ is no more involved in the right-hand side of first inequality. This inequality has important applications in the theory of ordinary differential equations in connection with proof of uniqueness of solution and its comparison criteria. It has been further extended by many authors to cover some more problems of ordinary differential equations. For more general case, let u^{ε} and v nonnegative piecewise continuous functions on $[0, +\infty)$ for which the inequality:

$$u^{\varepsilon}(t) \leq c(t) + \int_{t_0}^{t} u^{\varepsilon}(s)v(s)ds, \forall t \geq t_0,$$

holds, where t_0 is a nonnegative constant and c is nonnegative piecewise continuous function. Then,

$$u^{\varepsilon}(t) \leq c(t) + \int_{t_0}^{t} e^{\Gamma(t)-\Gamma(s)}c(s)v(s)ds, \forall t \geq t_0,$$

where $\Gamma(t) = \int_{t_0}^{t} v(\tau)d\tau$. As an extension, when $c(t)$ is bounded by c for all $t \geq\geq t_0 \geq 0$, for which the inequality

$$u^{\varepsilon}(t) \leq c + \int_{t_0}^{t} (u^{\varepsilon}(s)v(s) + w(s))ds, \forall t \geq t_0$$

holds, where w nonnegative piecewise continuous function on $[0, +\infty)$ Then,

$$u^{\varepsilon}(t) \leq (c + \int_{t_0}^{t} w(s)ds)e^{\int_a^t v(z)dz}, \forall t \geq t_0.$$

The somewhat more general extensions of the original Gronwall inequality can be found in [14]. The Gamidov's Lemma can be stated as follows (see [17]).

If

$$u^{\varepsilon}(t) \leq \ell(t) + c\int_0^t \phi(s)(u^{\varepsilon}(s))^p ds$$

where all functions are continuous and nonnegative on $[0, +\infty[$, $0 < p < 1$, $c \geq 0$, then there exists a constant $\kappa > 0$, such that

$$u^{\varepsilon}(t) \leq \ell(t) + c\kappa^p(\int_0^t \phi^{\frac{1}{1-p}}(s)ds)^{1-p}.$$

To obtain a result of exponential convergence and boundedness of solutions for the above system, we need the following new version of Gamidov's Lemma. Let $\varepsilon \in]0, \varepsilon^*[, \varepsilon^* > 0$.

Lemma 1.1: *If*

$$v^{\varepsilon}(t) \leq \ell + \int_0^t [ce^{(1-p)\gamma s}(v^{\varepsilon})^p(s) + \delta v^{\varepsilon}(s)]ds$$

where v *is continuous and nonnegative on* $[0,+\infty)$*;* $p \in (0,1)$*;* $\ell, c > 0$ *and* γ, δ *such that* $0 \leq \delta < \gamma$ *then*

$$v^{\varepsilon}(t) \leq 2^{\frac{p}{1-p}} \ell e^{\delta t} + \left(\frac{2^p c}{\gamma-\delta}\right)^{\frac{1}{1-p}} e^{\gamma t}.$$

Proof: Let

$$w_{\varepsilon}(t) = \ell + \int_0^t \left[c e^{(1-p)\gamma s}(v^{\varepsilon})^p(s) + \delta v^{\varepsilon}(s)\right] ds > 0, \forall t \geq 0$$

then, we have $w_{\varepsilon}(0) = \ell$ and $w_{\varepsilon}(t) \geq v^{\varepsilon}(t)$ for all $t \geq 0$. For each $\varepsilon > 0$, the function w_{ε} is differentiable at t and

$$\dot{w}_{\varepsilon}(t) = c e^{(1-p)\gamma t}(v^{\varepsilon})^p(t) + \delta v^{\varepsilon}(t)$$

then $\dot{w}_{\varepsilon}(t) \leq c e^{(1-p)\gamma t} w_{\varepsilon}^p(t) + \delta w_{\varepsilon}(t)$. Since $w(t) > 0$, it follows that

$$\frac{\dot{w}_{\varepsilon}(t)}{w_{\varepsilon}^p(t)} \leq c e^{(1-p)\gamma t} + \delta w_{\varepsilon}^{1-p}(t).$$

Consider now $u_{\varepsilon}(t) = w_{\varepsilon}^{1-p}(t), \forall t \geq 0$. It comes that

$$\dot{u}_{\varepsilon}(t) \leq c(1-p)e^{(1-p)\gamma t} + \delta(1-p)u_{\varepsilon}(t) \text{ and } u_{\varepsilon}(0) = \ell^{1-p}.$$

We denote by ρ the solution of the linear differential equation

$$\begin{cases} \dot{\rho}(t) = \delta(1-p)\rho(t) + c(1-p)e^{(1-p)\gamma t} \\ \rho(0) = \ell^{1-p}. \end{cases}$$

By the method of variation of constants, we get

$$\rho(t) = (\ell^{1-p} - \frac{c}{\gamma-\delta})e^{\delta(1-p)t} + \frac{c}{\gamma-\delta} e^{\gamma(1-p)t}.$$

It follows from the standard comparison principle that

$$u_{\varepsilon}(t) \leq \rho(t), \forall t \geq 0$$

which implies that

$$u_{\varepsilon}(t) \leq \ell^{1-p} e^{\delta(1-p)t} + \frac{c}{\gamma-\delta} e^{\gamma(1-p)t}.$$

Since $w_{\varepsilon}(t) = u_{\varepsilon}(t)^{\frac{1}{1-p}}$ and $\frac{1}{1-p} > 1$, then, we get

$$w_{\varepsilon}(t) \leq 2^{\frac{p}{1-p}} \ell e^{\delta t} + (\frac{2^p c}{\gamma-\delta})^{\frac{1}{1-p}} e^{\gamma t},$$

and consequently

$$v^{\varepsilon}(t) \leq 2^{\frac{p}{1-p}} \ell e^{\delta t} + (\frac{2^p c}{\gamma-\delta})^{\frac{1}{1-p}} e^{\gamma t}.$$

Our goal is to prove the boundedness of solutions for the following dynamical system:

$$\dot{x}(t) = A(t)x(t) + \psi(t, x(t), \varepsilon) \tag{1.3}$$

where $x(t,\varepsilon) \in \mathbb{R}^n$ is the system state, $A(t) \in \mathbb{R}^{n\times n}$, is matrice whose elements are continuous and bounded. The function $\psi(t,x,\varepsilon)$, $\varepsilon \in]0, \varepsilon^*[, \varepsilon^* > 0$, is smooth enouth and there exists a non negative constant $\psi_0(\varepsilon)$, such that

$$\|\psi(t, 0, \varepsilon)\| \leq \psi_0(\varepsilon), \forall t \geq 0, \varepsilon > 0.$$

to address this important problem, we suggest that it is convenient to separate the bounded and the convergence of solutions in two classes.

Let $\Phi(t,t_0)$ denotes the state transition matrix of $A(t)$. Thus, if the linear system is globally uniformly asymptotically stable then (see [25]),

$$\|\Phi(t,t_0)\| \leq ke^{-\gamma(t-t_0)}, \forall t \geq t_0 \geq 0, \tag{1.4}$$

k and γ are some noneagative constants.

Now, we prove the global uniform boundedness and convergence of solutions of (1.3) under the following assumption:

(H) There exist a nonnegative continuous function bounded function $\zeta(t)$ by a constant $\zeta > 0$ and a continuous positive function $\xi_\varepsilon(t)$, such that

$$\|\psi(t,x(t),\varepsilon)\| \leq \zeta(t)\|x\|^p + \xi_\varepsilon(t), 0 < p < 1, \forall x \in \mathbb{R}^n, \forall t \geq 0,$$

where ξ_ε satisfies

$$\xi_\varepsilon(t) \leq \rho(\varepsilon) < +\infty, \rho > 0, \forall t \geq 0.$$

Theorem 1.2: *Suppose that the matrix $A(t)$ is globally uniformly asymptotically stable and $(\mathcal{H})$ is satisfied, then the solutions of system (1.3) are globally uniformly bounded. Moreover, the solutions converge exponentially to a small neighborhood of the origin.*

Proof: The solution of the system (1.3) is given by:

$$x(t,\varepsilon) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,s)\psi(s,x(s),\varepsilon)ds.$$

Thus,

$$\|x(t,\varepsilon)\| \leq \|\Phi(t,t_0)\| \, \|x_0\| + \int_{t_0}^t \|\Phi(t,s)\| \, \|\psi(s,x(s),\varepsilon))\|ds.$$

From (1.4), it follows that

$$\|x(t,\varepsilon)\| \leq k\|x_0\|e^{-\gamma(t-t_0)} + k\int_{t_0}^t e^{-\gamma(t-s)}\|\psi(s,x(s),\varepsilon)\|ds, \forall t \geq t_0 \geq 0.$$

Since $(\mathcal{H})$ holds, one gets

$$\|x(t,\varepsilon)\| \leq \|x_0\|e^{-\gamma(t-t_0)} + k\int_{t_0}^t e^{-\gamma(t-s)}(\zeta\|x(s,\varepsilon)\|^p + \xi_\varepsilon(s))ds, \forall t \geq t_0 \geq 0.$$

Multiplying both sides by $e^{\gamma t}$, yields

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq k\|x_0\|e^{\gamma t_0} + k\int_{t_0}^t e^{\gamma s}\,\zeta\|x(s)\|^p + e^{\gamma s}\xi_\varepsilon(s)\,ds.$$

Thus,

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq k\|x_0\|e^{\gamma t_0} + k\int_{t_0}^t e^{\gamma s}\xi_\varepsilon(s)\,ds + k\zeta\int_{t_0}^t e^{(1-p)\gamma s}\,(e^{\gamma s}\|x(s,\varepsilon)\|)^p\,ds.$$

Let $u^\varepsilon(t) = e^{\gamma t}\|x(t,\varepsilon)\|$, it comes that

$$u^\varepsilon(t) \leq ku^\varepsilon(t_0) + k\int_{t_0}^t e^{\gamma s}\xi_\varepsilon(s)\,ds + k\zeta\int_{t_0}^t e^{(1-p)\gamma_1 s}\,(u^\varepsilon(s))^p\,ds.$$

Thus,

$$u^\varepsilon(t) \leq ku^\varepsilon(t_0) + k\int_{t_0}^t e^{\gamma s}\xi_\varepsilon(s)\,ds + k\zeta\int_{t_0}^t e^{(1-p)\gamma s}\,u^{\varepsilon p}(s)\,ds.$$

It follows that,

$$u(t) \leq ku(t_0) + \frac{\rho(\varepsilon)k}{\gamma}(e^{\gamma t} - e^{\gamma t_0}) + k\zeta \int_{t_0}^{t} e^{(1-p)\gamma s}\, u^{\varepsilon p}(s)\, ds.$$

Using Gamidov's Lemma, it comes that

$$u^{\varepsilon}(t) \leq ku^{\varepsilon}(t_0) + \frac{\rho(\varepsilon)k}{\gamma}(e^{\gamma t} - e^{\gamma t_0}) + k\zeta\kappa^{p}(\int_{t_0}^{t} e^{\gamma s} ds)^{1-p}.$$

So,

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq ke^{\gamma t_0}\|x(t_0,\varepsilon)\| + \frac{\rho(\varepsilon)k}{\gamma}(e^{\gamma t} - e^{\gamma t_0}) + \frac{k\zeta\kappa^{p}}{\gamma^{1-p}}[e^{\gamma t} - e^{\gamma t_0}]^{1-p}.$$

Therefore,

$$\|x(t,\varepsilon)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0,\varepsilon)\| + \frac{\rho(\varepsilon)k}{\gamma}(1 - e^{-\gamma(t-t_0)}) + \frac{k\zeta\kappa^{p}}{\gamma^{1-p}}e^{-\gamma t}[e^{\gamma t} - e^{\gamma t_0}]^{1-p}.$$

Finally we get,

$$\|x(t,\varepsilon)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0,\varepsilon)\| + \frac{\rho(\varepsilon)k}{\gamma} + \frac{k\zeta\kappa^{p}}{\gamma^{1-p}}.$$

Hence, the solutions of system (1.3) are globally uniformly bounded. Moreover, using the following estimation:

$$\|x(t,\varepsilon)\| - (\frac{\rho(\varepsilon)k}{\gamma} + \frac{k\zeta\kappa^{p}}{\gamma^{1-p}}) \leq ke^{-\gamma(t-t_0)}\|x(t_0,\varepsilon)\|,$$

and by taking initial conditions outside the ball B centered at the origin and of radius $(\frac{\rho(\varepsilon)k}{\gamma} + \frac{k\zeta\kappa^{p}}{\gamma^{1-p}})$, on can deduce that the solutions converge exponentially to B.

We give now an integral inequality variant of the prvious result. Let a be a positive constant, $u^{\varepsilon}(t)$ and $b(t)$, $t \in [t_0, t_1]$ be real-valued continuous positive functions, satisfying

$$u^{\varepsilon}(t) \leq a + 2\int_{t_0}^{t} b(s)(u^{\varepsilon}(s))^{\frac{1}{2}} ds, t \in [t_0, t_1]. \qquad (Iq1)$$

Then, for all $t \in [t_0, t_1]$, we have

$$u^{\varepsilon}(t) \leq (a^{\frac{1}{2}} + \int_{t_0}^{t} b(s) ds)^2. \qquad (Iq2)$$

Indeed, for some fixed $\theta > 0$, we define the differentiable function $v^{\varepsilon}(t)$ on $[t_0, t_1]$ by:

$$v^{\varepsilon}(t) = \theta + a + 2\int_{t_0}^{t} b(s)(u^{\varepsilon}(s))^{\frac{1}{2}} ds,$$

by computing the derivative of the above function and using the inequality $(Iq1)$ verified by $u^{\varepsilon}(t)$, one gets

$$\frac{d}{dt}v^{\varepsilon}(t) = 2b(t)(u^{\varepsilon}(t))^{\frac{1}{2}} \leq 2b(t)(v^{\varepsilon}(t))^{\frac{1}{2}}.$$

Since $v^{\varepsilon}(t) \geq \theta$, we can integrate the last inequality to obtain:

$$(v^{\varepsilon}(t))^{\frac{1}{2}} - (v^{\varepsilon}(t_0))^{\frac{1}{2}} \leq \int_{t_0}^{t} b(s) ds.$$

Thus,

$$u^{\varepsilon}(t) \leq v^{\varepsilon}(t) \leq ((a+\theta)^{\frac{1}{2}} + \int_{t_0}^{t} b(s)ds)^2.$$

Hence, we can conclude the new estimation on $u^{\varepsilon}(t)$ by taking the limit of θ to zero.

Using the above integral inequality $(Iq1)$, one can study some properties of solutions of perturbed systems of the form: $\dot{x}(t) = A(t)x(t) + \psi_{\varepsilon}(t, x(t))$, in the case when the linear part is globally uniformly asymptotically stable and the nonlinear part satisfies:

$$\|\psi_{\varepsilon}(s,x)\| \leq l(\varepsilon)\|x\|^{\frac{1}{2}}, \forall x \in \mathbb{R}^n, \forall t \geq 0, l(\varepsilon) > 0, \varepsilon > 0.$$

We can used also the same method as the one in Theorem 1.2 by considering:

$$x(t,\varepsilon) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)\psi_{\varepsilon}(s,x(s))ds.$$

Using the fact that,

$$\|x(t,\varepsilon)\| \leq \|\Phi(t,t_0)\| \, \|x_0\| + \int_{t_0}^{t} \|\Phi(t,s)\| \, \|\psi_{\varepsilon}(s,x(s))\|ds.$$

one gets the following inequality:

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq k\|x_0\|e^{\gamma t_0} + \int_{t_0}^{t} ke^{\gamma s}l(\varepsilon)\|x(s)\|^{\frac{1}{2}}ds,$$

which has the form of $(Iq1)$, so an estimation on the solutions as $(Iq2)$ can be obtained.

In the sequel, inspired by the idea of (see [10]), we will develop a Gronwall-Bellman-Bihari type inequality involving a small parameter which will be useful for studying a class of perturbed differential equations.

Lemma 1.3: *Let $x: [t_0, t_1] \to \mathbb{R}_+$ be a continuous function that satisfies the inequality:*

$$x(t) \leq M + \int_{t_0}^{t} \vartheta(s)\omega(x(s))ds, t \in [t_0, t_1],$$

where $M \geq 0, \vartheta: [t_0, t_1] \to \mathbb{R}_+$ is cotinuous and $\omega: \mathbb{R}_+ \to \mathbb{R}_+^*$ is continuous and monotone- increasing. Then the estimation

$$x(t) \leq \Phi^{-1}(\Phi(M) + \int_{t_0}^{t} \vartheta(s)ds), t \in [t_0, t_1]$$

holds, where $\Phi: \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi(u) := \int_0^u \frac{1}{\omega(s)}ds, u \in \mathbb{R}.$$

Now, we shall suppose the following assumption required for the boundedness of solutions.

(H') There exist an increasing nonnegative function ω, such that

$$\|\psi(t,x(t),\varepsilon)\| \leq \omega(\|x(s)\|), \forall x \in \mathbb{R}^n, \forall t \geq 0.$$

Theorem 1.4: *Suppose that the matrix $A(t)$ is globally uniformly asymptotically stable and ($\mathcal{H}'$) is satisfied, then the solutions of system (1.3) are globally uniformly bounded. Moreover, The solutions converge exponentially to a small neighborhood of the origin.*

Proof: The solution of the system(1.3) is given by:

$$x(t,\varepsilon) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)\psi(s,x(s),\varepsilon)ds.$$

Thus,

$$\|x(t,\varepsilon)\| \leq \|\Phi(t,t_0)\| \ \|x_0\| + \int_{t_0}^{t} \|\Phi(t,s)\| \ \|\psi(s,x(s),\varepsilon)\|ds.$$

Taking into account, the global uniform asymptotic stability of the nominal system $\dot{x}(t) =$A(t)$x(t)$, and the assumption ($\mathcal{H}'$), one gets,

$$\|x(t,\varepsilon)\| \leq k\|x_0\|e^{-\gamma(t-t_0)} + \int_{t_0}^{t} ke^{-\gamma(t-s)}\omega(\|x(s,\varepsilon)\|)ds.$$

Thus, using the fact that Ψ is a nonnegative increasing function, one obtains

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq k\|x_0\|e^{\gamma t_0} + \int_{t_0}^{t} ke^{\gamma s}\omega(e^{\gamma s}\|x(s,\varepsilon)\|)ds.$$

We set

$$u(t) = e^{\gamma t}\|x(t,\varepsilon)\|.$$

Then,

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} ke^{\gamma s}\omega(u(s))ds.$$

Using Lemma 1.3, it comes that

$$u(t) \leq \Phi^{-1}(\Phi(ku(t_0)) + \int_{t_0}^{t} ke^{\gamma s}ds),$$

where

$$\Phi(u) = \int_0^u \frac{1}{\omega(s)}ds.$$

The above inequality on $u(t)$ is equivalent to:

$$u(t) \leq ku(t_0) + \Phi^{-1}(\frac{k}{\gamma}(e^{\gamma t} - e^{\gamma t_0})).$$

Now, using that Φ^{-1} is an increasing function, it comes that

$$e^{\gamma t}\|x(t,\varepsilon)\| \leq k\|x_0\|e^{\gamma t} + \Phi^{-1}\left(\frac{k}{\gamma}(e^{\gamma t} - e^{\gamma t_0})\right).$$

Thus,

$$\|x(t,\varepsilon)\| \leq k\|x_0\|e^{-\gamma(t-t_0)} + e^{-\gamma t}\Phi^{-1}\left(\frac{k}{\gamma}(e^{\gamma t} - e^{\gamma t_0})\right).$$

Remark that, one has the following estimation

$$\Phi^{-1}\left(\frac{k}{\gamma}(e^{\gamma t} - e^{\gamma t_0})\right) \leq \frac{k}{\gamma}e^{\gamma t},$$

it is verified because $\Phi(x) \geq x$ or $\Psi \leq 1$.

Finally, we obtain

$$\|x(t,\varepsilon)\| \leq k\|x_0\|e^{-\gamma(t-t_0)} + \frac{k}{\gamma}.$$

Hence, the solutions of system (1.3) are globally uniformly bounded. Moreover, using the following estimation

$$\|x(t,\varepsilon)\| - \frac{k}{\gamma} \leq k\|x_0\| e^{-\gamma(t-t_0)},$$

and by taking initial conditions outside the ball B' centered at the origin and of radius $\frac{k}{\gamma}$, the solutions converge exponentially to B'.

We give now an application to control system to illustrate the applicability of the main result.

1.2 *Example*

Consider the following planar system:

$$\begin{cases} \dot{x}_1 = -x_1 - tx_2 + \frac{1}{(1+t^2)^2} \frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} \\ \dot{x}_2 = -x_2 + tx_1 + \frac{t}{(1+t^2)^2} \frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}} \end{cases} \tag{1.5}$$

which can be written as $\dot{x} = A(t)x + \psi(t,x,\varepsilon)$, where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state of the system,

$$A(t) = \begin{pmatrix} -1 & -t \\ t & -1 \end{pmatrix}, \psi(t,x,\varepsilon) = \begin{pmatrix} \psi_1(t,x,\varepsilon) \\ \psi_2(t,x,\varepsilon) \end{pmatrix} \text{ with } \begin{cases} \psi_1(t,x,\varepsilon) = \frac{\varepsilon}{(1+t^2)} + \frac{e^{-t}\sqrt{|x_1|}}{1+\sqrt{x_1^2+x_2^2}} \\ \psi_2(t,x,\varepsilon) = \frac{\varepsilon}{(1+t^2)} + \frac{e^{-t}\sqrt{|x_2|}}{1+\sqrt{x_1^2+x_2^2}} \end{cases}$$

Remark that, the fact that the function $\sqrt{|x_1|}$ and $\sqrt{|x_2|}$ are not Lipschitzian around zero does not pose a problem for the existence and uniqueness of the solutions because our study of the asymptotic bahavior of solutions is done outside a small neighborhood of the origin. A straightforward computation shows that the transition matrix R_A is given by

$$\Phi(t,t_0) = e^{-(t-t_0)} \begin{pmatrix} \cos\left(\frac{1}{2}(t^2 - t_0^2)\right) & -\sin\left(\frac{1}{2}(t^2 - t_0^2)\right) \\ \sin\left(\frac{1}{2}(t^2 - t_0^2)\right) & \cos\left(\frac{1}{2}(t^2 - t_0^2)\right) \end{pmatrix}.$$

It follows that $\mathrm{P}\Phi(t,t_0)\mathrm{P} = e^{-(t-t_0)}$. Moreover, the assumption $(\mathcal{H})$ is satisfied with $\zeta(t) = e^{-t}$, $\zeta = 1$, $\xi_\varepsilon(t) = \frac{\varepsilon}{(1+t^2)} \leq \rho(\varepsilon) = \varepsilon$. Thus, by application of Theorem 1.2, the solutions of system (1.3) are globally uniformly bounded and converge exponentially to a small neighborhood of the origin.

2. Conclusion

In this work some new sufficient conditions are proposed for the exponential stability in the sense that the solutions approach to a small ball centered at the origin. An example in dimensional two is given to demonstrate the design procedure and the effectiveness of the proposed approach.

Ethical approval: Compliance with ethical standards.

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Informed Consent: We confirm that informed consent was obtained from all human participants involved in the study.

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