

Existence results and optimization of fractional boundary value problems

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Abstract

We examine the solution to problem fractional with respect derivatives of Caputo. Our approach is mainly based on fractional calculus, particularly fractional derivatives and integrals, together with fixed point theory. First, we apply the Banach fixed point theorem to establish the existence and uniqueness of the solution. Then, we use the Schauder fixed point theorem to prove the existence of at least one solution. We also provide examples with optimization graphs to further demonstrate the validity of our results.

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1. Introduction

Fractional differential equations play major significance in numerous domains such as physics, control theory, neural networks, and epidemiology. Over time, a wide range of fractional differential and integral operators have been established and employed by mathematicians to serve particular needs, often relying on different invariant point theorems like Schauder's, Brouwer's, and Banach's contraction principle. details on the theory and application, see [1-8].

In 1967, Caputo proposed the problems emerging from the theory of viscoelasticity and proposed a novel definition of the fractional-order derivative. One significant advantage of the Caputo constructed lies in the fact that the boundary conditions for FDE of Caputo, see [9, 10].

Fifth-order problems emerge in several branches of mathematics and physics. A range of studies has applied different numerical techniques to tackle such problems, see [11-19].

In the article [20], the author studied on $(0,1)$ the following problem

$$\begin{cases} z^{(5)}(t) + \xi g(t)f(z(t)) = 0, & \xi \in \mathbb{R}_+, \\ z''(1) = z(0) = z'(0) = z''(0) = z'''(0) = 0. \end{cases}$$

In [21], the authors studied the nonlinear equation

$$m^{(5)}(t) = g(t, m(t)), t \in (0,1),$$

with the boundary conditions

$$\begin{cases} m(0) = m'(0) = m''(0) = m^{(4)}(1) = 0, \\ \zeta m'(1) + \xi m''(1) = 0, \zeta + \xi > 0, \end{cases},$$

and obtained existence results of at least one positive solution.

In the article [22], the authors interested in the existence of solution for the following problem

$$m^{(5)}(t) = -g(t, m(t)), t \in (0,1),$$

with

$$\begin{cases} m(1) = \xi m(\varepsilon), m^{(k)}(0) = 0, k = \overline{1,3}, \\ 0 < \varepsilon < 1, \xi \varepsilon^4 \neq 1, \end{cases}$$

where g is continuous function.

In [23], the authors by using lower and upper solution method and schauder fixed point theorem, studied the following:

$$\begin{cases} w^{(5)}(z) = -g(z, v(z), v'(z), v''(z)), z \in [0,1], \\ w(0) = w'(0) = w'(1) = w^{(3)}(1) = w^{(4)}(1) = 0. \end{cases}$$

In the paper [24], the authors studied the existence and nonexistence of monotone and convex positive solutions for the problem

$$\begin{cases} v^{(5)}(w) + g(w, v(w), v'(w), v''(w)) = 0, w \in (0,1), \\ v(0) = v''(0) = v''(1) = 0, \\ v^{(3)}(0) = \xi, u(0) = \zeta \int_{v_0}^{v_1} v(s)ds + \sum_{k=1}^n \gamma_k v(\varepsilon_k), \end{cases} \quad (1.1)$$

where $g: [0,1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuou function and $\xi > 0$, $\zeta, \varepsilon_k \geq 0$, also $0 \leq v_0 < \varepsilon_1 < \dots < \varepsilon_n \leq 1$ ($k = \overline{1, n}$).

In [25], the authors investigate the following

$$\begin{aligned} & \mathcal{D}^{\alpha_1} \mathcal{D}^{\alpha_2} \mathcal{D}^{\alpha_3} \mathcal{D}^{\alpha_4} \mathcal{D}^{\alpha_5} \mathcal{D}^{\alpha_6} x(t) \\ &= (t, \mathcal{D}^{\alpha_3} \mathcal{D}^{\alpha_4} \mathcal{D}^{\alpha_5} \mathcal{D}^{\alpha_6} x(t), x(t)) w_1 \\ &+ (t, \mathcal{D}^{\alpha_5} \mathcal{D}^{\alpha_6} x(t), x(t)) w_2 \\ &+ (t, I^\xi x(t), x(t)) w_3 + (t, x(t)) w_4, \end{aligned}$$

where

$$\begin{cases} x(0) = \varepsilon_0, \varepsilon_0 \in \mathbb{R}, \\ \mathcal{D}^{\alpha_5} \mathcal{D}^{\alpha_6} x(0) = \varepsilon_2, \varepsilon_2 \in \mathbb{R}, \\ \mathcal{D}^{\alpha_6} x(0) = \varepsilon_1, \varepsilon_1 \in \mathbb{R}, \end{cases} \quad \text{and} \quad \begin{cases} x(1) = \lambda_0, \lambda_0 \in \mathbb{R}, t \in I, \\ \mathcal{D}^{\alpha_5} \mathcal{D}^{\alpha_6} x(1) = \lambda_2, \lambda_2 \in \mathbb{R}, \\ \mathcal{D}^{\alpha_6} x(1) = \lambda_1, \lambda_1 \in \mathbb{R}, \end{cases}$$

where $(w_l)_{l=\overline{1,4}} \in \mathbb{R}$ and $0 < \alpha_k \leq 1$.

In this study, we discuss for all $t \in [0,1]$ the following eqation

$$\begin{aligned} & {}^c \mathcal{D}_{0+}^\gamma ({}^c \mathcal{D}_{0+}^\delta u(t)) \\ &= \varphi_1(t) G_1(t, u^{(4)}(t)) + \varphi_2(t) G_2(t, u^{(2)}(t)) + \varphi_3(t) G_3(t, u(t)), \end{aligned} \quad (1.2)$$

with boundary conditions

$$\begin{cases} u(1) = d_0, u'(0) = d_1, u''(0) = d_2, d_l \in \mathbb{R}, l = \overline{0,2}, \\ u(0) = \lambda_1 \int_0^1 u(t) dt, \lambda_1 \in \mathbb{R}, \\ u'(1) = \lambda_2 \int_0^1 u(t) dt, \lambda_2 \in \mathbb{R}, \end{cases} \quad (1.3)$$

where ${}^c \mathcal{D}_{0+}^\gamma$ and ${}^c \mathcal{D}_{0+}^\delta$ denote the Caputo fractional derivatives of orders $1 < \gamma \leq 2$ and $2 < \delta \leq 3$, respectively. Furthermore, $u^{(l)}(t)$, $l = 0, 2, 4$ represent the classical derivatives of the function u . Moreover, $\varphi_i \in C([0,1], \mathbb{R})$.

We arrange this work it follows: In Section two, we include a notation and definitions that is essential to the analysis of our problem (1.2)-(1.3). Section 3 is dedicated to the main problem (1.2)-(1.3). Moreover, we establish the existence and uniqueness of the solution using the Banach fixed-point theorem, and we demonstrate the existence of at least one solution via Schauder's theorem. Next, we present the examples 1 and 2 to confirm the our results.

2. Basic Preliminaries

In the following, we exhibit the property of FDEs, see [26-33].

Definition 2.1: Let $\eta \geq 0$ and let $f: [c, d] \rightarrow \mathbb{R}$ be a function. The Riemann–Liouville fractional integral of order η is defined by

$${}^{RL} \mathcal{J}_{c+}^\eta f(t) = \frac{1}{\Gamma(\eta)} \int_c^t (t-z)^{\eta-1} f(z) dz,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Corollary 2.2: Let $m_1, m_2 \in \mathbb{R}_+^*$. Then the semigroup property holds:

$${}^{RL}J_{c^+}^{m_1}({}^{RL}J_{c^+}^{m_2}f(t)) = {}^{RL}J_{c^+}^{m_1+m_2}f(t).$$

Definition 2.3: Let $\eta > 0$ and suppose that $f \in C^n([c, d], \mathbb{R})$. The Caputo fractional derivative of order η is defined by

$${}^c\mathcal{D}_{c^+}^\eta f(t) = \frac{1}{\Gamma(n-\eta)} \int_c^t (t-r)^{n-\eta-1} f^{(n)}(r) dr,$$

where $n = \lfloor \eta \rfloor + 1$.

Lemma 2.4: Let $\eta > 0$, $n \in \mathbb{N}^*$, and $(m_l)_{l=\overline{0, n-1}} \in \mathbb{R}$.

1. For every $t \in c, d]$, we have:

$$\begin{aligned} & {}^{RL}J_{c^+}^\eta({}^c\mathcal{D}_{c^+}^\eta f(t)) \\ &= f(t) + m_0 + m_1 t + m_2 t^2 + \dots + m_{n-1} t^{n-1}, \end{aligned}$$

where $n = \lfloor \eta \rfloor + 1$.

• If

$${}^c\mathcal{D}_{c^+}^\eta f(t) = 0,$$

then f is a polynomial of degree at most $n - 1$, that is

$$f(t) = m_0 + m_1 t + m_2 t^2 + \dots + m_{n-1} t^{n-1}.$$

Proposition 2.5: Let $m_1, m_2 \in \mathbb{R}_+^*$ such that $m_2 > m_1$. Then

$${}^c\mathcal{D}_{c^+}^{m_1}({}^{RL}J_{c^+}^{m_2}f(t)) = {}^{RL}J_{c^+}^{m_2-m_1}f(t).$$

Theorem 2.6: (Schauder's Fixed-Point Theorem) Let Φ be a non-empty, closed, convex, and bounded subset of a Banach space $\mathcal{X}$. If a continuous operator $Y: \Phi \rightarrow \Phi$ maps Φ into a relatively compact set, then Y has at least one fixed point in Φ .

Lemma 2.7: Let $2 < \delta \leq 3$, $1 < \gamma \leq 2$ and $g \in C([0, 1], \mathbb{R})$. We consider the following equation.

$${}^c\mathcal{D}_{0^+}^\gamma({}^c\mathcal{D}_{0^+}^\delta u(t)) = g(t), t \in (0, 1], \quad (2.1)$$

with conditions

$$u(0) = \int_0^1 \lambda_1 u(t) dt, d_0 = u(1), d_1 = u'(0), \quad (2.2)$$

and

$$u'(1) = \int_0^1 \lambda_2 u(t) dt, u''(0) = d_2. \quad (2.3)$$

Then, the solutions to problem (2.1)-(2.3) are given as follows:

$$\begin{aligned} & u(t) \\ &= \int_0^t \frac{(t-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} g(s) ds + (\delta+1)(\psi_1(\delta+1) - \psi_2)t^\delta \\ &+ (\psi_2 - \delta\psi_1)t^{\delta+1} + \lambda_1 \int_0^1 u(t) dt + d_1 t + \frac{d_2}{2} t^2, \end{aligned}$$

where

$$\begin{aligned}\psi_1 = & d_0 - \int_0^1 \frac{(1-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} g(s) ds - \lambda_1 \int_0^1 u(1) dt \\ & - d_1 - \frac{d_2}{2},\end{aligned}$$

and

$$\psi_2 = \lambda_2 \int_0^1 u(1) dt - \int_0^1 \frac{(1-s)^{\delta+\gamma-2}}{\Gamma(\delta+\gamma-1)} g(s) ds - d_1 - d_2.$$

Proof: Let $2 < \delta \leq 3$ and $1 < \gamma \leq 2$. From the Lemma 2.4, we get

$${}^c\mathcal{D}_{0+}^\delta u(t) = {}^{RL}\mathcal{J}_{0+}^\gamma g(t) + c_0 + c_1 t, \quad (c_i)_{i=0,1} \in \mathbb{R}.$$

By applying the operator ${}^{RL}\mathcal{J}_{0+}^\delta$ to this equation, we obtain

$$\begin{aligned}u(t) &= {}^{RL}\mathcal{J}_{0+}^{\delta+\gamma} g(t) + c_0 \left({}^{RL}\mathcal{J}_{0+}^\delta(1) \right) + c_1 \left({}^{RL}\mathcal{J}_{0+}^\delta(t) \right) \\ &+ \theta_0 + \theta_1 t + \theta_2 t^2,\end{aligned}$$

where $(\theta_i)_{i=0,2} \in \mathbb{R}$. We get

$$\begin{aligned}u(t) = & \int_0^t \frac{(t-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} g(s) ds + \frac{c_0}{\Gamma(\delta+1)} t^\delta \\ & + \frac{c_1}{\Gamma(\delta+2)} t^{\delta+1} + \theta_0 + \theta_1 t + \theta_2 t^2,\end{aligned}\tag{2.4}$$

and using the condition (2.2), we obtain

$$\theta_0 = \lambda_1 \int_0^1 u(t) dt.$$

Taking the derivative of the function given in (2.4) leads to

$$\begin{aligned}u'(t) &= \int_0^t \frac{(t-s)^{\delta+\gamma-2}}{\Gamma(\delta+\gamma-1)} g(s) ds + \frac{\delta c_0}{\Gamma(\delta+1)} t^{\delta-1} \\ &+ \frac{(\delta+1)c_1}{\Gamma(\delta+2)} t^\delta + \theta_1 + 2\theta_2 t.\end{aligned}\tag{2.5}$$

The condition (2.2), given

$$u'(0) = \theta_1 = d_1.$$

By differentiating the function (2.5), we get

$$\begin{aligned}u''(t) &= \frac{1}{\Gamma(\delta+\gamma-2)} \int_0^t (t-s)^{\delta+\gamma-3} g(s) ds \\ &+ \delta \left(\frac{(\delta-1)c_0}{\Gamma(\delta+1)} t^{\delta-2} + \frac{(\delta+1)c_1}{\Gamma(\delta+2)} t^{\delta-1} \right) + 2\theta_2.\end{aligned}$$

The condition (2.3), given

$$\theta_2 = \frac{d_2}{2}.$$

By substituting these values into the equation (2.4), we find

$$\begin{aligned}
& u(t) \\
&= \int_0^t \frac{(t-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} g(s) ds + \frac{c_0 t^\delta}{\Gamma(\delta+1)} + \frac{c_1 t^{\delta+1}}{\Gamma(\delta+2)} \\
&\quad + \lambda_1 \int_0^1 u(t) dt + d_1 t + \frac{d_2}{2} t^2.
\end{aligned}$$

Then

$$\begin{aligned}
& u(1) \\
&= \int_0^1 \frac{(1-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} g(s) ds + \frac{c_0}{\Gamma(\delta+1)} + \frac{c_1}{\Gamma(\delta+2)} \\
&\quad + \lambda_1 \int_0^1 u(t) dt + d_1 + \frac{d_2}{2} \\
&= d_0,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
& u'(1) \\
&= \frac{1}{\Gamma(\delta+\gamma-1)} \int_0^1 (1-s)^{\delta+\gamma-2} g(s) ds + \frac{\delta c_0}{\Gamma(\delta+1)} \\
&\quad + \frac{(\delta+1)c_1}{\Gamma(\delta+2)} + d_1 + d_2 \\
&= \lambda_2 \int_0^1 u(t) dt.
\end{aligned}$$

We obtain the system

$$\begin{cases} \frac{c_0}{\Gamma(\delta+1)} = \psi_1 - \frac{c_1}{\Gamma(\delta+2)}, \\ \frac{\delta c_0}{\Gamma(\delta+1)} = \psi_2 - \frac{(\delta+1)c_1}{\Gamma(\delta+2)}, \end{cases} \tag{2.7}$$

such that

$$\psi_1 = d_0 - \frac{1}{\Gamma(\delta+\gamma)} \int_0^1 (1-s)^{\delta+\gamma-1} g(s) ds - \lambda_1 \int_0^1 u(t) dt - d_1 - \frac{d_2}{2},$$

and

$$\psi_2 = \lambda_2 \int_0^1 u(1) dt - \frac{1}{\Gamma(\delta+\gamma-1)} \int_0^1 (1-s)^{\delta+\gamma-2} g(s) ds - d_1 - d_2.$$

After solving this system, we find the constants

$$c_0 = \Gamma(\delta+2)(\psi_1(\delta+1) - \psi_2),$$

and

$$c_1 = (\psi_2 - \delta\psi_1)\Gamma(\delta+2).$$

Substituting these value into the equation (2.4), we find the result.

3. Existence and Uniqueness of Solutions via Fixed-Point Theorems

For clarity, this section is divided into three subsections:

3.1 Hypotheses and Notation:

Let the space:

$$\mathcal{X} = \{u: u^{(l)} \in \mathcal{C}([0,1], \mathbb{R}), l = 0, 2, 4\},$$

where

$$\|u\|_{\mathcal{X}} = \|u\|_{\infty} + \|u^{(2)}\|_{\infty} + \|u^{(4)}\|_{\infty},$$

and

$$\|u^{(l)}\|_{\infty} = \sup_{t \in [0,1]} |u^{(l)}(t)|, l = 0, 2, 4.$$

We consider the operator $U: \mathcal{X} \rightarrow \mathcal{X}$, defined by

$$\begin{aligned} & Uu(t) \\ &= \frac{1}{\Gamma(\delta+\gamma)} \int_0^t (t-s)^{\delta+\gamma-1} [\varphi_1(s)G_1(s, u^{(4)}(s)) + \varphi_2(s)G_2(s, u^{(2)}(s)) \\ & \quad + \varphi_3(s)G_3(s, u(s))] ds + (\delta+1)(\psi_1(\delta+1) - \psi_2)t^{\delta} \\ & \quad + (\psi_2 - \delta\psi_1)t^{\delta+1} + \lambda_1 \int_0^1 u(t)dt + d_1t + \frac{d_2}{2}t^2, \end{aligned}$$

where

$$\begin{aligned} \psi_{1,u} &= d_0 - \frac{1}{\Gamma(\delta+\gamma)} \\ & \quad \times \int_0^1 (1-s)^{\delta+\gamma-1} [\varphi_1(s)G_1(s, u^{(4)}(s)) \\ & \quad + \varphi_2(s)G_2(s, u^{(2)}(s)) + \varphi_3(s)G_3(s, u(s))] ds \\ & \quad - \lambda_1 \int_0^1 u(t)dt - d_1 - \frac{d_2}{2}, \end{aligned}$$

and

$$\begin{aligned} \psi_{2,u} &= \lambda_2 \int_0^1 u(t)dt - \frac{1}{\Gamma(\delta+\gamma-1)} \\ & \quad \times \int_0^1 (1-s)^{\delta+\gamma-2} [\varphi_1(s)G_1(s, u^{(4)}(s)) \\ & \quad + \varphi_2(s)G_2(s, u^{(2)}(s)) + \varphi_3(s)G_3(s, u(s))] ds \\ & \quad - d_1 - d_2. \end{aligned}$$

Throughout this paper, we assume the following hypotheses

(H1) Let the functions $(G_i)_{i=\overline{1,3}}$ are jointly continuous.

(H2) For all $u_1, u_2 \in \mathbb{R}$ and $t \in [0,1]$, we have

$$|G_j(t, u) - G_j(t, v)| \leq |u(t) - v(t)|, j = \overline{1,3}.$$

We need to consider the quantities:

$$\begin{aligned} \Pi_1 &= \left(\frac{20}{\Gamma(\delta+\gamma+1)} + \frac{7}{\Gamma(\delta+\gamma)} \right) \sum_{i=1}^3 \|\varphi_i\|, \\ \Pi_2 &= \left(\frac{121}{\Gamma(\delta+\gamma-1)} + \frac{32}{\Gamma(\delta+\gamma-2)} \right) \sum_{i=1}^3 \|\varphi_i\|, \\ \Pi_3 &= \left(\frac{36}{\Gamma(\delta+\gamma-3)} + \frac{24}{\Gamma(\delta+\gamma-4)} \right) \sum_{i=1}^3 \|\varphi_i\|. \end{aligned}$$

and

$$\begin{aligned} \Lambda_1 &= \frac{20}{\Gamma(\delta+\gamma+1)} + \frac{5}{\Gamma(\delta+\gamma)} + \frac{1}{\Gamma(\delta+\gamma-1)} \\ & \quad + \frac{40}{\Gamma(\delta+\gamma-2)} + \frac{1}{\Gamma(\delta+\gamma-3)} + 19|d_0| + 25|d_1| \\ & \quad + 16|d_2|. \end{aligned}$$

3.2 Uniqueness of the Solution

Theorem 3.1: Assume that the conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ are valid. If

$$|\lambda_1| + |\lambda_2| < 1 - \sum_{l=1}^3 \Pi_l, \quad (3.1)$$

then, problem (1.2)-(1.3) has a unique solution.

Proof: For $(u, v) \in \mathcal{X}^2$ and $t \in [0, 1]$, we have

$$\begin{aligned} & |Uu(t) - Uv(t)| \\ & \leq \frac{1}{\Gamma(\delta+\gamma)} \int_0^1 (1-s)^{\delta+\gamma-1} |\varphi_3(t)G_3(t, u(t)) + \varphi_2(t)G_2(t, u^2(t)) \\ & \quad + \varphi_1(t)G_1(t, u^4(t)) - \varphi_1(t)G_1(t, v^4(t)) + \varphi_2(t)G_2(t, v^2(t)) \\ & \quad + \varphi_3(t)G_3(t, v(t))| ds \\ & \quad + |(\delta+1)[(\psi_{1,u} - \psi_{1,v})(\delta+1) - (\psi_{2,v} - \psi_{2,u})]| \\ & \quad + |\delta(\psi_{1,v} - \psi_{1,u}) - (\psi_{2,v} - \psi_{2,u})|. \end{aligned}$$

We have

$$|\psi_{1,u} - \psi_{1,v}| \leq \left(\frac{\sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma+1)} + \lambda_1 \right) \|u - v\|,$$

and

$$|\psi_{2,u} - \psi_{2,v}| \leq \left(\frac{\sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma)} + \lambda_2 \right) \|u - v\|.$$

Thus

$$\begin{aligned} & |Uu(t) - Uv(t)| \\ & \leq \frac{\sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma+1)} \|u - v\| + \left(\frac{19 \sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma+1)} + |\lambda_1| \right) \|u - v\| \\ & \quad + \left(\frac{7 \sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma)} + |\lambda_2| \right) \|u - v\| \\ & \leq (\Pi_1 + |\lambda_1| + |\lambda_2|) \|u - v\|. \end{aligned} \quad (3.2)$$

On other hand

$$\begin{aligned} & Uu^{(2)}(t) \\ & = \int_0^t \frac{(t-s)^{\delta+\gamma-3}}{\Gamma(\delta+\gamma-2)} [\varphi_3(t)G_3(t, u(t)) + \varphi_2(t)G_2(t, u^2(t)) \\ & \quad + \varphi_1(t)G_1(t, u^4(t))] ds \\ & \quad + (\delta+1)(\psi_{1,u}(\delta+1) - \psi_{2,u})\delta(\delta-1)t^{\delta-2} \\ & \quad + \delta(\delta+1)(\psi_{2,u} - \delta\psi_{1,u})t^{\delta-1} + d_2, \end{aligned}$$

and

$$\begin{aligned} & |Uu^{(2)}(t) - Uv^{(2)}(t)| \\ & \leq \frac{\sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma-1)} \|u - v\| + \frac{120 \sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma-1)} \|u - v\| \\ & \quad + \frac{32 \sum_{i=1}^3 \|\varphi_i\|}{\Gamma(\delta+\gamma-2)} \|u - v\| \\ & \leq \Pi_2 \|u - v\|. \end{aligned} \quad (3.3)$$

Also

$$\begin{aligned}
& Uu^{(4)}(t) \\
&= \int_0^t \frac{(t-s)^{\delta+\gamma-5}}{\Gamma(\delta+\gamma-4)} [\varphi_3(t)G_3(t, u(t)) + \varphi_2(t)G_2(t, u^2(t)) \\
&\quad + \varphi_1(t)G_1(t, u^4(t))]ds \\
&\quad + \delta(\delta-2)(\delta^2-1)(\delta-3)(\psi_{1,u}(\delta+1) - \psi_{2,u})t^{\delta-4} \\
&\quad + (\delta-2)(\delta^2-1)\delta(\psi_{2,u} - \delta\psi_{1,u})t^{\delta-3},
\end{aligned}$$

thus

$$|Uu^{(4)}(t) - Uv^{(4)}(t)| \leq \Pi_3 \|u - v\|. \quad (3.4)$$

Using the formula (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
& \|Uu - Uv\|_X \\
& \leq [(\sum_{l=1}^3 \Pi_l) + |\lambda_1| + |\lambda_2|] \|u - v\|.
\end{aligned}$$

Hence, by condition (3.1), the operator U becomes a contraction mapping.

3.2 Existence of at Least One Solution

Theorem 3.2: Suppose that Hypothesis $(\mathcal{H}1)$ holds, and that the following condition is satisfied

$$0 < \sum_{k=1}^3 |\varphi_k(t)G_k(t, u_k)| < M,$$

where M denote a positive real number. Then, problem (1.2) has at least one solution on $[0, 1]$.

Proof: Let $(u_n)_{n \in \mathbb{N}}$ be a sequence, where

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_\infty = 0.$$

Thanks to the continuity of the functions $(G_i)_{i=1,3}$, we can easily determine the following limits:

$$\begin{aligned}
& |Uu_n(t) - Uu(t)| \rightarrow 0, n \rightarrow 0, \\
& |Uu_n^{(2)}(t) - Uu^{(2)}(t)| \rightarrow 0, n \rightarrow 0,
\end{aligned}$$

and

$$|Uu_n^{(4)}(t) - Uu^{(4)}(t)| \rightarrow 0, n \rightarrow 0,$$

then

$$\|Uu_n - Uu\|_X \rightarrow 0, n \rightarrow 0,$$

thus, U is continuous. Let Ω_r be the set defined by

$$\Omega_r = \{u \in C([0, 1], \mathbb{R}), \|u\|_X \leq \sigma\},$$

such that

$$M\Lambda_1 \leq \frac{\sigma}{2}, 4|\lambda_1| + |\lambda_2| \leq \frac{1}{10}. \quad (3.5)$$

We have

$$\begin{aligned}
& |Uu(t)| \\
&= \frac{1}{\Gamma(\delta+\gamma)} \int_0^t (t-s)^{\delta+\gamma-1} |\varphi_1(s)G_1(s, u^{(4)}(s)) + \varphi_2(s)G_2(s, u^{(2)}(s)) \\
&\quad + \varphi_3(s)G_3(s, u(s))| ds + ((\delta+1)^2 + \delta)|\psi_{1,u}| + (\delta+2)|\psi_{2,u}| \\
&\quad + |\lambda_1| \|u\| + |d_1| + \frac{|d_2|}{2},
\end{aligned}$$

where

$$|\psi_{1,u}| \leq \frac{M}{\Gamma(\delta+\gamma+1)} + |\lambda_1| \|u\| + |d_0| + |d_1| + \frac{|d_2|}{2},$$

and

$$|\psi_{2,u}| \leq \frac{M}{\Gamma(\delta+\gamma)} + |\lambda_2| \|u\| + |d_1| + |d_2|.$$

Thus

$$\begin{aligned}
|Uu(t)| \leq & 5 \left(4 \frac{M}{\Gamma(\delta+\gamma+1)} + \frac{M}{\Gamma(\delta+\gamma)} \right) \\
& + 5(4|\lambda_1| + |\lambda_2|) \text{PuP} + 19|d_0| \\
& + 25|d_1| + 15|d_2|.
\end{aligned}$$

In a similar way

$$|Uu^{(2)}(t)| \leq \frac{M}{\Gamma(\delta+\gamma-1)} + \frac{16M}{\Gamma(\delta+\gamma-2)} + |d_2|,$$

and

$$|Uu^{(4)}(t)| \leq \frac{M}{\Gamma(\delta+\gamma-3)} + 24 \frac{M}{\Gamma(\delta+\gamma-2)}.$$

From (3.5), we get

$$\begin{aligned}
\|Uu\|_X &\leq M\Lambda_1 + 5(4|\lambda_1| + |\lambda_2|)\sigma \\
&\leq \sigma,
\end{aligned}$$

then $U(\Omega_r)$ is bounded. Let $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, we get

$$\begin{aligned}
& |Uu(t_2) - Uu(t_1)| \\
&\leq \frac{1}{\Gamma(\delta+\gamma)} \int_0^{t_1} |(t_2-s)^{\delta+\gamma-1} - (t_1-s)^{\delta+\gamma-1}| |\varphi_1(s)G_1(s, u^{(4)}(s)) \\
&\quad + \varphi_2(s)G_2(s, u^{(2)}(s)) + \varphi_3(s)G_3(s, u(s))| ds \\
&\quad + \frac{1}{\Gamma(\delta+\gamma)} \int_{t_1}^{t_2} |(t_2-s)^{\delta+\gamma-1}| |\varphi_1(s)G_1(s, u^{(4)}(s)) \\
&\quad + \varphi_2(s)G_2(s, u^{(2)}(s)) + \varphi_3(s)G_3(s, u(s))| ds \\
&\quad + (\delta+1)|\psi_{1,u}(\delta+1) - \psi_{2,u}| |t_2^\delta - t_1^\delta| \\
&\quad + |\psi_{2,u} - \delta\psi_{1,u}| |t_2^{\delta+1} - t_1^{\delta+1}| + |d_1| |t_2 - t_1| + \frac{|d_2|}{2} |t_2^2 - t_1^2| \\
&\leq \mathcal{J}_1 + \mathcal{J}_2 + \hat{o}(t_1, t_2),
\end{aligned}$$

where

$$\mathcal{J}_1 = \frac{M}{\Gamma(\delta+\gamma)} \int_0^{t_1} |(t_2-s)^{\delta+\gamma-1} - (t_1-s)^{\delta+\gamma-1}| ds \rightarrow 0 \text{ as } t_2 \rightarrow t_1,$$

and

$$\mathcal{J}_2 = \frac{M|t_2-t_1|^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \rightarrow 0 \text{ as } t_2 \rightarrow t_1,$$

also

$$F(t_1, t_2) \rightarrow 0 \text{ as } t_2 \rightarrow t_1,$$

thus

$$|Uu(t_2) - Uu(t_1)| \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

In a similar way

$$\begin{cases} |Uu^{(2)}(t_2) - Uu^{(2)}(t_1)| \rightarrow 0 \text{ as } t_2 \rightarrow t_1, \\ |Uu^{(4)}(t_2) - Uu^{(4)}(t_1)| \rightarrow 0 \text{ as } t_2 \rightarrow t_1. \end{cases}$$

Finally, U is equicontinuous, by the Arzelà–Ascoli Theorem, the operator U is compact. Therefore, the Schauder Fixed Point Theorem guarantees the existence of at least one solution.

Example 1: Let $t \in [0,1]$, we consider

$$\begin{cases} D^{1,50}(D^{2,50}u(t))(t) = \sqrt{t}u''(t) + t^4u^{(4)}(t) + \frac{1}{2}u(t), \\ u(0) = 2 \int_0^1 u(t)dt, u'(0) = 1, u(1) = 0, \\ u'(1) = \frac{1}{2} \int_0^1 u(t)dt, u''(0) = \frac{1}{2}; \end{cases} \quad (3.6)$$

In this example, we have

$$\begin{cases} \lambda_2 = 2, d_1 = 1, d_2 = \frac{1}{4}, \gamma = (1, 50), \delta = (2, 50), \\ \varphi_3(t) = \frac{1}{2}, \varphi_1(t) = t^4, \varphi_2(t) = \sqrt{t}, \\ G_3(t, u(t)) = u(t), G_2(t, u^{(2)}(t)) = u''(t), G_1(t, u^{(4)}(t)) = u^{(4)}(t). \end{cases}$$

Hence, 3.6 possesses a unique solution over $[0,1]$.

Example 2: We consider for all $t \in [0,1]$, the problem the following

$$\begin{cases} D^{1,98}(D^{2,99}u(t))(t) = (\cos t)u''(t) + (\sin t)u(t) + (\sin^2 t)u^4(t), \\ u(1) = \frac{\sqrt{3}}{2}, u(0) = \exp(\pi) \int_0^1 u(t)dt, u'(0) = \pi, \\ u'(1) = \frac{1}{1-\pi} \int_0^1 u(t)dt, u''(0) = \frac{1}{\sqrt{5}+2}. \end{cases} \quad (3.7)$$

We have

$$\lambda_1 = \exp(\pi), d_1 = \pi, d_2 = \frac{1}{\sqrt{5}+2}, \gamma = (1, 98), \delta = (2, 99),$$

and

$$\varphi_3(t) = \sin^2 t, \varphi_1(t) = \sin t, \varphi_2(t) = \cos t,$$

Notation: This graph provides a representation of a numerical approximation solution to (3.7). We notice that the composition $D^{1,98}(D^{2,99}u)$ is taken in this order, the nonlocal conditions couple the whole solution domain.

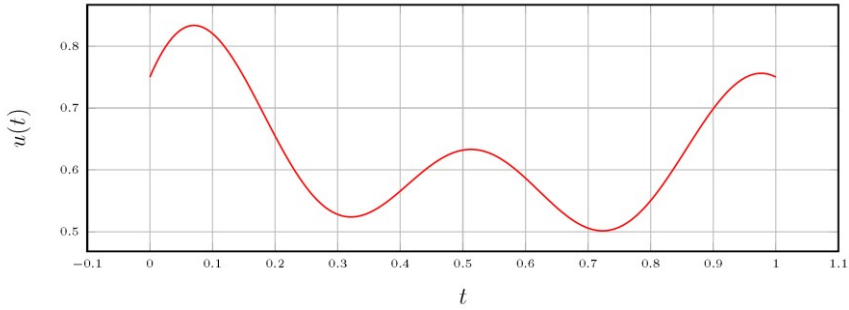


Figure 1
Approximation of $u(t)$ for Problem (3.7)

Example 3: We consider the following problem on the interval $[0,1]$:

$$D_C^{1,98}(D_C^{2,99}u(t)) = u^4(t)(\sin^2(t)) + u(t)(\sin t) + (u''(t))^2(\cos t),$$

with conditions:

$$u(1) = \frac{\sqrt{3}}{2}, u'(1) = \frac{1}{1 - \pi \int_0^1 u(s) ds}, u''(0) = \frac{1}{\sqrt{5}+2},$$

$$u'(0) = \pi, u(0) = e^\pi \int_0^1 u(s) ds.$$

One can verify that this example satisfies all conditions of Theorem 3.1.

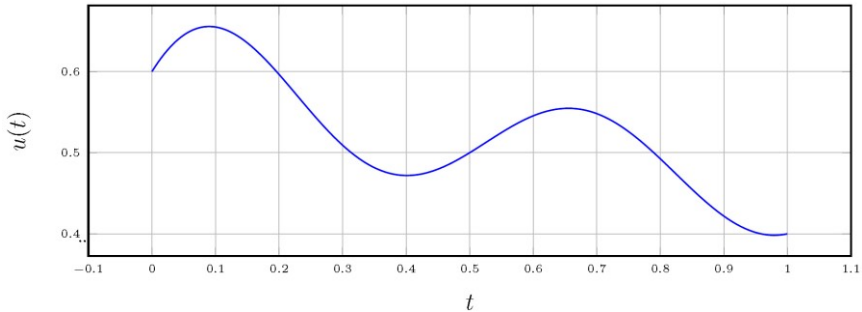


Figure 2
Approximation solution of $u(t)$ for Example 3.

Discussion

The graphs (see Figures 1 and 2) represent approximations of solutions for example 2 and example 3. They were produced by discretizing the fractional derivatives.

Conclusion

The comparison of the two graphs reveals that modifying the integral condition significantly affects the solution. The first condition introduces a strong nonlinear feedback that can cause stiffness and sharper variations in the solution,

but the modified condition smooths this interaction, resulting in a more stable and less oscillatory solution. This shows how changes in nonlocal boundary conditions can influence the qualitative nature of fractional boundary-value problems.

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