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Existence and UML-stability results for fractional Duffing equation with sequential Caputo-Riemann-Liouville derivatives

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Abstract

In this article, we study a Duffing problem include two sequential fractional derivatives, we prove the existence and uniqueness results using some classical fixed point theorems. The UML-stability of the proposed problem is examined. We conclude our work by presenting an application.

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1. Introduction and fractional calculus

Differential equations include fractional calculus are considered as active field of research due to their applications in numerous areas including computer science biological sciences and dynamics, we refer to the works [16, 17, 24] and the references therein. Many interesting topics concerning research for differential problems with fractional derivatives are devoted to the existence, Ulam-stability (U-stability) and UML-stability of the solutions, see [2, 6, 11, 18,

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22, 29]. Recently several authors have interested in the existence and stability of fractional boundary value problem, we refer to the monographs [1, 12, 14, 22, 27, 28]. In this paper, we study the sequential Duffing problem of arbitrary orders. The classical form of the equation [5]

$$D^2 z(\iota) + \lambda D^1 z(\iota) = \phi(\iota) - \varphi(\iota, z(\iota)), 0 \leq \iota \leq 1, \lambda > 0,$$

with the boundary value condition: $z(0) = d_1,\ D^1 z(0) = d_2,\ d_i \in \mathbb{R}, i = 1,2$, this equation has many applications in chaotic phenomena, for more information, we cite the works [3, 4, 7, 20, 21]. In the last years, some authors have studied the above problem with fractional derivative, for instance see [8, 10, 13, 15, 23, 26]. The following Duffing problem with sequential fractional derivatives is the subject of the paper

$$_cD^\vartheta\left[{}^{R,L}D^\delta z(\iota)\right] = \phi(\iota) - \Delta\varphi(\iota, z(\iota), {}^{R.L}D^\beta z(\iota)) - \psi(\iota, z(\iota), I^\alpha z(\iota)), \quad (1)$$

with initial values ${}^{R.L}D^\delta z(0) = \theta, z(1) - I^\gamma z(\omega) = 0, z(0) = 0$ for $1 < \vartheta \leq 2, 0 < \beta < \delta \leq 1, \alpha, \gamma \geq 0, \Delta > 0,\ \theta \in \mathbb{R},\ 0 < \eta < 1,\ \iota \in \Xi, \Xi = [0, T]$, where ${}_CD^\vartheta, {}^{RL}D^\delta$ denote the Caputo and Riemann-Liouville fractional derivatives, $\varphi, \psi: \Xi \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi: \Xi \to \mathbb{R}$ are given continuous functions. ${}_CD^\vartheta$ [16, 17], is defined by

$$_cD^\vartheta[z(\iota)] = I^{n-\vartheta}\left[z^{(n)}(\iota)\right], \vartheta > 0,$$

and ${}^{R,L}D^\delta$ [16, 17], is defined by

$${}^{RL}D^\delta[z(\iota)] = \left(\frac{d}{dt}\right)^n (I^{n-\vartheta}[z(\iota)]).$$

The R-L fractional integral [16, 17] of order $\vartheta > 0$, stated as

$$I^\vartheta[z(\iota)] = \frac{1}{\Gamma(\vartheta)} \int_0^\iota (\iota - s)^{\vartheta-1} z(s) ds, \iota > 0.$$

where $\Gamma(.)$ is the Euler gamma function.

Lemma 1: [16, 25] *For $\vartheta > 0$, we have*

$$I^\vartheta\left[{}_cD^\vartheta z(\iota)\right] - z(\iota) = \sum_{i=0}^{n-1} c_i^i \iota,$$

where $c_i \in \mathbb{R}$.

Lemma 2: [16, 25] *For $z \in C(0,T) \cap L^1(0,T)$ and ${}^{RL}D^\delta z \in C(0,T) \cap L^1(0,T)$, we can write*

$$I^\delta\left[{}^{R,L}D^\delta z(\iota)\right] - z(\iota) =_{i=1}^n c_i \iota^{\delta - i}, n = [\delta] + 1.$$

Theorem 3: [19] *For $\iota \in \Xi'$, we can write*

$$y(\iota) - b(\iota) \leq \sum_{i=1}^n a_i(\iota) \int_0^\iota (\iota - s)^{\varepsilon_i - 1} y(s) ds,$$

with $\varepsilon_i > 0$, a_i $(i = 1,2,\ldots,n)$ are the bounded and monotonic increasing functions on $\Xi' = [0,T)$, then

$$y(\iota) - b(\iota) \leq \sum_{j=1}^\infty \left(\sum_{1',2',\ldots,j'=1}^n \frac{\prod_{i=1}^j \left[a_{i'}(t)\Gamma(\varepsilon_{i'})\right]}{\Gamma\left(\sum_{i=1}^j \varepsilon_{i'}\right)} \int_0^\iota (\iota - s)^{\sum_{i=1}^j \varepsilon_{i'} - 1} b(s) ds \right).$$

Remark 4: For $n=2$, if $\varepsilon_1,\varepsilon_2>0, a_1,a_2\geq 0, b(\iota)$ is nonnegative and locally integrable on Ξ' and $z(\iota)$ is nonnegative and locally integrable on Ξ' with

$$y(\iota)-b(\iota)\leq a_1\int_0^\iota (\iota-s)^{\varepsilon_1-1}y(s)ds+a_2\int_0^\iota (\iota-s)^{\varepsilon_2-1}y(s)ds,$$

then

$$y(\iota)-b(\iota)\ \leq\ \sum_{j=1}^{\infty}\left(\frac{(a_1\Gamma(\varepsilon_1))^j}{\Gamma(j\varepsilon_1)}\int_0^\iota (\iota-s)^{j\varepsilon_1-1}k(s)ds\right.$$
$$\left.+\frac{(a_2\Gamma(\varepsilon_2))^j}{\Gamma(j\varepsilon_2)}\int_0^\iota (\iota-s)^{j\varepsilon_2-1}k(s)ds\right).$$

Remark 5: Under the conditions of Remark 6, let $b(\iota)$ is a nondecreasing function on Ξ'. Then we have

$$y(\iota)\leq b(\iota)\big(E_{\varepsilon_1}[a_1\Gamma(\varepsilon_1)\iota^{\varepsilon_1}]+E_{\varepsilon_2}[a_2\Gamma(\varepsilon_2)\iota^{\varepsilon_2}]\big),$$

where E_ε [18] is defined by: $E_\varepsilon[x]=\sum_{j=1}^{\infty}\frac{x^\varepsilon}{\Gamma(j\varepsilon+1)}, x\in\mathbb{C}$.

Let $Z=\{z\colon z\in C(\Xi,\mathbb{R})\text{and}_{RL}D^\beta z\in C(\Xi,\mathbb{R})\}$, the space $(Z,\|.\|_Z)$ is a Banach space,.with the norm

$$\|z\|_Z=\|z\|+\left\|^{RL}D^\beta z\right\|,$$

where

$$\|z\|=\sup_{\iota\in\Xi}|z(\iota)|\text{and}\left\|z^{RL}D^\beta\right\|=\sup_{\iota\in\Xi}\left|^{RL}D^\beta z(\iota)\right|.$$

We define the stability for (1). For $\varpi>0$, we give:

$$\left|{}^{C}D^{\vartheta}\left[^{RL}D^\beta y(\iota)\right]-\left(\phi(\iota)-\Delta_1\chi_y(\iota)-{}_2\chi_y(\iota)\right)\right|\leq\varpi,\iota\in\Xi, \tag{2}$$

and

$$\left|{}^{C}D^{\vartheta}\left[^{RL}D^\beta y(\iota)\right]-\left(\phi(\iota)-\Delta_1\chi_y(\iota)-{}_2\chi_y(\iota)\right)\right|\leq\varpi\chi(t),t\in\Xi, \tag{3}$$

where $\chi\in C(\Xi,\mathbb{R}^+)$ and ${}_1\chi_y(\iota)=\varphi(\iota,y(\iota),^{RL}D^\beta y(\iota))$ and ${}_2\chi_y(\iota)=\psi\big(\iota,y(\iota),I^\alpha y(\iota)\big)$.

Definition 6: The fractional Duffing equation (1) is UHML-stability with respect to $E_{\vartheta+\delta}$ if $\forall\varpi>0$ and for any $y\in Z$ of (2), there exists $z\in Z$ of (1) and $\exists\sigma>0$, with

$$|y(\iota)-z(\iota)|\leq\sigma\varpi E_{\vartheta+\delta}[\iota],\iota\in\Xi.$$

Definition 7: The fractional Duffing equation (1) is UHRML-stable, with respect to $\chi E_{\vartheta+\delta}$ if $\forall\varpi>0$ and for all $y\in Z$ of (3), there exists $z\in Z$ of (1) $\exists\sigma_\chi>0$, with

$$|y(\iota)-z(\iota)|\leq\sigma_\chi\varpi\chi(\iota)E_{\vartheta+\delta}[t],\iota\in\Xi.$$

Remark 8: $y \in Z$ is a solution of (2) iff $g: \Xi \to \mathbb{R}$ which depend on z, with $|g(t)| \leq \varpi, \iota \in \Xi$ and

$$^{C}D^{\vartheta}\left[^{RL}D^{\beta}y(\iota)\right] = \left(\phi(t) - \Delta_1\chi_y(\iota) -_2\chi_y(\iota)\right) + g(\iota), \iota \in \Xi.$$

2. Existence theorems

Lemma 9: *For a given* $h(\iota) \in C(\Xi, \mathbb{R})$*, the problem*

$$\begin{cases} {}^{C}D^{\vartheta}\left[^{R.L}D^{\delta}z(\iota) - h(\iota) = 0, \iota \in \Xi\right] \\ {}^{R.L}D^{\delta}z(0) = z(1) = I^{\gamma}z(\omega), z(0) = 0, \\ 1 < \vartheta \leq 2, 0 < \delta < 1, 0 < \omega < 1, \theta \in \mathbb{R}, \end{cases} \tag{4}$$

has a unique solution

$$\begin{aligned} z(\iota) &= \frac{1}{\Gamma(\vartheta+\delta)}\int_0^{\iota}(\iota-s)^{\vartheta+\delta-1}h(s)ds \\ &+ \frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta+\gamma)}\int_0^{\omega}(\omega-s)^{\vartheta+\delta+\gamma-1}h(s)ds \\ &- \frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta)}\int_0^{T}(T-s)^{\vartheta+\delta-1}h(s)ds \\ &+ \frac{\theta\Gamma(\delta+\gamma+2)}{\Pi}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{T^{\delta}}{\Gamma(\delta+1)}\right)t^{\delta+1} + \frac{\theta}{\Gamma(\delta+1)}\iota^{\delta}, \end{aligned} \tag{5}$$

where $\Pi := \Gamma(\delta+2)\omega^{\delta+\gamma+1} - T^{\delta+1}\Gamma(\delta+\gamma+2)$, $\omega^{\delta+\gamma+1} \neq \frac{\Gamma(\delta+\gamma+2)}{T^{\delta+1}\Gamma(\delta+2)}$.

Proof: Using Lemma 1, we can write

$$^{R.L}D^{\delta}z(\iota) = \frac{1}{\Gamma(\vartheta)}\int_0^{\iota}(\iota-s)^{\vartheta-1}h(s)ds + c_0 + c_1\iota, \tag{6}$$

where $c_0, c_1 \in \mathbb{R}$. Now, applying Lemma 2, we get

$$\begin{aligned} z(\iota) &= \frac{1}{\Gamma(\vartheta+\delta)}\int_0^{\iota}(\iota-s)^{\vartheta+\delta-1}h(s)ds + c_0\frac{1}{\Gamma(\delta+1)}\iota^{\delta} \\ &+ c_1\frac{1}{\Gamma(\delta+2)}\iota^{\delta+1} + c_2\iota^{\delta-1}, \end{aligned} \tag{7}$$

where $c_2 \in \mathbb{R}$. Since $z(0) = 0$, we have $c_2 = 0$, and by $^{R.L}D^{\delta}z(0) = \theta$, we obtain $c_0 = \theta$.

Now, using $z(T) = I^{\gamma}z(\eta)$, we get

$$\begin{aligned} c_1 = \frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)}{\Pi}\Bigg[&\frac{1}{\Gamma(\vartheta+\delta+\gamma)}\int_0^{\omega}(\omega-s)^{\vartheta+\delta+\gamma-1}h(s)ds \\ &- \frac{1}{\Gamma(\vartheta+\delta)}\int_0^{T}(T-s)^{\vartheta+\delta-1}h(s)ds + \theta\left(\frac{\eta^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{T^{\delta}}{\Gamma(\delta+1)}\right)\Bigg]. \end{aligned}$$

By placing coefficients c_0, c_1 and c_2 in (7) we get (4).

We define $O: Z \to Z$ by:

$$\begin{aligned} Oz(\iota) &= \frac{1}{\Gamma(\vartheta+\delta)}\int_0^{\iota}(\iota-s)^{\vartheta+\delta-1}\left(\phi(s) - \Delta_1\chi_z(s) -_2\chi_z(s)\right)ds \\ &+ \frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta+\gamma)}\int_0^{\omega}(\omega-s)^{\vartheta+\delta+\gamma-1}\left(\phi(s) - \Delta_1\chi_z(s) -_2\chi_z(s)\right)ds \end{aligned}$$

$$-\frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta)}\int_0^T (T-s)^{\vartheta+\delta-1}\big(\phi(s)-\Delta_1\chi_z(s)-_2\chi_z(s)\big)ds$$
$$+\frac{\theta\Gamma(\delta+\gamma+2)}{\Pi}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}-\frac{T^{\delta}}{\Gamma(\delta+1)}\right)\iota^{\delta+1}+\frac{\theta}{\Gamma(\delta+1)}\iota^{\delta}, \tag{8}$$

and

$${}^{RL}D^{\beta}Oz(\iota)$$
$$=\frac{1}{\Gamma(\vartheta+\delta-\beta)}\int_0^{\iota} (\iota-s)^{\vartheta+\delta-\beta-1}\big(\phi(s)-\Delta_1\chi_z(s)-_2\chi_z(s)\big)ds$$
$$+\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)\iota^{\delta-\beta+1}}{\Pi\Gamma(\vartheta+\delta+\gamma)\Gamma(\delta-\beta+2)}\int_0^{\omega} (\omega-s)^{\vartheta+\delta+\gamma-1}\big(\phi(s)-\Delta_1\chi_z(s)-_2\chi_z(s)\big)ds$$
$$-\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)\iota^{\delta-\beta+1}}{\Pi\Gamma(\vartheta+\delta)\Gamma(\delta-\beta+2)}\int_0^{T} (T-s)^{\vartheta+\delta-1}\big(\phi(s)-\Delta_1\chi_z(s)-_2\chi_z(s)\big)ds \tag{9}$$
$$+\frac{\theta\Gamma(\delta+\gamma+2)\Gamma(\delta+2)}{\Pi\Gamma(\delta-\beta+2)}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}-\frac{T^{\delta}}{\Gamma(\delta+1)}\right)\iota^{\delta-\beta+1}+\frac{\theta}{\Gamma(\delta-\beta+1)}\iota^{\delta-\beta},$$

We give the notations:

$$\nabla_1:=\frac{T^{\vartheta+\delta}}{\Gamma(\vartheta+\delta+1)}+\frac{\Gamma(\delta+\gamma+2)T^{\delta+1}\omega^{\vartheta+\delta+\gamma}}{|\Pi|\Gamma(\vartheta+\delta+\gamma+1)}+\frac{\Gamma(\delta+\gamma+2)T^{\vartheta+\delta+1}}{|\Pi|\Gamma(\vartheta+\delta+1)},$$
$$\Lambda_1:=\frac{|\theta|T^{\delta}}{\Gamma(\delta+1)}+\frac{|\theta|\Gamma(\delta+\gamma+2)}{|\Pi|}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^{\delta}}{\Gamma(\delta+1)}\right)T^{\delta+1},$$
$$\nabla_2:=\frac{T^{\vartheta+\delta-\beta}}{\Gamma(\vartheta+\delta-\beta+1)}+\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\delta+1}\omega^{\vartheta+\delta+\gamma}}{|\Pi|\Gamma(\vartheta+\delta+\gamma+)\Gamma(\delta-\beta+1)} \tag{10}$$
$$+\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\vartheta+2\delta-\beta+1}}{|\Pi|\Gamma(\vartheta+\delta+1)\Gamma(\delta-\beta+2)},$$
$$\Lambda_2:=\frac{|\theta|\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\delta-\beta+1}}{|\Pi|\Gamma(\delta-\beta+2)}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^{\delta}}{\Gamma(\delta+1)}\right)+\frac{|\theta|T^{\delta-\beta}}{\Gamma(\delta-\beta+1)}.$$

Now, we prove that O has a fixed point.

Theorem 10: *If* $\varphi,\psi:\Xi\times\mathbb{R}^2\to\mathbb{R}$, $\phi:\Xi\to\mathbb{R}$ *are continuous functions and let*

(H_1): $\exists$ $\mu_1,\mu_2>0$ *which for all* $t\in\Xi$ *and* $y_i,z_i\in\mathbb{R}, i=1,2,$ *the following inequalities hold true*

$$|\varphi(\iota,z_1,z_2)-\varphi(\iota,y_1,y_2)| \le \mu_1(|z_1-y_1|+|z_2-y_2|),$$

$$|\psi(\iota,z_1,z_2)-\psi(\iota,y_1,y_2)| \le \mu_2(|z_1-y_1|+|z_2-y_2|).$$

If

$$\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)<(\nabla_1+\nabla_2)^{-1}-\Delta\mu_1, \tag{11}$$

is valid, then (1) has one and only one solution on Ξ.

Proof: Define

$$\varepsilon\ge\frac{N(\Delta+2)+\Lambda_1+\Lambda_2}{1-\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right](\nabla_1+\nabla_2)}, N=\max\{N_1,N_2,N_3\},$$

where $N_i<\infty$ such that

$$N_1=\sup_{t\in\Xi}|\varphi(\iota,0{,}0)|, N_2=\sup_{t\in\Xi}|\psi(\iota,0{,}0)| \text{ and } N_3=\sup_{t\in\Xi}|\phi(\iota)|.$$

We prove that $TB_\varepsilon\subset B_\varepsilon$, $B_\varepsilon=\{z\in Z:\|z\|_Z\le\varepsilon\}$. Using (H_1), we can write

$$\begin{aligned}
\left|{}_1\chi_z(\iota)\right| &= \left|\varphi\left(\iota,z(\iota),{}^{RL}D^\beta z(\iota)\right)\right|\\
&\le \left|\varphi\left(\iota,z(\iota),{}^{RL}D^\beta z(t)\right)-\varphi(\iota,0{,}0)\right|+|\varphi(\iota,0{,}0)|\\
&\le \mu_1\left(|z(\iota)|+\left|{}^{RL}D^\delta z(\iota)\right|\right)+N_1\le\mu_1\left(\|z\|+\left\|{}^{RL}D^\delta z\right\|\right)+N_1\\
&\le \mu_1\|w\|_W+N_1\le\mu_1\varepsilon+N,\\
\left|{}_2\chi_z(\iota)\right| &= \left|\psi\left(\iota,z(\iota),I^\alpha z(\iota)\right)\right|\\
&\le \left|\psi\left(\iota,z(\iota),I^\alpha z(\iota)\right)-\psi(\iota,0{,}0)\right|+|\psi(\iota,0{,}0)|\\
&\le \mu_2(|z(\iota)|+|I^\alpha z(\iota)|)+N_2\le\mu_2\left(\|z\|+\frac{\|z\|}{\Gamma(\alpha+1)}\right)+N_2\\
&\le \mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\|z\|_Z+N_2\le\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\varepsilon+N,
\end{aligned}\tag{12}$$

For $z\in B_\varepsilon$ and by (12), we have

$$\begin{aligned}
|Oz(\iota)| &\le \frac{1}{\Gamma(\vartheta+\delta)}\int_0^\iota(\iota-s)^{\vartheta+\delta-1}|\phi(s)-\Delta_1\chi_z(s)-{}_2\chi_z(s)|ds\\
&\quad+\frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{|\Pi|\Gamma(\vartheta+\delta+\gamma)}\int_0^\omega(\omega-s)^{\vartheta+\delta+\gamma-1}|\phi(s)-\Delta_1\chi_z(s)-{}_2\chi_z(s)|ds\\
&\quad+\frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{|\Pi|\Gamma(\vartheta+\delta)}\int_0^T(T-s)^{\vartheta+\delta-1}|\phi(s)-\Delta_1\chi_z(s)-{}_2\chi_z(s)|ds\\
&\quad+\frac{|\theta|}{\Gamma(\delta+1)}\iota^\delta+\frac{|\theta|\Gamma(\delta+\gamma+2)}{|\Pi|}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^\delta}{\Gamma(\delta+1)}\right)\iota^{\delta+1}\\
&\le \left(\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\varepsilon+N(\Delta+2)\right)\\
&\quad\left[\frac{T^{\vartheta+\delta}}{\Gamma(\vartheta+\delta+1)}+\frac{\Gamma(\delta+\gamma+2)T^{\delta+1}\omega^{\vartheta+\delta+\gamma}}{|\Pi|\Gamma(\vartheta+\delta+\gamma+1)}+\frac{\Gamma(\delta+\gamma+2)T^{\vartheta+\delta+1}}{|\Pi|\Gamma(\vartheta+\delta+1)}\right]\\
&\quad+\frac{|\theta|\Gamma(\delta+\gamma+2)}{|\Pi|}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^\delta}{\Gamma(\delta+1)}\right)T^{\delta+1}+\frac{|\theta|T^\delta}{\Gamma(\delta+1)}\\
&= \left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\nabla_1\varepsilon+N(\Delta+2)\nabla_1+\Lambda_1,
\end{aligned}$$

which implies that

$$\|Oz\|\le\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\nabla_1\varepsilon+N(\Delta+2)\nabla_1+\Lambda_1.$$

Using (9) and (10), we can write

$$\begin{aligned}
&\left\|{}^{RL}D^\delta Oz\right\|\\
&\le \left[\frac{T^{\vartheta+\delta-\beta}}{\Gamma(\vartheta+\delta-\beta+1)}+\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\delta+1}\omega^{\vartheta+\delta+\gamma}}{\Pi\Gamma(\vartheta+\delta+\gamma+)\Gamma(\delta-\beta+1)}\right.\\
&\left.+\frac{\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\vartheta+2\delta-\beta+1}}{\Pi\Gamma(\vartheta+\delta+1)\Gamma(\delta-\beta+2)}\right]\left(\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\varepsilon+N(\Delta+2)\right)
\end{aligned}$$

$$+\frac{|\theta|\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\delta-\beta+1}}{\Pi\Gamma(\delta-\beta+2)}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^{\delta}}{\Gamma(\delta+1)}\right)+\frac{|\theta|T^{\delta-\beta}}{\Gamma(\delta-\beta+1)}$$
$$=\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\nabla_2\varepsilon+N(\Delta+2)\nabla_2+\Lambda_2.$$

From the definition of $\|.\|_Z$, we have

$$\begin{aligned}\|O(z)\|_Z &= \|O(z)\|+\left\|{}^{RL}D^{\delta}Oz(t)\right\| \\ &\leq \left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right](\nabla_1+\nabla_2)\varepsilon+N(\Delta+2)(\nabla_1+\nabla_2) \\ &\quad +\Lambda_1+\Lambda_2 \leq \varepsilon,\end{aligned}$$

which implies that $OB_\varepsilon\subset B_\varepsilon$. Now for $y,z\in B_\varepsilon, \iota\in\Xi$, we get

$$|Oz(\iota)-Oy(\iota)|$$
$$\leq \frac{1}{\Gamma(\vartheta+\delta)}\int_0^{\iota}(\iota-s)^{\vartheta+\delta-1}\left(\Lambda\left|{}_1\chi_z(s)-{}_1\chi_y(s)\right|+\left|{}_2\chi_z(s)-{}_2\chi_y(s)\right|\right)ds$$
$$+\frac{\Gamma(\delta+\gamma+2)t^{\delta+1}}{\Pi\Gamma(\vartheta+\delta+\gamma)}\int_0^{\omega}(\omega-s)^{\vartheta+\delta+\gamma-1}\left(\Lambda\left|{}_1\chi_z(s)-{}_1\chi_y(s)\right|+\left|{}_2\chi_z(s)-{}_2\chi_y(s)\right|\right)ds$$
$$+\frac{\Gamma(\delta+\gamma+2)t^{\delta+1}}{\Pi\Gamma(\vartheta+\delta)}\int_0^{T}(T-s)^{\vartheta+\delta-1}\left(\Lambda\left|{}_1\chi_z(s)-{}_1\chi_y(s)\right|+\left|{}_2\chi_z(s)-{}_2\chi_y(s)\right|\right)ds.$$

So, applying (H_1), we get

$$\|Oz-Oy\|\leq\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\nabla_1\|z-y\|_Z, \tag{13}$$

also, we have

$$\left\|{}^{RL}D^{\delta}Oz-{}^{RL}D^{\beta}Oy\right\|\leq\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right]\nabla_2\|z-y\|_Z. \tag{14}$$

From (13) and (14), we can write

$$\begin{aligned}\|Oz-Oy\|_Z &= \|Oz-Oy\|+\left\|{}^{RL}D^{\delta}Oz-{}^{RL}D^{\beta}Oy\right\| \\ &\leq \left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right](\nabla_1+\nabla_2)\|z-y\|_Z,\end{aligned}$$

Thus, the proof is achieved.

Now we are in a position to prove the existence of at least one solutions for (1) by using Lemma of [9].

Theorem 11: *Suppoe that $\varphi,\psi:\Xi\times\mathbb{R}^2\to\mathbb{R}$ and $\phi:\Xi\to\mathbb{R}$ are continuous. Assume that:*

(H_2): Let $\rho_1,\rho_2,\rho_1,\rho_2\in\mathbb{R}_+$ and $\rho_0,\rho_0\in\mathbb{R}_+^*$ which for real numbers $z_1,z_2\in\mathbb{R}$, the following inequalities are satisfied

$$|\varphi(\iota,z_1,z_2)|\leq\rho_0+\rho_1|z_1|+\rho_2|z_2|,$$

and

$$|\psi(\iota,z_1,z_2)|\leq\varrho_0+\varrho_1|z_1|+\rho_2|z_2|.$$

(H_3): Let $\beta_\phi>0$, with

$$|\phi(\iota)|\leq\beta_\phi,\forall\iota\in\Xi$$

If

$$\Delta(\rho_1+\rho_2)+(\varrho_1+\varrho_2)<(\nabla_1+\nabla_2)^{-1}. \tag{15}$$

Then the Duffing problem (1) admits one solution at least.

Proof: We begin by showing that O is completely continuous (O is continuous).

For $F \subset Z$ and $L_i>0, i=1,2$, we get

$$|\varphi(\iota,y,z)|\le L_1, |\psi(\iota,y,z)|\le L_2, \text{for each } y,z\in F.$$

Then for any $z\in F$, we have

$$\begin{aligned}|Oz(\iota)| \le\ & \frac{1}{\Gamma(\vartheta+\delta)}\int_0^\iota (\iota-s)^{\vartheta+\delta-1}|\phi(s)-\Delta_1\chi_z(s){-}_2\chi_z(s)|ds\\ &+\frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta+\gamma)}\int_0^\omega (\omega-s)^{\vartheta+\delta+\gamma-1}|\phi(s)-\Delta_1\chi_z(s){-}_2\chi_z(s)|ds\\ &+\frac{\Gamma(\delta+\gamma+2)\iota^{\delta+1}}{\Pi\Gamma(\vartheta+\delta)}\int_0^T (T-s)^{\vartheta+\delta-1}|\phi(s)-\Delta_1\chi_z(s){-}_2\chi_z(s)|ds\\ &+\frac{\theta\Gamma(\delta+\gamma+2)}{\Pi}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^\delta}{\Gamma(\delta+1)}\right)\iota^{\delta+1}+\frac{\theta\iota^\delta}{\Gamma(\delta+1)},\end{aligned}$$

using (H_3), we can write

$$\|O(z)\|\le\left[\beta_\phi+\Delta L_1+L_2\right]\nabla_1+\Lambda_1.$$

On the other hand, we have

$$\left\|{}^{RL}D^\delta O(z)\right\|\le\left[\beta_\phi+\Delta L_1+L_2\right]\nabla_2+\Lambda_2.$$

Hence for $z\in$, we obtain $\|O\|_Z<\infty$.

Now, we prove that O is equi-continuous sets of Z. Let $z\in F$ and $0\le\iota_1<\iota_2\le 1$, then we can write

$$\begin{aligned}&|Oz(\iota_2)-Oz(\iota_1)|\\ &\le\frac{(\beta_\phi+\Delta L_1+L_2)}{\Gamma(\vartheta+\delta+1)}\left[(\iota_2-\iota_1)^{\theta+\delta}+\left|\iota_2^{\theta+\delta}-\iota_1^{\theta+\delta}\right|\right]\\ &+(\beta_\phi+\Delta L_1+L_2)\left[\frac{\Gamma(\delta+\gamma+2)\omega^{\vartheta+\delta+\gamma}}{|\Pi|\Gamma(\vartheta+\delta+\gamma+1)}+\frac{\Gamma(\delta+\gamma+2)T^{\vartheta+\delta}}{|\Pi|\Gamma(\vartheta+\delta+1)}\right]\left|\iota_2^{\delta+1}-\iota_1^{\delta+1}\right|\\ &+\frac{|\theta|\Gamma(\delta+\gamma+2)}{|\Pi|}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}+\frac{T^\delta}{\Gamma(\delta+1)}\right)\left|\iota_2^{\delta+1}-\iota_1^{\delta+1}\right|\\ &+\frac{|\theta|}{\Gamma(\delta+1)}\left|\iota_2^\delta-\iota_1^\delta\right|,\end{aligned}$$

and

$$\begin{aligned}&\left|{}^{RL}D^\delta Oz(\iota_2){-}{}^{RL}D^\beta Oz(\iota_2)\right|\\ &\le\frac{(\beta_\phi+\Delta L_1+L_2)}{\Gamma(\vartheta+\delta-\beta+1)}\left[(\iota_2-\iota_1)^{\vartheta+\delta-\beta}+\left|\iota_2^{\vartheta+\delta-\beta}-\iota_1^{\vartheta+\delta-\beta}\right|\right]\\ &+\Bigg[(\beta_\phi+\Delta L_1+L_2)\frac{(k+\Lambda N_1+N_2)\Gamma(\delta+\gamma+2)\Gamma(\delta+2)\omega^{\vartheta+\delta+\gamma}}{|\Pi|\Gamma(\vartheta+\delta+\gamma+1)\Gamma(\delta-\beta+2)}\\ &+\frac{(\beta_\phi+\Delta L_1+L_2)\Gamma(\delta+\gamma+2)\Gamma(\delta+2)T^{\vartheta+\delta}}{|\Pi|\Gamma(\vartheta+\delta+1)\Gamma(\delta-\beta+2)}\Bigg]\left|\iota_2^{\delta-\beta+1}-\iota_1^{\delta-\beta+1}\right|\\ &+\frac{|\theta|\Gamma(\delta+\gamma+2)\Gamma(\delta+2)}{|\Pi|\Gamma(\delta-\beta+2)}\left(\frac{\omega^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}-\frac{T^\delta}{\Gamma(\delta+1)}\right)\left|\iota_2^{\delta-\beta+1}-\iota_1^{\delta-\beta+1}\right|\\ &+\frac{|\theta|}{\Gamma(\delta-\beta+1)}\left|\iota_2^{\delta-\beta}-\iota_1^{\delta-\beta}\right|.\end{aligned}$$

Therefore, the operator O is equicontinuous.

Finally, we verify that Σ defined by

$$\Sigma = \{z \in Z : z = \sigma O(z), 0 < \sigma < 1\},$$

is bounded. For $w \in \Sigma$ and $\iota \in \Xi,$ we have

$$z(\iota) = \sigma O z(\iota).$$

By $(H_2),$ we can write

$$\begin{aligned} &|z(\iota)| \\ &\leq \left[\beta_{\phi} + \Delta\rho_0 + \rho_0 + [\Delta(\rho_1 + \rho_2) + (\rho_1 + \rho_2)]\|z\|_Z\right]\nabla_1 + \Lambda_1. \end{aligned}$$

and

$$\begin{aligned} &\left|{}^{RL}D^{\delta} w(\iota)\right| \\ &\leq \left[\beta_{\phi} + \Delta\rho_0 + \rho_0 + [\Delta(\rho_1 + \rho_2) + (\rho_1 + \rho_2)]\|z\|_Z\right]\nabla_2 + \Lambda_2. \end{aligned}$$

In view of the above estimates,we get

$$\begin{aligned} \|z\|_Z \leq\ & [\Delta(\lambda_1 + \lambda_2) + (\gamma_1 + \gamma_2)](\nabla_1 + \nabla_2)\|z\|_Z \\ & + \left[\beta_{\phi} + \Lambda\lambda_0 + \gamma_0\right](\nabla_1 + \nabla_2) + \Lambda_1 + \Lambda_2, \end{aligned}$$

which imply that

$$\|z\|_Z \leq \frac{[\beta_{\phi} + \Lambda\lambda_0 + \gamma_0](\nabla_1 + \nabla_2) + \Lambda_1 + \Lambda_2}{1 - [\Delta(\lambda_1 + \lambda_2) + (\gamma_1 + \gamma_2)](\nabla_1 + \nabla_2)}.$$

Consequently, F is bounded.

We deduce that O admits one fixed point at least, which is a solution of (1).

3. UML-stability theorems

In the following theorems, we will prove the UHML-stability and UHRML-stability of (1).

Theorem 12: *Let $\varphi, \psi \in C(\Xi \times \mathbb{R}^2, \mathbb{R}), \phi \in C(\Xi, \mathbb{R})$ and assume that (H_1) satisfies. Then the Duffing equation (1) is UHML-stable.*

Proof: Let $y \in Z$ be a solution of (2). Then, by applying Lemma 11, we have

$$y(\iota) = I^{\vartheta+\delta} h_y(\iota) + \frac{b_0}{\Gamma(\delta+1)}\iota^{\delta} + \frac{b_1}{\Gamma(\delta+2)}\iota^{\delta+1} + b_2\iota^{\delta-1},$$

and

$$\begin{aligned} &\left|y(\iota) - I^{\vartheta+\delta} h_y(\iota) - \frac{b_0}{\Gamma(\delta+1)}\iota^{\delta} - \frac{b_1}{\Gamma(\delta+2)}\iota^{\delta+1} - b_2\iota^{\delta-1}\right| \\ &\leq \frac{\varpi}{\Gamma(\vartheta+\delta+1)}\iota^{\vartheta+\delta} \leq \frac{\varpi}{\Gamma(\vartheta+\delta+1)}. \end{aligned} \tag{16}$$

From these relation, it follows

$$\begin{aligned} |y(\iota) - z(\iota)| =\ & \Big|y(\iota) - I^{\vartheta+\delta} h_y(\iota) - \frac{b_0}{\Gamma(\delta+1)}\iota^{\delta} - \frac{b_1}{\Gamma(\delta+2)}\iota^{\delta+1} - b_2\iota^{\delta-1} \\ & + I^{\vartheta+\delta}\left[h_y(t) - h_z(t)\right]\Big| \\ \leq\ & \left|y(\iota) - I^{\vartheta+\delta} h_y(\iota) - \frac{b_0}{\Gamma(\delta+1)}\iota^{\delta} - \frac{b_1}{\Gamma(\delta+2)}\iota^{\delta+1} - b_2\iota^{\delta-1}\right| \\ & + \left|I^{\vartheta+\delta}\left[h_y(\iota) - h_z(\iota)\right]\right|, \end{aligned}$$

where

$$h_y(t) = \phi(t) - \Lambda\varphi_y^*(\iota) - \psi_y^*(\iota),$$

and

$$h_z(\iota) = \phi(\iota) - \Lambda\varphi_z^*(\iota) - \psi_z^*(\iota),$$

So, by (H_1), we get

$$\begin{aligned} \left|^{\vartheta+\delta}\left[h_y(\iota) - h_z(\iota)\right]\right| \leq & \frac{\Delta\mu_1}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y - z\|_Z ds \\ & + \frac{\mu_2}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y - z\|_Z ds. \end{aligned} \tag{17}$$

Thanks tot (16) and (17), we can state that

$$\begin{aligned} & \|y(s) - z(s)\|_Z \\ & \leq \frac{\varpi}{\Gamma(\vartheta+\delta+1)} + \frac{\Delta\mu_1}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds \\ & + \frac{\mu_2}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds. \end{aligned}$$

Then, thanks to Remark 4 and Remark 5, conclude that

$$\|y - z\|_Z \leq \frac{\varpi}{\Gamma(\vartheta+\delta+1)} \left(E_{\vartheta+\delta}\left[\Delta\mu_1\iota^{\vartheta+\delta}\right] + E_{\vartheta+\delta}\left[\mu_2\iota^{\vartheta+\delta}\right]\right).$$

Theorem 13: *Let $\varphi, \psi : \Xi \times \mathbb{R}^2 \to \mathbb{R}, \phi : \Xi \to \mathbb{R}$ be continuous. Assume (H_1) satisfies. If $\exists\ \chi \in C(\Xi, \mathbb{R}_+)$ is increasing and $\exists\ \sigma_\chi > 0$ such that*

$$I^{\vartheta+\delta}[\chi(\iota)] \leq \sigma_\chi \chi(\iota), \iota \in \Xi. \tag{18}$$

Then (1) is UHRML-stable with respect to $\chi E_{\vartheta+\delta}$.

Proof: Let $y \in Z$ be a solution of (3). Using Remark 10, we get

$$\begin{aligned} & \left|y(\iota) - I^{\vartheta+\delta} h_y(\iota) - \frac{b_0}{\Gamma(\delta+1)} \iota^\delta - \frac{b_1}{\Gamma(\delta+2)} \iota^{\delta+1} - b_2 \iota^{\delta-1}\right| \\ & \leq \frac{\varpi}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \chi(s) ds. \end{aligned}$$

and

$$z(\iota) = I^{\vartheta+\delta} h_z(\iota) + \frac{c_0}{\Gamma(\delta+1)} \iota^\delta + \frac{c_1}{\Gamma(\delta+2)} \iota^{\delta+1} + c_2 \iota^{\delta-1}.$$

Then

$$\begin{aligned} & \|y(s) - z(s)\|_Z \\ & \leq \frac{\varpi}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \chi(s) ds + \frac{\Delta\mu_1}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds \\ & + \frac{\mu_2}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds. \end{aligned}$$

By (18), we get

$$\begin{aligned} & \|y(s) - z(s)\|_Z \leq \varpi\pi_\chi \chi(\iota) + \frac{\Delta\mu_1}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds \\ & + \frac{\mu_2}{\Gamma(\vartheta+\delta)} \int_0^\iota (\iota - s)^{\vartheta+\delta-1} \|y(s) - z(s)\|_Z ds. \end{aligned}$$

Now, using Remark 4 and Remark 5, we have

$$\|y-z\|_Z \leq \varpi\sigma_\chi \chi(\iota)\left(E_{\vartheta+\delta}\left[\Delta\mu_1\iota^{\vartheta+\delta}\right]+E_{\vartheta+\delta}\left[\mu_2\iota^{\vartheta+\delta}\right]\right).$$

So, the Duffing equation (1) is UHRML-stable.

4. Example

We give the Duffing equation

$$\begin{cases} {}_C D^{\frac{4}{3}}[{}^{RL}D^{\frac{1}{2}}z(\iota)]+\frac{1}{25e^{\pi}}\left(\frac{\sin z(\iota)}{2\sqrt{\pi}(120+\iota)}+\frac{\left|{}^{RL}D^{\frac{1}{3}}z(\iota)\right|}{2\sqrt{\pi}(120+\iota)\left(\left|{}^{RL}D^{\frac{1}{3}}z(\iota)\right|\right)}\right) \\ \quad +\frac{e^{-3\iota}}{240}\left(\cos z(\iota)+I^{\frac{\sqrt{5}}{2}}z(\iota)+\frac{1}{7}\right)=\frac{3}{5}\iota^{e^{2t+1}} \\ z(0)=R,\ {}^{RL}D^{\frac{1}{2}}z(0)=\frac{\ln 2}{5},\ z(1)=I^{\frac{e}{2}}z\left(\frac{1}{5}\right), \end{cases} \tag{19}$$

and

$$\begin{cases} \left|{}_C D^{\frac{4}{3}}\left[{}^{RL}D^{\frac{1}{2}}z(\iota)\right]-\frac{3}{5}\iota^{e^{2t+1}}-\frac{1}{25e^{\pi}}\varphi\left(\iota,z(\iota),{}^{RL}D^{\frac{1}{3}}z(\iota)\right)-\psi\left(\iota,z(\iota),I^{\frac{\sqrt{5}}{2}}z(\iota)\right)\right|\leq\varpi, \\ \left|{}_C D^{\frac{4}{3}}\left[{}^{RL}D^{\frac{1}{2}}z(\iota)\right]-\frac{3}{5}t^{e^{2t+1}}-\frac{1}{25e^{\pi}}\varphi\left(\iota,z(\iota),{}^{RL}D^{\frac{1}{3}}z(\iota)\right)-\psi\left(\iota,z(\iota),I^{\frac{\sqrt{5}}{2}}z(\iota)\right)\right|\leq\varpi F(t), \end{cases}$$

where

$$\begin{cases} \varphi\left(\iota,z(\iota),{}^{RL}D^{\frac{1}{3}}z(\iota)\right)=\frac{\sin z(\iota)}{2\sqrt{\pi}(120+\iota)}+\frac{\left|{}^{RL}D^{\frac{1}{3}}z(\iota)\right|}{2\sqrt{\pi}(120+\iota)\left(1+\left|{}^{RL}D^{\frac{1}{3}}z(\iota)\right|\right)}, \\ \psi\left(\iota,z(\iota),I^{\frac{\sqrt{5}}{2}}z(\iota)\right)=\frac{e^{-3\iota}}{240}\left(\cos z(\iota)+I^{\frac{\sqrt{5}}{2}}z(\iota)+\frac{1}{7}\right), \end{cases}$$

and $\vartheta=\frac{4}{3}, \delta=\frac{1}{2}, \Delta=\frac{1}{25e^{\pi}}, \beta=\frac{1}{3}, \alpha=\frac{\sqrt{5}}{2}, \omega=\frac{1}{5}, \gamma=\frac{e}{2}, \theta=\frac{\ln 2}{5}$.

We have

$$\nabla_1=1.162, \nabla_2=1.4674.$$

For all $t\in[0,1]$ and $y_j, z_j\in\mathbb{R}, j=1,2,$ we have

$$\left|\varphi\left(\iota,z_1(\iota),z_2(\iota)\right)-\varphi\left(\iota,y_1(\iota),y_2(\iota)\right)\right|\leq\frac{1}{240\sqrt{\pi}}\left(|z_1-y_1|+|z_2-y_2|\right),$$

and

$$\left|\psi\left(\iota,z_1(\iota),z_2(\iota)\right)-\psi\left(\iota,y_1(\iota),y_2(\iota)\right)\right|\leq\frac{1}{240}\left(|z_1-y_1|+|z_2-y_2|\right).$$

Hence condition (H_1) holds with $\mu_1=\frac{1}{240\sqrt{\pi}}$ and $\mu_2=\frac{1}{240}$ respectively. Thus condition

$$\left[\Delta\mu_1+\mu_2\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\right](\nabla_1+\nabla_2)=0.02134\,3<1,$$

is satisfied. Then (19) admits one solution on $[0,1]$ and is UHML-stable with

$$\|y-z\|_Z \leq \frac{\varpi}{\Gamma\left(\frac{11}{6}\right)}\left(E_{\frac{11}{6}}\left[\frac{1}{25e^{\pi}}.\frac{1}{240\sqrt{\pi}}\iota^{\frac{11}{6}}\right]+E_{\frac{11}{6}}\left[\frac{1}{240}\iota^{\frac{11}{6}}\right]\right), 0\leq\iota\leq 1.$$

Let $\chi(\iota)=\iota^{\frac{1}{2}}$,

$$I^{\frac{4}{3}+\frac{1}{2}}\left[\iota^{\frac{1}{2}}\right]=\frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{10}{3}\right)}\iota^{\frac{1}{2}+\frac{4}{3}+\frac{1}{2}}\leq\frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{10}{3}\right)}\iota^{\frac{1}{2}}=\sigma_\chi\chi(\iota).$$

Hence (18) is satisfied, then by theorem 13 the problem (19) is UHRML-stable with

$$\|y-z\|_Z \leq \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{10}{3}\right)}\varpi\iota^{\frac{1}{2}}\left(E_{\frac{11}{6}}\left[\frac{1}{25e^{\pi}}.\frac{1}{240\sqrt{\pi}}\iota^{\frac{11}{6}}\right]+E_{\frac{11}{6}}\left[\frac{1}{240}\iota^{\frac{11}{6}}\right]\right), 0\leq\iota\leq 1.$$

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